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ON THE TOLERANCE AVERAGING OF HEAT CONDUCTION FOR PERIODIC HEXAGONAL-TYPE COMPOSITES

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The paper deals periodic composites which two-dimensional material structure is determined by the anisotropic conductivity matrix invariant with respect to the pair of translations such that the periodic cell determined by them coincide with the shapely hexagon. Moreover, it will be assumed that the material structure is invariant over the π/3-rotations with the centers of every hexagon as the origin of the rotation. The main aim of this paper is to prove that the tolerance model of such composite, obtained by the orthogonalization method, is isotropic.

Keywords: Tolerance averaging, periodicity cell, shapely hexagon, orthogonalization method

1. ITRODUCTION

Throughout this contribution we shall deal with the microheterogeneous periodic hexagonal-type composite solid made of the perfectly bonded constituents. The behaviour of these solids will be restricted to the heat conduction problem based on the Fourier heat conduction law and will be inveatigated in the framework of the well known two-dimensional parabolic equation which, under denotations $\nabla = [\partial_1, \partial_2, 0]$, $\partial = [0, 0, \partial_3]$, $x = [x_1, x_2, 0]$, $z = x_3$, can be written in the form

$$c\partial_t w - \nabla \cdot (\mathbf{A}\nabla w) - \partial \cdot (a\partial w) + f = 0 \tag{1.1}$$

Symbol $w = w(\cdot)$ stand here for a temperature field defined in $\Omega \subset R^2$, $f = f(\cdot)$ be the known density of heat sources. In the above equation $c = c(\cdot)$ is the heat flux and

$$\mathbf{K}(x,z)=\begin{bmatrix}\mathbf{A}(x,z) & [0,0]^T\\ [0,0] & a(z)\end{bmatrix}$$

is the conductivity martrix. We shall also assume the heat flux continuity conditions in normal directions on the interfaces Γ, $\Gamma\subset\Omega$, between the constituents of the considered composite.

The well-known fact is that due to the discontinuous and highly oscillating form of functional coefficients $c(\cdot), A(\cdot)$, the direct application of (1.1) to the analysis of special problems in most cases is difficult. That is why the mentioned above heat conduction problem is usually replaced, under some additional assumption, by some other problems with more regular coefficients. The characteristing feature of these assumptions is that microstructure of considered material is characterized by a certain scalar parameter $\lambda>0$. In this case conductivity matrix A in (1.1) depends on λ, $\mathbf{K}=\mathbf{K}_\lambda$. In many cases, in which the local averaged temperature u can be introduced, the tolerace averaging technique approach is very useful to the averaging of the heat conduction problem (1.1). The tolerance averaged model obtained on this way consists of the system of differential equations with more regular coefiicients for averaged temperature $u=u(\cdot)\in SV_0^1(\Omega)$ and fluctuation amplitudes $w^A=w^A(\cdot,t)\in SV_0^1(\Omega)$, which are new basic unknowns. Just introduced functional space $SV_0^1(\Omega)$ is a certain new space consisting of slowly-varying functions. For particulars the reader is reffered to [1,2].

2. FORMULATION OF THE MODELLING EQUATIONS

There are known two methods via which the tolerace averaged model can be obtained. The first one is the method based of a new concept of extended stationary action principle introduced in [1]. This method have been resulted in many applications dealing functionally graded materials. To the periodic problems the orthogonalization metod explained in [2] usually have been applied. Since the subject of this paper is a certain periodic material structures we shall restrict ourselves to the tolerance model equations obtained by orthogonalization metod. To thos end following procedure explained in [2] we look for the temperature foield in the form

$$w(x,z,t)=u(x,z,t)+g^A(x)W^A(x,z,t) \tag{2.1}$$

Where $u(x,t)=\langle c\rangle^{-1}\langle cw\rangle(x,t)$ is the averaged temperature field and $W^A(x,t)$ are extra unknowns which are usually referred to as *fluctuation amplitudes.* Here and in the sequel $\langle\cdot\rangle$ stand for the averaged operator, [2], superscripts denoted by capital latins $A,B,...$ run over $1,2,...,N$, where N is a number of fluctuation amplitudes. *Shape functions* $g^A(x)$, caused by the periodic structure of the composite, should be periodic and satisfy some additional conditions like $\langle g^A \rangle = 0$ and $g^A \in O(\lambda)$, cf. [2]. Foolowing [2] the system of tolerance averaged equations will be rewritten in the form

