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MODELLING OF THIN PERIODIC PLATES SUBJECTED TO LARGE DEFLECTIONS

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The objects under consideration are thin linear-elastic plates with periodic structure in planes parallel to the plate midplane, subjected to large (of the order of plate thickness) deflections. The main aim is to propose a mathematical model describing geometrically nonlinear problems of such plates, which is based on the tolerance averaging technique, cf. Woźniak et al. [3]. Results calculated for a special static problem by the tolerance model are compared with results obtained within the known tolerance linear model of thin plates.

Keywords: periodic plates, nonlinear bending, tolerance modelling

1. INTRODUCTION

In many engineering problems we deal with plates which are made of isotropic materials, but as a result of changing thickness or using two or more materials with different elastic properties their behaviour is similar to behaviour of anisotropic or orthotropic ones with discontinuities of geometric or/and material properties, cf. Fig. 1. It leads to governing equations of these plates, which have non-continuous, highly oscillating, functional coefficients. Exact solutions to these equations are very difficult to obtain. Therefore, various simplified approaches, introducing effective plate properties, are proposed. Amongst them there have to be mentioned models based on the asymptotic homogenization, e.g. homogenized model of periodic plates proposed by Kohn and Vogelius [2]. Unfortunately, governing equations of these models usually neglect the effect of the microstructure size on the plate behaviour. In this paper, in order to describe this effect, *the tolerance modelling approach* is applied.

The aim of this contribution is twofold: to derive governing equations of *the nonlinear tolerance model* of thin periodic plates subjected to large deflections, which take into account the effect of the microstructure size, and to compare some numerical results obtained by this model with those obtained by *the linear tolerance model* of thin plates.

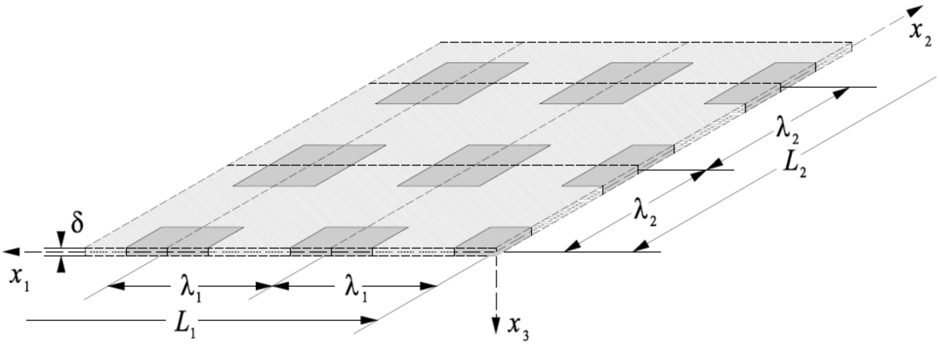

Fig. 1. A fragment of a thin periodic plate

2. FUNDAMENTAL EQUATIONS

Let $Ox_1x_2x_3$ be an orthogonal Cartesian coordinate system in the physical space. The time coordinate is denoted by t. Subscripts i, j, k, l run over 1, 2, 3 and α, β, γ, ω run over 1, 2. Setting $\mathbf{x}=(x_1,x_2)$ and $z=x_3$, it is assumed that the undeformed plate occupies the region $\Omega\equiv\{(\mathbf{x},z):-\delta(\mathbf{x})/2\leq z\leq\delta(\mathbf{x})/2,\ \mathbf{x}\in\Pi\}$, with midplane Π and the plate thickness $\delta(\cdot)$.

Periodic plates consist of many repetitive elements called *periodicity cells,* having identical geometric and material properties. Our considerations concern plates with a periodic structure along two directions. Then the periodicity cell can be defined as a plane region $\text{Ì}\equiv[-\lambda_1/2,\lambda_1/2]\times[-\lambda_2/2,\lambda_2/2]$, with λ_1, λ_2 being the cell dimensions along the x_1- and x_2-axis. The diameter of the periodicity cell, given by $\lambda=[(\lambda_1)^2+(\lambda_2)^2]^{1/2}$, is called *the microstructure parameter*. It is assumed that $\max(\delta)<<\lambda<<\min(L_1,L_2)$, where L_1, L_2 are characteristic dimensions of the plate along the x_1- and x_2-axis. For this reason each periodicity cell can be treated as a thin plate. Here and further the partial derivative with respect to a space coordinate is denoted by $\partial_\alpha=\partial/\partial x_\alpha$, and the derivative with respect to time t is denoted by an overdot.

The considerations are based on the well-known nonlinear theory of thin plates (cf. the book edited by Woźniak [4]). Denote a plate midplane deflection by $w(\mathbf{x},t)$, the in-plane displacements along the x_α-axes by $u_{0\alpha}(\mathbf{x},t)$, $\mathbf{x}\in\Pi$, $t\in(t_0,t_1)$, the mass density of the plate material per unit area by

$$\mu(\mathbf{x})=\int_{-\delta(\mathbf{x})/2}^{\delta(\mathbf{x})/2}\rho(\mathbf{x})dz\ ,$$

the elastic moduli tensor by a_{ijkl}, the components of shell and bending stiffnesses tensors by

$$b_{\alpha\beta\gamma\omega}(\mathbf{x})=\int_{-\delta(\mathbf{x})/2}^{\delta(\mathbf{x})/2} c_{\alpha\beta\gamma\omega}(\mathbf{x})dz\,,\quad d_{\alpha\beta\gamma\omega}(\mathbf{x})=\int_{-\delta(\mathbf{x})/2}^{\delta(\mathbf{x})/2} c_{\alpha\beta\gamma\omega}(\mathbf{x})z^2dz\,,$$

where $c_{\alpha\beta\gamma\omega}=a_{\alpha\beta\gamma\omega}-a_{\alpha\beta33}a_{33\gamma\omega}/a_{3333}$, and the total loadings in the z-axis by $q(\mathbf{x},t)$. Neglecting terms involving tangent and rotational inertia of the plate and loads tangent to the plate midplane, we obtain:

(i) the strain-displacement relations:

$$\begin{aligned}&E_{\alpha\beta}=E_{0\alpha\beta}+z\kappa_{\alpha\beta},\\&E_{0\alpha\beta}=\tfrac{1}{2}(\partial_\beta u_{0\alpha}+\partial_\alpha u_{0\beta}+\partial_\alpha w\partial_\beta w),\quad \kappa_{\alpha\beta}=-\partial_{\alpha\beta}w,\end{aligned}\tag{2.1}$$

