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FREE VIBRATIONS OF THIN PLATES WITH TRANSVERSALLY GRADED STRUCTURE

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In this note vibrations of thin composite plates with a smooth and a slow gradation of macroscopic properties are considered. Plates of this kind have transversally graded macrostructure. In this paper certain averaged mathematical models of these plates are proposed. In an example, these models are applied to obtain fundamental free vibrations frequencies of a plate band, using the finite differences method.

Key words: thin plates, transversally graded structure, free vibrations

1. PRELIMINARIES

Objects under consideration are thin plates, which have on the macrolevel functionally graded structure in planes parallel to the plate midplane. However, they have tolerance-periodic microstructure, cf. Fig. 1. The microstructure size is assumed to be very small compared to characteristic length of the plate in the midplane.

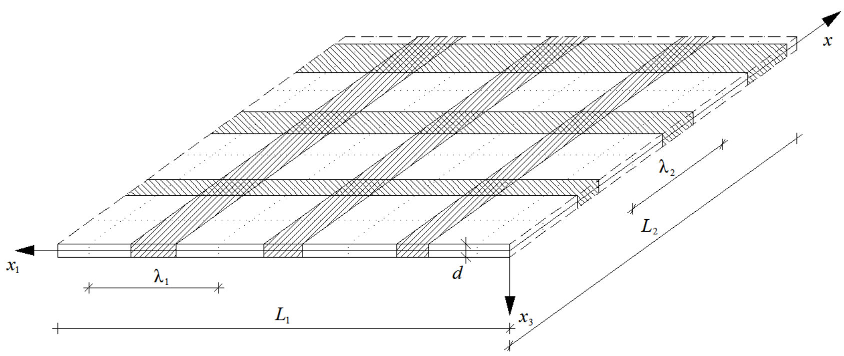

Fig. 1. A fragment of a thin plate with functionally (transversally) graded macrostructure

These plates are described by partial differential equations with highly oscillating, tolerance-periodic, non-continuous coefficients. Unfortunately, these equations are not a good tool to analyse special engineering problems. Hence, various averaged models are formulated, which describe plates of this kind by equations with smooth, slowly-varying coefficients. These plates can be treated to be made of *a functionally graded material*, cf. the book [12]. Plates of this kind can be called *transversally graded plates*.

Functionally graded materials and structures are usually described in the framework of approaches used to analyse macroscopically homogeneous media, e.g. periodic. Some of these methods are presented in the book [12]. Between them we have to mention models based on the asymptotic homogenization, cf. the book [7]. These models are applied for periodic plates in the paper [8]. However, the effect of the microstructure size is omitted in the governing equations of these models.

In order to take into account this effect we apply *the tolerance averaging technique* (cf. the books [13,9].). Applications of this technique to the modelling of dynamic problems of various periodic structures are shown in a series of papers, e.g. [1], [2,3], [10], [14]. The tolerance modelling is also adopted for nonstationary problems of functionally graded structures, e.g. for thin tolerance-periodic plates in [4]; for thin shallow tolerance-periodic shells in [6]; for skeletonal functionally graded shells in [11]; for annular plates with longitudinally graded structure in [15]; for transversally and longitudinally graded plates in [5].

The aims of this paper are two. Firstly, the tolerance and asymptotic models of the transversally graded plates are presented. Secondly, an application of the asymptotic model to analyse free vibration frequencies for a transversally graded plate band is shown. Results are obtained using the finite differences method.

2. MODELLING FOUNDATIONS

Let $Ox_1x_2x_3$ be the orthogonal Cartesian coordinate system and t be the time coordinate. Subscripts i,k,l run over $1,2,3$ and α,β,δ run over $1,2$. Denote $\mathbf{x}\equiv(x_1,x_2)$ and $z\equiv x_3$. Let the undeformed plate occupy the region $\Omega\equiv\{(\mathbf{x},z):-d(\mathbf{x})/2\le z\le d(\mathbf{x})/2,\mathbf{x}\in\Pi\}$, where Π is the midplane and $d(\cdot)$ is the plate thickness. By ∂_α denote derivatives of x_α, and also $\partial_{\alpha\ldots\delta}\equiv\partial_\alpha\ldots\partial_\delta$. Let $\Omega\equiv[-\lambda_1/2,\lambda_1/2]\times[-\lambda_2/2,\lambda_2/2]$ be the "basic cell" on Ox_1x_2 plane, where λ_1, λ_2 are cell length dimensions along the x_1-, the x_2-axis, respectively. The diameter of cell Ω, called *the microstructure parameter*, is denoted by $\lambda\equiv[(\lambda_1)^2+(\lambda_2)^2]^{1/2}$. This parameter satisfies condition $d_{max}<<\lambda<<\max(L_1,L_2)$. It is assumed that $d(\cdot)$