$$\begin{aligned}
&\langle \mathbf{A} \rangle \nabla^2 u + \langle a \rangle \partial^2 u - \langle c \rangle \partial_t u + \langle \mathbf{A} \cdot \nabla g^A \rangle \nabla W^A = \langle f \rangle \\
&\langle c g^A g^B \rangle \partial_t W^B - \langle a g^A g^B \rangle \partial^2 W^B + \langle \nabla g^A \cdot \mathbf{A} \cdot \nabla g^B \rangle W^B + \\
&+ \langle \nabla g^A \cdot \mathbf{A} \rangle \nabla u = -\langle f g^A \rangle
\end{aligned} \tag{2.2}$$

Under the additional assumption that unknowns depends exclusively on x the above system ofequations reduces to the form

$$\begin{aligned}
&\langle \mathbf{A} \rangle \nabla^2 u - \langle c \rangle \partial_t u + \langle \mathbf{A} \cdot \nabla g^A \rangle \nabla W^A = \langle f \rangle \\
&\langle c g^A g^B \rangle \partial_t W^B + \langle \nabla g^A \cdot \mathbf{A} \cdot \nabla g^B \rangle W^B + \langle \nabla g^A \cdot \mathbf{A} \rangle \nabla u = -\langle f g^A \rangle
\end{aligned} \tag{2.3}$$

investigated in [3]. Following [3] we shall restrict considerations to the case of honeycomb-type composites satisfying two following assumptions:

Assumption 1. The material structure of the anisotropic conductor is invariant under $2\pi/3$ – rotations with respect to the center of a regular periodicity cell.

Assumption 2. The sequence $g^1,...,g^N$ of the shape functions is invariant under $2\pi/3$ – rotations with respect to the center of a regular periodicity cell.

We are to show that under the above assumptions tolerance equations (2.3) can be transformed to the equivalent system of equations with the isotopic coefficients.

3. ISOTROPIC PROPERTIES OF MODEL EQUATIONS

To transform tolerance equations (2.3) to the isotropic form we are to outline the line of approach similar to this presented in [3]. Firstly, we shall rewrite decomposition (2.1) in the form

$$w(x,z,t) = u(x,z,t) + g_r^a(x) W_r^a(x,z,t) \tag{3.1}$$

where indices a, r run over the sequences 1,2,,,,n and 1,2,3, respectively. In the decomposition mentioned above decomposition

$$g_{r+1}^a(x) = g_r^a(Qx) \tag{3.2}$$

where Q denote the matrix of rotation over $\pi/3$ and W_r^a are fluctuation amlitudes W^A also renumerated on the same way. Hence N=3n.
Now, instead of fluctuation amplitud W_r^a we shall introduce new amplitudes

$$U^a = W_1^a + W_2^a + W_3^a, \quad \mathbf{v}^a = \mathbf{t}^r W_r^a \tag{3.3}$$

where $\mathbf{t}^r$ are three vectors determines the basic hexagon. Formula (1.7) represents an invertible linear transformation between W_r^a and $U^a, \mathbf{v}^a$. Let us introduce the following averaged coefficients

$$\begin{aligned}
&\mathbf{A}_2^{ab} = \langle \nabla g_r^a \cdot \mathbf{A} \cdot \nabla g_s^b \rangle \mathbf{t}^r \otimes \mathbf{t}^s, \ \mathbf{B}^a = \langle \mathbf{A} \cdot \nabla g_r^a \rangle \otimes \mathbf{t}^r, \\
&\mathbf{C}_2^{ab} = \langle c g_r^a g_s^b \rangle \mathbf{t}^r \otimes \mathbf{t}^s, \quad \mathbf{K}_2^{ab} = \langle a g_r^a g_s^b \rangle \mathbf{t}^r \otimes \mathbf{t}^s, \\
&\bar{c}_2^{ab} = \langle c(g_1^a + g_2^a + g_3^a)(g_1^b + g_2^b + g_3^b) \rangle, \\
&\bar{a}_2^{ab} = \langle \nabla(g_1^a + g_2^a + g_3^a) \cdot \mathbf{A} \cdot \nabla(g_1^b + g_2^b + g_3^b) \rangle, \\
&f^a = \langle f(g_1^a + g_2^a + g_3^a) \rangle, \ \mathbf{f}^a = \langle f g_r^a \rangle \mathbf{t}^r,
\end{aligned} \tag{3.4}$$