(ii) the stress-strain relations:

$$S_{\alpha\beta}=c_{\alpha\beta\gamma\omega}E_{\gamma\omega},\tag{2.2}$$

(iii) the equilibrium equations expressed in terms of displacements $u_{0\alpha}$ and deflection w:

$$\begin{aligned}&\tfrac{1}{2}\partial_\beta[b_{\alpha\beta\gamma\omega}(\partial_\omega u_{0\gamma}+\partial_\gamma u_{0\omega}+\partial_\omega w\partial_\gamma w)]=0,\\&\tfrac{1}{2}\partial_\beta[b_{\alpha\beta\gamma\omega}(\partial_\omega u_{0\gamma}+\partial_\gamma u_{0\omega}+\partial_\omega w\partial_\gamma w)\partial_\alpha w]-\partial_{\alpha\beta}(d_{\alpha\beta\gamma\omega}\partial_{\gamma\omega}w)-\mu\ddot{w}-q=0.\end{aligned}\tag{2.3}$$

Let us introduce the action functional in the form:

$$\begin{aligned}&\mathrm{A}(w(\cdot),u_{0\alpha}(\cdot))=\\&=\int_\Pi\int_{t_0}^{t_1}\Lambda(\mathbf{y},w(\mathbf{y},t),\dot{w}(\mathbf{y},t),\partial_\alpha w(\mathbf{y},t),\partial_{\alpha\beta}w(\mathbf{y},t),u_{0\alpha}(\mathbf{y},t),\partial_\beta u_{0\alpha}(\mathbf{y},t))\,dtd\mathbf{y},\end{aligned}\tag{2.4}$$

with the lagrangean given by

$$\begin{aligned}\Lambda=\tfrac{1}{2}[&-\tfrac{1}{4}b_{\alpha\beta\gamma\omega}(\partial_\gamma u_{0\omega}+\partial_\omega u_{0\gamma}+\partial_\gamma w\partial_\omega w)(\partial_\beta u_{0\alpha}+\partial_\alpha u_{0\beta}+\partial_\alpha w\partial_\beta w)-\\&-d_{\alpha\beta\gamma\omega}\partial_{\gamma\omega}w\partial_{\alpha\beta}w+\mu\dot{w}\dot{w}]+qw,\end{aligned}\tag{2.5}$$

for which the Euler-Lagrange equations take the form:

$$\begin{aligned}&\frac{\partial\Lambda}{\partial u_{0\alpha}}-\partial_\beta\frac{\partial\Lambda}{\partial(\partial_\beta u_{0\alpha})}=0,\\&\frac{\partial\Lambda}{\partial w}-\partial_\alpha\frac{\partial\Lambda}{\partial(\partial_\alpha w)}+\partial_{\alpha\beta}\frac{\partial\Lambda}{\partial(\partial_{\alpha\beta}w)}-\frac{\partial}{\partial t}\frac{\partial\Lambda}{\partial\dot{w}}=0.\end{aligned}\tag{2.6}$$

Applying the principle of stationary action, i.e. (2.6) and (2.5), we obtain the fundamental system of equations of the nonlinear theory of plates (2.3), with highly oscillating, periodic, non-continuous functional coefficients $b_{\alpha\beta\gamma\omega}(\mathbf{x})$, $d_{\alpha\beta\gamma\omega}(\mathbf{x})$ and $\mu(\mathbf{x})$.

3. TOLERANCE MODELLING

To obtain averaged equations of periodic plates subjected to large deflections we apply the tolerance averaging technique, cf. the book edited by Woźniak et al. [3]. In the tolerance modelling some basic concepts, such as: an averaging operator, a tolerance-periodic function, a highly oscillating function, a slowly-varying function, defined and explained in this book are used. Some of them are recalled below.

3.1 Basic concepts

Let Ì($\mathbf{x}$)=$\mathbf{x}$+Ì, $\Pi_\Omega = \{\mathbf{x}\in \Pi : \Omega(\mathbf{x}) \subset \Pi\}$, be a cell at $\mathbf{x}\in \Pi_\Omega$. The *averaging operator* for an arbitrary integrable function f is defined by

$$< f >(\mathbf{x}) = \frac{1}{\lambda_1\lambda_2} \int\limits_{\Omega(\mathbf{x})} f(y_1, y_2) dy_1 dy_2, \quad \mathbf{x}\in \Pi_\Omega. \tag{3.1}$$

It can be shown that for periodic function f of $\mathbf{x}$, its averaged value calculated from (3.1) is constant.

Let $\partial^k f$ be the k-th gradient of function $f=f(\mathbf{x})$, $\mathbf{x}\in\Pi$, $k=0,1,\ldots,\alpha$, $\alpha\geq 0$, $\partial^k f\equiv f$ and $\tilde{f}^{(k)}(\cdot,\cdot)$ be a function defined in $\overline{\Pi}\times R^m$.

Function $f\in H^\alpha(\Pi)$ is called *the tolerance-periodic function* (with respect to cell Ì and tolerance parameter d), $f\in TP_d^\alpha(\Pi,\Omega)$, if for $k=0,1,\ldots,\alpha$, the following conditions hold:

(1°) $(\forall \mathbf{x}\in\Pi)(\exists \tilde{f}^{(k)}(\mathbf{x},\cdot)\in H^0(\Omega))[\| \partial^k f|_{\Pi_\mathbf{x}}(\cdot) - \tilde{f}^{(k)}(\mathbf{x},\cdot)\|_{H^0(\Pi_\mathbf{x})} \leq d]$,

(2°) $\int_{\Omega(\cdot)} \tilde{f}^{(k)}(\cdot,\mathbf{z}) d\mathbf{z}\in C^0(\overline{\Pi})$.

Function $\tilde{f}^{(k)}(\mathbf{x},\cdot)$ is referred as to *the periodic approximation of* $\partial^k f$ *in* Ì($\mathbf{x}$), $\mathbf{x}\in\Pi$, $k=0,1,\ldots,\alpha$.

Function $f\in H^\alpha(\Pi)$ is called *the slowly-varying function* (with respect to cell Ì and tolerance parameter d), $F\in SV_d^\alpha(\Pi,\Omega)$, if

(1°) $f\in TP_d^\alpha(\Pi,\Omega)$,

(2°) $(\forall \mathbf{x}\in\Pi)[\tilde{F}^{(k)}(\mathbf{x},\cdot)|_{\Omega(\mathbf{x})} = \partial^k F(\mathbf{x}),\ k=0,\ldots,\alpha]$.

It means that periodic approximation $\tilde{F}^{(k)}$ of $\partial^k F(\cdot)$ in Ì($\mathbf{x}$) is a constant function for every $\mathbf{x}\in\Pi$.

Function $\phi\in H^{\alpha}(\Pi)$ is called *the highly oscillating function* (with respect to cell Ì and tolerance parameter d), $\phi\in HO_{d}^{\alpha}(\Pi,\Omega)$, if

(1°) $\phi\in TP_{\delta}^{\alpha}(\Pi,\Omega)$,

(2°) $(\forall\mathbf{x}\in\Pi)\,[\tilde{\phi}^{(k)}(\mathbf{x},\cdot)|_{\Omega(\mathbf{x})}=\partial^{k}\tilde{\phi}(\mathbf{x}),\ k=0,1,...,\alpha]$.