can be a tolerance-periodic function in **x** and all material and inertial properties of the plate, as a mass density $\rho=\rho(\cdot,z)$ and elastic moduli $a_{ijkl}=a_{ijkl}(\cdot,z)$, are tolerance-periodic functions in **x** and even functions in z. Let $w(\mathbf{x},t)$ ($\mathbf{x}\in\overline{\Pi}$, $t\in(t_0,t_1)$) be a plate midplane deflection and $a_{\alpha\beta\gamma\delta}$, $a_{\alpha\beta33}$, a_{3333} be the non-zero components of the elastic moduli tensor. Denote $c_{\alpha\beta\gamma\delta}\equiv a_{\alpha\beta\gamma\delta}-a_{\alpha\beta33}a_{\gamma\delta33}(a_{3333})^{-1}$.

The mean plate properties (being tolerance-periodic functions in **x**) – mass density μ and bending stiffnesses $B_{\alpha\beta\gamma\delta}$ – are defined in the form:

$$\mu\equiv\int_{-d/2}^{d/2}\rho dz,\qquad B_{\alpha\beta\gamma\delta}\equiv\int_{-d/2}^{d/2}z^2c_{\alpha\beta\gamma\delta}dz. \tag{2.1}$$

From the well-known assumptions of the Kirchhoff-type plates theory we obtain the partial differential equation of the fourth order for deflection $w(\mathbf{x},t)$ for transversally graded plates

$$\partial_{\alpha\beta}(B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w)+\mu\ddot{w}=0, \tag{2.2}$$

where coefficients are highly oscillating, non-continuous, tolerance-periodic functions. Because external loadings are neglected in equation (2.2) it describes free vibrations of transversally graded plates under consideration.

3. INTRODUCTORY CONCEPTS

In the modelling some concepts defined in the book [9] are used. Some of these concepts are reminded below. For tolerance-periodic plates some of them were also presented in [4].

A cell at $\mathbf{x}\in\Pi_\Omega$ is denoted by $\Omega(\mathbf{x})\equiv\mathbf{x}+\Omega$, $\Pi_\Omega=\{\mathbf{x}\in\Pi:\ \Omega(\mathbf{x})\subset\Pi\}$. The known *averaging operator* for an integrable function f is defined by

$$<f>(\mathbf{x})=\tfrac{1}{\lambda_1\lambda_2}\int_{\Omega(\mathbf{x})}f(y_1,y_2)dy_1dy_2,\quad \mathbf{x}\in\Pi_\Omega. \tag{3.1}$$

If f is a tolerance-periodic function in **x**, its averaged value calculated from (3.1) is a slowly-varying function in **x**.

Let us denote the k–th gradient of function $f=f(\mathbf{x}),\mathbf{x}\in\Pi,\ k=0,1,...,\alpha$, ($\alpha\geq0$), by $\partial^k f$; $\partial^0 f\equiv f$; and $\tilde{f}^{(k)}(\cdot,\cdot)$ be a function defined in $\overline{\Pi}\times R^m$.

Function $f\in H^\alpha(\Pi)$ is *the tolerance-periodic function,* $f\in TP^\alpha_\delta(\Pi,\Omega)$, if for $k=0,1,...,\alpha$ the following conditions are satisfied

(1°) $(\forall\mathbf{x}\in\Pi)\,(\exists\tilde{f}^{(k)}(\mathbf{x},\cdot)\in H^0(\Omega))\ [\|\partial^k f\big|_{\Pi_\mathbf{x}}(\cdot)-\tilde{f}^{(k)}(\mathbf{x},\cdot)\|_{H^0(\Pi_\mathbf{x})}\leq\delta]$,

(2°) $\int_{\Omega(\cdot)}\tilde{f}^{(k)}(\cdot,\mathbf{z})d\mathbf{z}\in C^0(\overline{\Pi})$.

Function $\tilde{f}^{(k)}(\mathbf{x},\cdot)$ is called *the periodic approximation of* $\partial^k f$ *in* $\Omega(\mathbf{x}),\mathbf{x}\in\Pi$, $k=0,1,...,\alpha$.