where $\mathbf{t}^2 = Q\mathbf{t}^1$, $\mathbf{t}^3 = Q\mathbf{t}^2$, $\mathbf{t}^1 = Q\mathbf{t}^3$, are three vectors determines the basic hexagon. Rather simple manipulations yield to

$$\begin{aligned}
&\nabla \cdot \langle \mathbf{A} \rangle \cdot \nabla u + \mathbf{B}^a : \nabla \mathbf{v}^a - \langle c \rangle \partial_t u = \langle f \rangle \\
&\mathbf{C}_2^{ab} \cdot \dot{\mathbf{V}}^b - \mathbf{K}_2^{ab} \cdot \partial^2 \mathbf{v}^b + \mathbf{A}_2^{ab} \cdot \mathbf{v}^b + (\mathbf{B}^a)^T \cdot \nabla u = -\mathbf{f}^a \\
&\bar{c}_2^{ab} \dot{U}^b + \bar{a}_2^{ab} U^b = f^a
\end{aligned} \tag{3.5}$$

It can be proved that

$$\begin{aligned}
&\langle \mathbf{A} \rangle = k\mathbf{1}, \quad \mathbf{A}_2^{ab} = \hat{a}_2^{ab}\mathbf{1} + \breve{a}_2^{ab} \in, \ \mathbf{B}^a = \hat{b}^a \mathbf{1} + \breve{b}^a \in, \\
&\mathbf{C}_2^{ab} = \hat{c}_2^{ab}\mathbf{1} + \breve{c}_2^{ab} \in, \ \mathbf{K}_2^{ab} = \hat{k}_2^{ab}\mathbf{1} + \breve{k}_2^{ab} \in,
\end{aligned} \tag{3.6}$$

for

$$\begin{aligned}
&k = 0.5\langle tr\mathbf{A} \rangle, \ \hat{b}^a = \frac{3}{4}\langle (tr\mathbf{A})\nabla g_r^a \rangle \otimes \mathbf{t}^r, \ \breve{b}^a = \frac{3}{4}\langle (tr\mathbf{A})\nabla g_r^a \rangle \otimes \tilde{\mathbf{t}}^r, \\
&\hat{a}_2^{ab} = \mathbf{t}^r \cdot \langle \nabla g_r^a (tr\mathbf{A}) \nabla g_s^b \rangle \cdot \mathbf{t}^s, \ \breve{a}_2^{ab} = \frac{3}{8}\mathbf{t}^r \cdot \langle \nabla g_r^a (tr\mathbf{A}) \nabla g_s^b \rangle \cdot \tilde{\mathbf{t}}^s, \\
&\hat{c}_2^{ab} = \langle c g_r^a g_s^b \rangle \delta^{rs}, \ \breve{c}_2^{ab} = \langle c g_r^a g_s^b \rangle \in^{rs}, \ \hat{k}_2^{ab} = \langle a g_r^a g_s^b \rangle \delta^{rs}, \ \breve{k}_2^{ab} = \langle a g_r^a g_s^b \rangle \in^{rs}
\end{aligned} \tag{3.7}$$

Hence system of equations (3.5) is isotropic.

4. CONCLUDING REMARKS

Characteristic features of equations (3.5) can be listed as follows.

1° Model equations (3.5) are isotropic and hence we have proved the main thesis of this note that the averaged heat transfer response of the honeycomb-type rigid conductor is isotrpic even material properties of the conductor are anisotropic.

2° The obtained equations (3.5) include a special case in which every cell of the honeycomb-type considered conductor has a threefold axis of symmetry. Similar result in elastodynamic was obtained In [4].

3° As it has been mentioned at the end of Section 2, in the special case in which basic unknowns do not depend on *z* results obtained In this note reduces to results obtained in [3].

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UŚREDNIANIE TOLERANCYJNE PRZEWODNICTWA CIEPŁA W KOMPOZYTACH O PERIODYCZNEJ STRUKTURZE HEKSAGONALNEJ

S t r e s z c z e n i e

W pracy zaproponowano równoważną postać tolerancyjnie uśrednionego modelu przewodnictwa ciepła kompozytu o mikroperiodycznej strukturze materialnej typu plastra miodu. Założono niezmienniczość tej mikrostruktury względem obrotów o kąt $2\pi/3$ a także taką samą niezmienniczość stosowanych funkcji kształtu. W otrzymanej nowej postaci model ten ma współczynniki izotropowe.