Moreover, for every $F\in SV_{d}^{\alpha}(\Pi,\Omega)$ function $f=\phi F\in TP_{d}^{\alpha}(\Pi,\Omega)$ satisfies condition

(3°) $\tilde{f}^{(k)}(\mathbf{x},\cdot)|_{\Omega(\mathbf{x})}=F(\mathbf{x})\partial^{k}\tilde{\phi}(\mathbf{x})|_{\Omega(\mathbf{x})},\ k=1,...,\alpha$.

If α=0 then we denote $\tilde{f}\equiv\tilde{f}^{(0)}$.

By $h(\cdot)$ denote a highly oscillating function, $h\in HO_{d}^{\alpha}(\Pi,\Omega)$, defined on $\overline{\Pi}$, continuous together with gradient $\partial^{1}h$. Its gradient $\partial^{2}h$ is a piecewise continuous and bounded. Function $h(\cdot)$ is called *the fluctuation shape function* of the 2-nd kind, if it depends on λ as a parameter and satisfies conditions:

(1°) $\partial^{k}h\in O(\lambda^{\alpha-k})$ for $k=0,1,...,\alpha,\ \alpha=2,\ \partial^{0}h\equiv h$,

(2°) $<h>(\mathbf{x})\approx 0$ for every $\mathbf{x}\in\Pi_{\Omega}$.

Set of all fluctuation shape functions of the 2-nd kind is denoted by $FS_{d}^{2}(\Pi,\Omega)$. Condition (2°) can be replaced with $<\mu h>(\mathbf{x})\approx 0$ for every $\mathbf{x}\in\Pi_{\Omega}$, where $\mu>0$ is a certain tolerance-periodic function.

3.2 Fundamental assumptions

Following the book – edited by Woźniak, Michalak and Jędrysiak [5], and using the above concepts we introduce modelling assumptions.

The first assumption is *the micro-macro decomposition*, in which we assume for deflection $w(\mathbf{x},t)$ and in-plane displacements $u_{0\alpha}(\mathbf{x},t)$:

$$\begin{aligned}
&w(\mathbf{x},t)=W(\mathbf{x},t)+h^{A}(\mathbf{x})V^{A}(\mathbf{x},t), && A=1,...,N,\\
&u_{0\alpha}(\mathbf{x},t)=U_{\alpha}(\mathbf{x},t),\\
&W(\cdot,t),\ V^{A}(\cdot,t)\in SV_{d}^{2}(\Pi,\Omega), && \mathbf{x}\in\Pi,\ \ t\in(t_{0},t_{1}),\\
&U_{\alpha}\in SV_{d}^{1}(\Pi,\Omega), && h^{A}(\cdot)\in FS_{d}^{2}(\Pi,\Omega).
\end{aligned}\tag{3.2}$$

The basic kinematic unknowns $W(\cdot,t)$ and $U_{\alpha}(\cdot,t)$ are called *the macrodeflection* and *the in-plane macrodisplacements*, respectively, $V^{A}(\cdot,t)$ are additional basic kinematic unknowns, called *the fluctuation amplitudes*; $h^{A}(\cdot)$ are the known fluctuation shape functions.

The tolerance averaging approximation is the second modelling assumption, in which it is assumed that in the course of modelling terms $O(d)$ are negligibly small, e.g. in formulas:

$$\begin{aligned}
&< f >(\mathbf{x}) = < \bar{f} >(\mathbf{x}) + O(d), \qquad < fF >(\mathbf{x}) = < f >(\mathbf{x})F(\mathbf{x}) + O(d),\\
&< f\partial_\alpha(h^A F) >(\mathbf{x}) = < f\partial_\alpha h^A >(\mathbf{x})F(\mathbf{x}) + O(d),\\
&\mathbf{x}\in\Pi;\ \alpha = 1,2;\ A = 1,\ldots,N;\ 0 < d << 1;\\
&f \in TP_d^2(\Pi,\Omega),\ F\in SV_d^2(\Pi,\Omega),\ h^A\in FS_d^2(\Pi,\Omega).
\end{aligned} \tag{3.3}$$

We also assume a decomposition of the transversal load $q(\mathbf{x},t)$ in the form $q(\mathbf{x},t) = q^0(\mathbf{x},t) + \tilde{q}(\mathbf{x},t)$, where $q^0 = < q >$ is the slowly-varying averaged load, $q^0 \in SV_d^0(\Pi,\Omega)$, and $\tilde{q}$ is the oscillating part of the load, $\tilde{q} \in HO_d^1(\Pi,\Omega)$, and $< \tilde{q} > \cong 0$.

4. MODELLING PROCEDURE

4.1 Averaged description

Now, we apply the tolerance modelling to the action functional. Substituting the micro-macro decomposition (3.2) to formula (2.4) and using the averaging operator (3.1), bearing in mind assumptions (3.3), we obtain the tolerance averaged action functional

$$\begin{aligned}
&\mathrm{A}_h(W(\cdot),U_\alpha(\cdot),V^A(\cdot)) =\\
&= \int_\Pi\int_{t_0}^{t_1} < \Lambda_h >(\mathbf{y},W,\partial_\alpha W,\partial_{\alpha\beta}W,\dot{W},U_\alpha,\partial_\beta U_\alpha,V^A,\dot{V}^A)\,dt\,d\mathbf{y}.
\end{aligned} \tag{4.1}$$

Denoting

$$\begin{aligned}
&B_{\alpha\beta\gamma\omega} \equiv < b_{\alpha\beta\gamma\omega} >,\quad \hat{B}^{AB}_{\alpha\omega} \equiv < b_{\alpha\beta\gamma\omega}\partial_\gamma h^A \partial_\beta h^B > \lambda^{-2},\\
&\tilde{B}^{AB}_{\alpha\beta} \equiv < b_{\alpha\beta\gamma\omega}\partial_\gamma h^A \partial_\omega h^B > \lambda^{-2},\\
&\bar{B}^{ABCD} \equiv < b_{\alpha\beta\gamma\omega}\partial_\gamma h^A \partial_\omega h^B \partial_\alpha h^C \partial_\beta h^D > \lambda^{-4},\\
&D_{\alpha\beta\gamma\omega} \equiv < d_{\alpha\beta\gamma\omega} >,\quad \tilde{D}^A_{\alpha\beta} \equiv < d_{\alpha\beta\gamma\omega}\partial_{\gamma\omega} h^A >,\\
&\bar{D}^{AB} \equiv < d_{\alpha\beta\gamma\omega}\partial_{\gamma\omega} h^A \partial_{\alpha\beta} h^B >,\\
&m \equiv < \mu >,\quad \tilde{m}^{AB} \equiv < \mu h^A h^B > \lambda^{-4},\quad Q \equiv < q >,\quad \tilde{Q}^A \equiv < q h^A > \lambda^{-2},
\end{aligned} \tag{4.2}$$

the tolerance averaged lagrangean (2.8) takes the form

$$\begin{aligned}
&< \Lambda_h > = -\tfrac{1}{2}\{\tfrac{1}{2}\lambda^2 \tilde{B}^{AB}_{\alpha\beta} V^A V^B (\partial_\beta U_\alpha + \partial_\alpha U_\beta + \partial_\alpha W \partial_\beta W) +\\
&+ \tfrac{1}{4} B_{\alpha\beta\gamma\omega}(\partial_\beta U_\alpha + \partial_\alpha U_\beta + \partial_\alpha W \partial_\beta W)(\partial_\omega U_\gamma + \partial_\gamma U_\omega + \partial_\gamma W \partial_\omega W) +\\
&+ \lambda^2 \hat{B}^{AB}_{\alpha\omega} V^A V^B \partial_\alpha W \partial_\omega W + \tfrac{1}{4}\lambda^4 \bar{B}^{ABCD} V^A V^B V^C V^D +\\
&+ D_{\alpha\beta\gamma\omega}\partial_{\alpha\beta} W \partial_{\gamma\omega} W + 2\tilde{D}^A_{\alpha\beta} V^A \partial_{\alpha\beta} W + \bar{D}^{AB} V^A V^B -\\
&- m\dot{W}\dot{W} - \lambda^4 \tilde{m}^{AB} \dot{V}^A \dot{V}^B\} + QW + \lambda^2 \tilde{Q}^A V^A.
\end{aligned} \tag{4.3}$$

4.2 Model equations

Let us apply the principle of stationary action to the averaged action functional (4.1). The tolerance averaged lagrangean (4.3) has to satisfy the following system of Euler-Lagrange equations:

$$\begin{aligned}
&\frac{\partial < \Lambda_h >}{\partial U_\alpha} - \partial_\beta \frac{\partial < \Lambda_h >}{\partial(\partial_\beta U_\alpha)} = 0,\\
&\frac{\partial < \Lambda_h >}{\partial W} - \partial_\beta \frac{\partial < \Lambda_h >}{\partial(\partial_\beta W)} + \partial_{\alpha\beta} \frac{\partial < \Lambda_h >}{\partial(\partial_{\alpha\beta})} - \frac{\partial}{\partial t}\frac{\partial < \Lambda_h >}{\partial \dot{W}} = 0,\\
&\frac{\partial < \Lambda_h >}{\partial V^A} - \frac{\partial}{\partial t}\frac{\partial < \Lambda_h >}{\partial \dot{V}^A} = 0.
\end{aligned} \tag{4.4}$$

Combining (4.3) with (4.4) we obtain the following system of N+3 equations: two equations for the in-plane macrodisplacements $U_\alpha(\cdot,t)$, one equation for the macrodeflection $W(\cdot,t)$, and N equations for the fluctuation amplitudes $V^A(\cdot,t)$:

$$\begin{aligned}
&\tfrac{1}{2} B_{\alpha\beta\gamma\omega}\partial_\beta(\partial_\omega U_\gamma + \partial_\gamma U_\omega + \partial_\gamma W \partial_\omega W) + \tfrac{1}{2}\lambda^2 \tilde{B}^{AB}_{\alpha\beta}\partial_\beta(V^A V^B) = 0,\\
&\tfrac{1}{2} B_{\alpha\beta\gamma\omega}(\partial_\omega U_\gamma + \partial_\gamma U_\omega + \partial_\gamma W \partial_\omega W)\partial_{\alpha\beta} W + \tfrac{1}{2}\lambda^2 \tilde{B}^{AB}_{\alpha\beta} V^A V^B \partial_{\alpha\beta} W +\\
&+ \lambda^2 \hat{B}^{AB}_{\alpha\omega}\partial_\alpha(V^A V^B \partial_\omega W) - D_{\alpha\beta\gamma\omega}\partial_{\alpha\beta\gamma\omega} W - \tilde{D}^A_{\alpha\beta}\partial_{\alpha\beta} V^A - m\ddot{W} + Q = 0,\\
&\tfrac{1}{2}\lambda^2 \tilde{B}^{AB}_{\alpha\beta}(\partial_\beta U_\alpha + \partial_\alpha U_\beta + \partial_\alpha W \partial_\beta W) V^B + \lambda^2 \hat{B}^{AB}_{\alpha\omega}\partial_\alpha W \partial_\omega W V^B +\\
&+ \tfrac{1}{2}\lambda^4 \bar{B}^{ABCD} V^B V^C V^D + \tilde{D}^A_{\alpha\beta}\partial_{\alpha\beta} W + \bar{D}^{AB} V^B + \lambda^4 \tilde{m}^{AB}\ddot{V}^B - \lambda^2 \tilde{Q}^A = 0.
\end{aligned} \tag{4.5}$$

The above equations of the nonlinear tolerance model have constant coefficients and describe the effect of the microstructure size on the overall dynamic plate behaviour by terms involving parameter λ. Solutions to these equations have to be considered together with the boundary conditions for: the in-plane macrodisplacements $U_\alpha(\cdot,t)$, the macrodeflection $W(\cdot,t)$, and the fluctuation amplitudes $V^A(\cdot,t)$, and have a physical sense only if the following conditions hold for every time $t \in (t_0,t_1)$:

$$W(\cdot,t) \in SV^2_d(\Pi,\Omega),\ V^A(\cdot,t) \in SV^2_d(\Pi,\Omega),\ U_\alpha(\cdot,t) \in SV^1_d(\Pi,\Omega). \tag{4.6}$$

Let us recall equations of *the linear tolerance model* of thin periodic plates, which under denotations (4.2) can be written as following:

$$\begin{aligned}
&D_{\alpha\beta\gamma\omega}\partial_{\alpha\beta\gamma\omega} W + \tilde{D}^A_{\alpha\beta}\partial_{\alpha\beta} V^A + m\ddot{W} = Q,\\
&\tilde{D}^A_{\alpha\beta}\partial_{\alpha\beta} W + \bar{D}^{AB} V^B + \lambda^4 \tilde{m}^{AB}\ddot{V}^B = \lambda^2 \tilde{Q}^A,
\end{aligned} \tag{4.7}$$

cf. Jędrysiak [1], Woźniak, Michalak and Jędrysiak [5].

5. EXAMPLES OF APPLICATION

5.1 Formulation of the problem

The object under consideration is a simply supported rectangular plate with constant thickness δ and length dimensions L_1 and L_2 along the x_1- and x_2-axis, respectively. The plate is made of two isotropic materials, having Young's moduli E' and E'', and Poisson's ratio ν' and ν'', periodically distributed along the x_1- and x_2-axis. For the sake of simplicity we consider a static bending problem. Hence, the known load q, unknown macrodisplacements, macrodeflection and fluctuation amplitudes are now functions of the space coordinates x_1, x_2 only, i.e. $q=q(\cdot)$, $U_\alpha=U_\alpha(\cdot)$, $W=W(\cdot)$, $V^A=V^A(\cdot)$. Moreover, we assume only one fluctuation shape function in the form

$$h = h^1 = \lambda^2 \cos\frac{2\pi x_1}{\lambda_1}\cos\frac{2\pi x_2}{\lambda_2}\,, \tag{5.1}$$

satisfying the condition $<h>=0$. Let $q(\mathbf{x})$ be a slowly-varying function in $\mathbf{x}$. Hence,

$$\tilde{Q} \equiv 0\,. \tag{5.2}$$

Under the above-mentioned assumptions denotations (4.2) can be written in the form:

$$\begin{aligned}
&B_{\alpha\beta\gamma\omega} \equiv < b_{\alpha\beta\gamma\omega} >, && D_{\alpha\beta\gamma\omega} \equiv < d_{\alpha\beta\gamma\omega} >,\\
&\tilde{B}_{\alpha\beta} \equiv < b_{\alpha\beta\gamma\omega}\partial_\gamma h \partial_\omega h > \lambda^{-2}, && \hat{B}_{\alpha\omega} \equiv < b_{\alpha\beta\gamma\omega}\partial_\gamma h \partial_\beta h > \lambda^{-2},\\
&\bar{B} \equiv < b_{\alpha\beta\gamma\omega}\partial_\gamma h \partial_\omega h \partial_\alpha h \partial_\beta h > \lambda^{-4}, && \tilde{D}_{\alpha\beta} \equiv < d_{\alpha\beta\gamma\omega}\partial_{\gamma\omega} h >,\\
&\bar{D} \equiv < d_{\alpha\beta\gamma\omega}\partial_{\gamma\omega} h \partial_{\alpha\beta} h > .
\end{aligned} \tag{5.3}$$

Thus, tolerance averaged lagrangean (4.3) takes the form

$$\begin{aligned}
< \Lambda_h > =& -\tfrac{1}{2}\{D_{\alpha\beta\gamma\omega}\partial_{\alpha\beta}W\partial_{\gamma\omega}W + 2\tilde{D}_{\alpha\beta}\partial_{\alpha\beta}WV + \bar{D}V^2 +\\
&+ \tfrac{1}{4}B_{\alpha\beta\gamma\omega}(\partial_\beta U_\alpha + \partial_\alpha U_\beta + \partial_\alpha W \partial_\beta W)(\partial_\omega U_\gamma + \partial_\gamma U_\omega + \partial_\gamma W \partial_\omega W) +\\
&+ \tfrac{1}{2}\lambda^2 \tilde{B}_{\alpha\beta}(\partial_\beta U_\alpha + \partial_\alpha U_\beta + \partial_\alpha W \partial_\beta W)V^2 + \lambda^2 \hat{B}_{\alpha\omega}\partial_\alpha W \partial_\omega W V^2 + \tfrac{1}{4}\lambda^4 \bar{B} V^4\} +\\
&+ QW,
\end{aligned} \tag{5.4}$$

whereas equations (4.6) have the following form:

$$
\begin{aligned}
&\tfrac{1}{2} B_{\alpha\beta\gamma\omega}\partial_\beta(\partial_\omega U_\gamma + \partial_\gamma U_\omega + \partial_\gamma W \partial_\omega W) + \tfrac{1}{2}\lambda^2 \tilde{B}_{\alpha\beta}\partial_\beta(V^2) = 0,\\
&\tfrac{1}{2} B_{\alpha\beta\gamma\omega}(\partial_\omega U_\gamma + \partial_\gamma U_\omega + \partial_\gamma W \partial_\omega W)\partial_{\alpha\beta} W - D_{\alpha\beta\gamma\omega}\partial_{\alpha\beta\gamma\omega} W - \tilde{D}_{\alpha\beta}\partial_{\alpha\beta} V +\\
&+\tfrac{1}{2}\lambda^2 \tilde{B}_{\alpha\beta}\partial_{\alpha\beta} W V^2 + \lambda^2 \hat{B}_{\alpha\omega}\partial_\alpha(\partial_\omega W V^2) + Q = 0, \qquad (5.5)\\
&\tfrac{1}{2}\lambda^2 \tilde{B}_{\alpha\beta}(\partial_\beta U_\alpha + \partial_\alpha U_\beta + \partial_\alpha W \partial_\beta W)V + \lambda^2 \hat{B}_{\alpha\omega}\partial_\alpha W \partial_\omega W V + \tfrac{1}{2}\lambda^4 \bar{B} V^3 +\\
&+\tilde{D}_{\alpha\beta}\partial_{\alpha\beta} W + \bar{D} V = 0.
\end{aligned}
$$

On the other hand, *the linear tolerance model* of thin periodic plates, subjected to the static load with assumptions (5.1) and (5.2), is described by the equations:

$$
\left(D_{\alpha\beta\gamma\omega} - \frac{\tilde{D}_{\gamma\omega}\tilde{D}_{\alpha\beta}}{\bar{D}} \right)\partial_{\alpha\beta\gamma\omega} W = Q, \quad V = -\frac{\tilde{D}_{\alpha\beta}}{\bar{D}}\partial_{\alpha\beta} W \,, \qquad (5.6)
$$

both independent of *the microstructure parameter*.

It can be seen that equations (5.5) stand a system of coupled nonlinear differential equations, solutions to which are very difficult to obtain. In contrast, system (5.6) can be solved equation by equation, but it neglects the effect of the microstructure size.

5.2 Solutions to the tolerance models

Let us consider a simply supported square plate (L_1=L_2=L), made of two different isotropic materials. The periodicity cell is also square, and is defined as Ì≡[−λ/2,λ/2]×[−λ/2,λ/2], cf. Fig. 2.

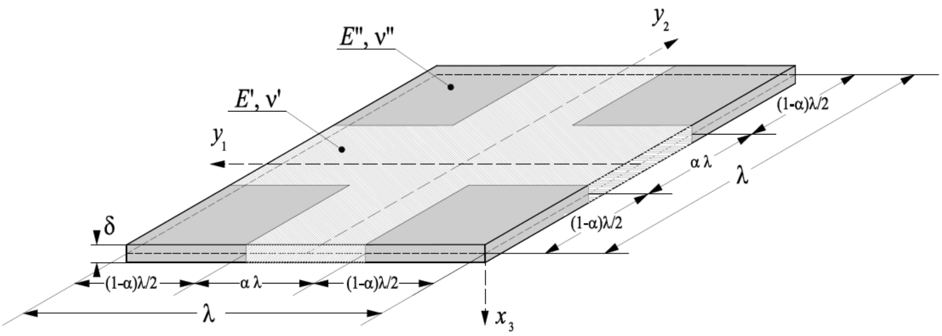

Fig. 2. A basic periodicity cell

It is assumed that the Young's modulus is given by

$$E(\mathbf{x})=\begin{cases} E' & \text{if} \quad \mathbf{x}\in[-\lambda/2,-\alpha\lambda/2)\times[-\alpha\lambda/2,\alpha\lambda/2]\cup \\ & \cup[-\alpha\lambda/2,\alpha\lambda/2]\times[-\lambda/2,\lambda/2]\cup \\ & \cup(\alpha\lambda/2,\lambda/2]\times[-\alpha\lambda/2,\alpha\lambda/2], \\ E''=\varepsilon E' & \text{if} \quad \mathbf{x}\in[-\lambda/2,-\alpha\lambda/2)\times[-\lambda/2,-\alpha\lambda/2)\cup \\ & \cup[-\lambda/2,-\alpha\lambda/2)\times(\alpha\lambda/2,\lambda/2]\cup \\ & \cup(\alpha\lambda/2,\lambda/2]\times[-\lambda/2,-\alpha\lambda/2)\cup \\ & \cup(\alpha\lambda/2,\lambda/2]\times(\alpha\lambda/2,\lambda/2], \end{cases}$$

where α is a dimensionless parameter describing distribution of material properties in the periodicity cell, cf. Fig. 2. However, Poisson's ratio is assumed to be the same for both materials, i.e. $\nu'=\nu''=\nu$. The fluctuation shape function (5.1) is given by $h=\lambda^2\cos(2\pi x_1/\lambda)\cos(2\pi x_2/\lambda)$. We assume that the load is given by the formula

$$q^0(x_1,x_2)=q_0\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}.$$

Solutions $W(\cdot)$, $U_\alpha(\cdot)$, $V^A(\cdot)$ have to satisfy boundary conditions for the simply supported plate with immovable edges, i.e. $W=\partial_{11}W=0$ for $x_1=0$, L; $W=\partial_{22}W=0$ for $x_2=0$, L; $V=\partial_{11}V=0$ for $x_1=0$, L, $V=\partial_{22}V=0$ for $x_2=0$, L, $U_1=U_2=0$ for $x_1=0$, L and for $x_2=0$, L. Therefore, denoting $\xi_m=m\pi/L$, $\zeta_n=n\pi/L$, solutions to (5.5) can be assumed in the form:

$$\begin{aligned} W(x_1,x_2)&=A_W^{mn}\sin\xi_m x_1\sin\zeta_n x_2, \\ U_\alpha(x_1,x_2)&=A_{U_\alpha}^{mn}\sin\xi_m x_1\sin\zeta_n x_2, \\ V(x_1,x_2)&=A_V^{mn}\sin\xi_m x_1\sin\zeta_n x_2, \end{aligned} \tag{5.7}$$

where A_W^{mn}, $A_{U_\alpha}^{mn}$, A_V^{mn} are new unknown constant parameters. Our investigations are restricted to the first approximations of (5.7). From the symmetry we conclude that $W(x_1,x_2)$ and $V(x_1,x_2)$ are even functions of x_1 and x_2, while $U_1(x_1,x_2)$ is an odd function of x_1 and $U_2(x_1,x_2)$ is an odd function of x_2. Hence, solutions (5.7) take the form as:

$$\begin{aligned} W(x_1,x_2)&=A_W\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}, \\ U_1(x_1,x_2)&=A_{U_1}\sin\frac{2\pi x_1}{L}\sin\frac{\pi x_2}{L}, \\ U_2(x_1,x_2)&=A_{U_2}\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L}, \\ V(x_1,x_2)&=A_V\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}. \end{aligned} \tag{5.8}$$

To find amplitudes A_W, $A_{U\alpha}$, A_V, the Ritz method can be used, cf. Timoshenko and Woinowsky-Krieger [6]. The conditions of the Ritz method take the form:

$$\frac{\partial V_{\max}}{\partial A_{U_1}} = 0\,, \quad \frac{\partial V_{\max}}{\partial A_{U_2}} = 0\,, \quad \frac{\partial V_{\max}}{\partial A_W} = 0\,, \quad \frac{\partial V_{\max}}{\partial A_w} = 0\,, \tag{5.9}$$

where $V_{\max}$ is the maximal strain energy of the plate. Using notations:

$$\begin{aligned}
&C_0 = 2\alpha - \alpha^2 + \varepsilon(\alpha^2 - 2\alpha + 1),\\
&C_1 = \lambda^2\{\frac{1-\cos(4\alpha\pi)}{2}(1-\varepsilon) + 4\pi^2[(2\alpha-\alpha^2) + (\alpha^2 - 2\alpha + 1)\varepsilon]\},\\
&C_4 = \lambda^4\{[\tfrac{63}{8}\pi^2 + 36\pi^4(2\alpha-\alpha^2) + 6\pi^3\sin(4\alpha\pi)(1-\alpha) +\\
&+\tfrac{1}{8}\pi^2\cos(8\alpha\pi) - 8\pi^2\cos(4\alpha\pi)](1-\varepsilon) + 36\pi^4\varepsilon\},\\
&C_5 = 4[1-\cos(2\alpha\pi)](1-\varepsilon),\\
&C_{10} = \{16\pi^4(2\alpha-\alpha^2) + 16\pi^3\sin(2\alpha\pi)(1-\alpha) +\\
&-2\pi^2[1-\cos(4\alpha\pi)]\}(1-\varepsilon) + 16\pi^4\varepsilon,\\
&C_{12} = \{16\pi^4(2\alpha-\alpha^2) - 16\pi^3\sin(2\alpha\pi)(1-\alpha) +\\
&-2\pi^2[1-\cos(4\alpha\pi)]\}(1-\varepsilon) + 16\pi^4\varepsilon,
\end{aligned} \tag{5.10}$$

coefficients (5.3) can be written as

$$\begin{aligned}
&B_{\alpha\beta\gamma\omega} = C_0 b'_{\alpha\beta\gamma\omega}, \quad \lambda^2\tilde{B}_{\alpha\beta} = C_1(b'_{\alpha\beta11} + b'_{\alpha\beta22}), \quad \lambda^2\hat{B}_{\beta\omega} = C_1(b'_{1\beta1\omega} + b'_{2\beta2\omega}),\\
&\lambda^4\overline{B} = C_4(b'_{1111} + b'_{2222}), \quad D_{\alpha\beta\gamma\omega} = C_0 d'_{\alpha\beta\gamma\omega}, \quad \tilde{D}_{\alpha\beta} = C_5(d'_{\alpha\beta11} + d'_{\alpha\beta22}),\\
&\overline{D} = C_{10}(d'_{1111} + 2d'_{1122} + d'_{2222}) + 4C_{12}d'_{1212},
\end{aligned} \tag{5.11}$$

and the maximal strain energy $V_{\max}$ takes the form

$$\begin{aligned}
&V_{\max} = \tfrac{1}{4}QA_W L^2 - \frac{C_0}{L^2}\frac{\delta E'}{(1-\nu^2)}\{\tfrac{1}{4}(\tfrac{5}{3}-\nu)\pi^2 L(A_{U_1} + A_{U_2})(A_W)^2 +\\
&+\tfrac{1}{16}(9-\nu)\pi^2L^2[(A_{U_2})^2 + (A_{U_1})^2] + \tfrac{8}{9}A_{U_1}A_{U_2}L^2(1+\nu) + \tfrac{5}{128}\pi^4(A_W)^4\} -\\
&-\frac{1}{L^2}\frac{\delta^3 E'}{(1-\nu^2)}\{\tfrac{1}{48}[C_{12}(1-\nu) + C_{10}(1+\nu)]L^4(A_V)^2 -\\
&-\tfrac{1}{24}\pi^2 C_5 A_W A_V L^2(1+\nu) + \tfrac{1}{24}\pi^4 C_0(A_W)^2\} +\\
&+\lambda^2\frac{\delta E'}{(1-\nu^2)}\{\tfrac{1}{3}C_1 L(1+\nu)(A_{U_1} + A_{U_2}) - \tfrac{3}{32}C_1\pi^2(A_W)^2 - \tfrac{1}{16}C_4\lambda^2L^2\}(A_V)^2.
\end{aligned} \tag{5.12}$$

Applying conditions (5.9) to polynomial (5.12) we obtain the system of four algebraic equations:

– two of the form

$$\begin{aligned}
&\frac{C_0}{L^2}\frac{\delta E'}{(1-\nu^2)}[\tfrac{1}{4}(\tfrac{5}{3}-\nu)\pi^2 L(A_W)^2+\tfrac{1}{8}(9-\nu)\pi^2L^2A_{U_1}+\tfrac{8}{9}A_{U_2}L^2(1+\nu)]-\\
&-\lambda^2\frac{\delta E'}{(1-\nu^2)}\tfrac{1}{3}C_1L(1+\nu)(A_V)^2=0,\\
&\frac{C_0}{L^2}\frac{\delta E'}{(1-\nu^2)}[\tfrac{1}{4}(\tfrac{5}{3}-\nu)\pi^2 L(A_W)^2+\tfrac{1}{8}(9-\nu)\pi^2L^2A_{U_2}+\tfrac{8}{9}A_{U_1}L^2(1+\nu)]-\\
&-\lambda^2\frac{\delta E'}{(1-\nu^2)}\tfrac{1}{3}C_1L(1+\nu)(A_V)^2=0,
\end{aligned}\tag{5.13}$$

which are linear in parameters A_{U_1} and A_{U_2}, and

– two of the form

$$\begin{aligned}
&\tfrac{1}{4}QL^2-\frac{C_0}{L^2}\frac{\delta E'}{(1-\nu^2)}[\tfrac{1}{2}(\tfrac{5}{3}-\nu)\pi^2L(A_{U_1}+A_{U_2})A_W+\tfrac{5}{32}\pi^4(A_W)^3]-\\
&-\lambda^2\frac{\delta E'}{(1-\nu^2)}\tfrac{3}{16}C_1\pi^2(A_V)^2A_W+\\
&+\frac{1}{L^2}\frac{\delta^3E'}{(1-\nu^2)}[\tfrac{1}{24}\pi^2C_5A_VL^2(1+\nu)-\tfrac{1}{12}\pi^4C_0A_W]=0,\\
&\lambda^2\frac{\delta E'}{(1-\nu^2)}[\tfrac{2}{3}C_1L(1+\nu)(A_{U_1}+A_{U_2})-\tfrac{3}{16}C_1\pi^2(A_W)^2-\tfrac{1}{8}C_4\lambda^2L^2]A_V-\\
&-\frac{1}{L^2}\frac{\delta^3E'}{(1-\nu^2)}\{\tfrac{1}{24}[C_{12}(1-\nu)+C_{10}(1+\nu)]L^4A_V-\tfrac{1}{24}\pi^2C_5A_WL^2(1+\nu)\}=0.
\end{aligned}\tag{5.14}$$

Amplitudes A_{U_1} and A_{U_2} can be derived from equations (5.13):

$$A_{U_1}=A_{U_2}=\frac{\tfrac{1}{4}(\nu-\tfrac{5}{3})\pi^2(A_W)^2+\tfrac{1}{3}\lambda^2L^2\dfrac{C_1}{C_0}(1+\nu)(A_V)^2}{[\tfrac{1}{8}(9-\nu)\pi^2+\tfrac{8}{9}(1+\nu)]L}.\tag{5.15}$$

Substituting obtained formulas for A_{U_1} and A_{U_2} to (5.14) we arrive at two nonlinear algebraic equations for constants A_W and A_V, which can be solved numerically for every special case.

In contrast, approximate solutions for *the linear tolerance model* of thin periodic plates, governed by equations (5.6), can be obtained by substituting $(5.8)_1$ and $(5.8)_4$ into (5.6). It leads to a system of linear algebraic equations for constants A_W, A_V. Solving these equations we arrive at:

$$\begin{aligned}
A_W&=\frac{1}{2}\frac{QL^4}{\pi^4}\frac{12(1-\nu^2)}{E'\delta^3}\frac{(1+\nu)C_{10}+(1-\nu)C_{12}}{2C_0[C_{10}(1+\nu)+C_{12}(1-\nu)]-(1+\nu)^2(C_5)^2},\\
A_V&=\frac{1}{2}\frac{QL^2}{\pi^2}\frac{12(1-\nu^2)}{E'\delta^3}\frac{(1+\nu)C_5}{2C_0[C_{10}(1+\nu)+C_{12}(1-\nu)]-(1+\nu)^2(C_5)^2}.
\end{aligned}\tag{5.16}$$

Obtained formulas can be now applied to a numerical analysis.

5.3 Calculational results

Assuming solutions of the considered problem in form (5.8), calculated constants A_W and A_V are values of macrodeflection $W(x_1,x_2)$ and fluctuation amplitude $V(x_1,x_2)$ at point $(x_1,x_2)=(L/2,L/2)$.

In calculations we assume that: the Young's modulus E'=200 GPa, the Poisson's ratio ν=0.3; the length L=1.0 m, the thickness $\delta=5\times10^{-3}$ m, the amplitude of the load is q_0=5 kPa.

Figures 3-7 show some numerical results.

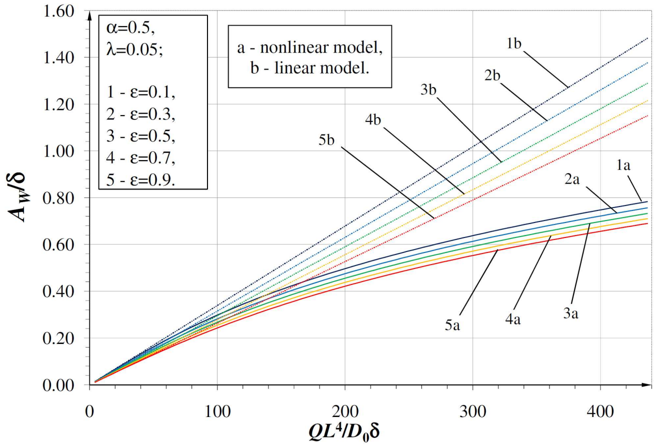

Fig. 3. Dimensionless ratio A_W/δ versus $QL^4/\delta D_0$, $D_0=E'\delta^3/12(1-\nu^2)$

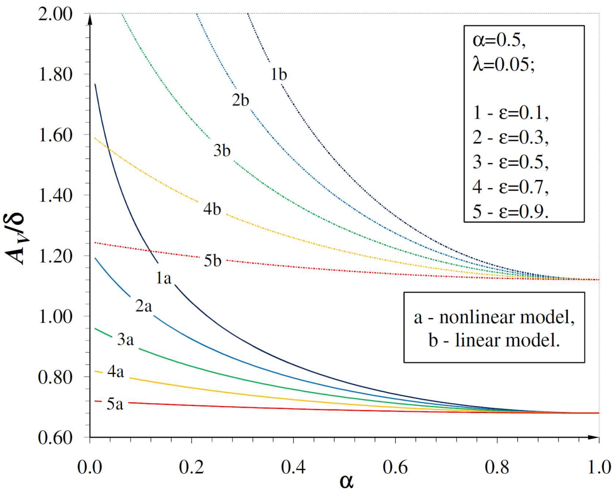

Fig. 4. Dimensionless ratio A_W/δ versus dimensionless parameter α

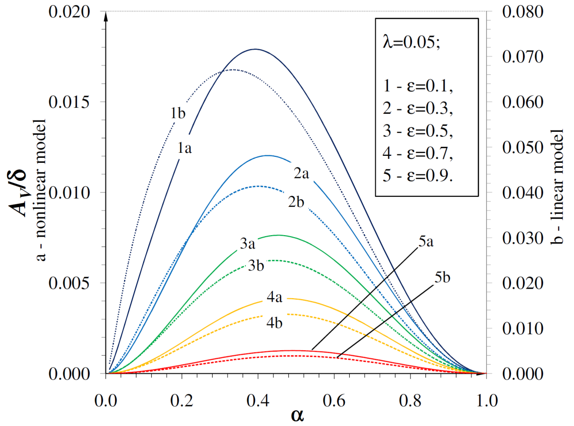

Fig. 5. Dimensionless ratio A_V/δ versus dimensionless parameter α

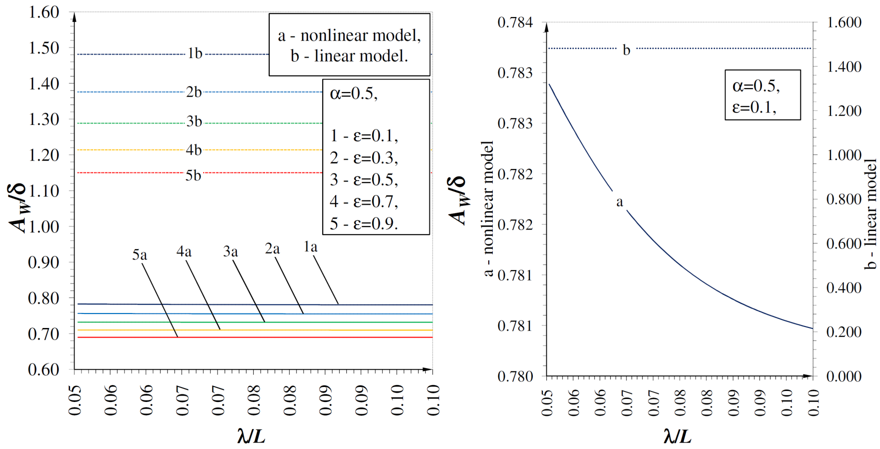

Fig. 6. Dimensionless ratio A_W/δ versus dimensionless ratio λ/L

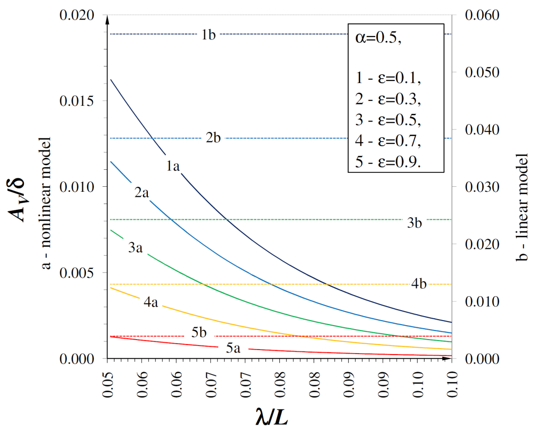

Fig. 7. Dimensionless ratio A_V/δ versus dimensionless ratio λ/L

6. FINAL REMARKS

In this note *the tolerance modelling technique* has been used to obtain governing equations with constant coefficients of *the tolerance model for thin periodic plates subjected to large deflections* instead of equations with highly oscillating, periodic, non-continuous coefficients.

Analyzing results presented as the diagrams in Figs. 3-7 it can be observed that:

- differences between the macrodeflection within the nonlinear tolerance model and linear tolerance model for deflections smaller than 1/5δ are less than 10% and increase with the increasing of the load, (Fig. 3);
- these differences increase with the decreasing values of the parameters α and ε, i.e. with the decreasing of the plate longitudinal and flexural rigidity, (Fig. 4);
- differences between the values of fluctuation amplitudes depend also on the value of the load and on the parameters α and ε, but the differences are lesser (in the case of the increasing load) or more distinct (in the case of the decreasing stiffness) than it is in the case of macrodeflections, (Fig. 5);
- Figs. 6 and 7 show that both the macrodeflection and the fluctuation amplitude considered within the framework of the nonlinear tolerance model are depended on ratio λ/L, while those obtained from the linear tolerance model are independent of λ.

The nonlinear tolerance model equations describe the effect of the microstructure size, while the linear model neglects this effect.

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MODELOWANIE CIENKICH PŁYT PERIODYCZNYCH
O DUŻYCH UGIĘCIACH

S t r e s z c z e n i e

W pracy rozpatrywane są cienkie, liniowo-sprężyste płyty o budowie periodycznej w płaszczyznach równoległych do płaszczyzny środkowej. Zagadnienia statyki i dynamiki tego rodzaju płyt w zakresie dużych ugięć opisane są układem równań różniczkowych nieliniowych o silnie oscylujących, periodycznych, nieciągłych współczynnikach (por. książka pod red. Woźniaka i in. [3]). W celu otrzymania równań o stałych współczynnikach zastosowano tu technikę tolerancyjnego uśredniania, omówioną w książce pod red. Woźniaka, Michalaka i Jędrysiaka [5]. Zaproponowano nieliniowy model tolerancyjny, opisujący nieliniowo-geometryczne zagadnienia cienkich płyt periodycznych. Model ten zastosowano do wyznaczenia ugięć dla danego obciążenia, a otrzymane wyniki porównano z wynikami uzyskanymi w ramach liniowego modelu tolerancyjnego.