Function $F\in H^{\alpha}(\Pi)$ is *the slowly-varying function*, $F\in SV^{\alpha}_{\delta}(\Pi,\Omega)$, if

(1°) $F\in TP^{\alpha}_{\delta}(\Pi,\Omega)$,

(2°) $(\forall\mathbf{x}\in\Pi)\,[\tilde{F}^{(k)}(\mathbf{x},\cdot)|_{\Omega(\mathbf{x})}=\partial^k F(\mathbf{x}),\quad k=0,\ldots,\alpha]$.

Function $\phi\in H^{\alpha}(\Pi)$ is *the highly oscillating function*, $\phi\in HO^{\alpha}_{\delta}(\Pi,\Omega)$, if

(1°) $\phi\in TP^{\alpha}_{\delta}(\Pi,\Omega)$,

(2°) $(\forall\mathbf{x}\in\Pi)\,[\tilde{\phi}^{(k)}(\mathbf{x},\cdot)|_{\Omega(\mathbf{x})}=\partial^k\tilde{\phi}(\mathbf{x}),\quad k=0,1,...,\alpha]$.

(3°) $\forall F\in SV^{\alpha}_{\delta}(\Pi,\Omega)\ \ \exists f\equiv\phi F\in TP^{\alpha}_{\delta}(\Pi,\Omega)$

$\tilde{f}^{(k)}(\mathbf{x},\cdot)|_{\Omega(\mathbf{x})}=F(\mathbf{x})\partial^k\tilde{\phi}(\mathbf{x})|_{\Omega(\mathbf{x})},\quad k=1,\ldots,\alpha$.

For $\alpha=0$ let us denote $\tilde{f}\equiv\tilde{f}^{(0)}$.

Let $h(\cdot)$ be defined on $\overline{\Pi}$ a highly oscillating function, $h\in HO^{2}_{\delta}(\Pi,\Omega)$, continuous together with gradient $\partial^1 h$. However, gradient $\partial^2 h$ is a piecewise continuous and bounded. Function $h(\cdot)$ is *the fluctuation shape function* of the 2-nd kind, $FS^{2}_{\delta}(\Pi,\Omega)$, if it depends on λ as a parameter and conditions hold:

(1°) $\partial^k h\in O(\lambda^{\alpha-k})$ for $k=0,1,\ldots,\alpha$, $\alpha=2$, $\partial^0 h\equiv h$,

(2°) $<\mu h>(\mathbf{x})\approx 0$ for every $\mathbf{x}\in\Pi_{\Omega}$,

where $\mu>0$ is a certain tolerance-periodic function.

4. MODELLING ASSUMPTIONS

Following the monography [9] and using the introductory concepts we can formulate two fundamental modelling assumptions.

The first of them is *the micro-macro decomposition*, in which it is assumed that the plate deflection w can be decomposed in the form:

$$w(\mathbf{x},t)=W(\mathbf{x},t)+h^{A}(\mathbf{x})V^{A}(\mathbf{x},t),\quad A=1,\ldots,N,\quad \mathbf{x}\in\Pi,\qquad(4.1)$$

where $W(\cdot,t),V^{A}(\cdot,t)\in SV^{2}_{\delta}(\Pi,\Omega)$ (for every t) are basic kinematic unknowns, and $h^{A}(\cdot)\in FS^{2}_{\delta}(\Pi,\Omega)$. Function $W(\cdot,t)$ is called *the macrodeflection*; $V^{A}(\cdot,t)$ are called *the fluctuation amplitudes*; however, $h^{A}(\cdot)$ are the known fluctuation shape functions.

The tolerance averaging approximation is the second modelling assumption, in which terms $O(\delta)$ are assumed to be negligibly small in the course of modelling, e.g. in formulas:

$$
\begin{aligned}
&<\phi>(\mathbf{x}) = <\bar{\phi}>(\mathbf{x}) + O(\delta),\\
&<\phi F>(\mathbf{x}) = <\phi>(\mathbf{x})F(\mathbf{x}) + O(\delta),\\
&<\phi\partial_\alpha(h^A F)>(\mathbf{x}) = <\phi\partial_\alpha h^A>(\mathbf{x})F(\mathbf{x}) + O(\delta),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{x}\in\Pi;\ \alpha=1,2;\ A=1,\ldots,N;\ 0<\delta<<1;\\
&\phi\in TP^2_\delta(\Pi,\ ),\ F\in SV^2_\delta(\Pi,\ ),\ h^A\in FS^2_\delta(\Pi,\ ).
\end{aligned}
$$

5. TOLERANCE MODELLING

Following the book [9] the modelling procedure will be outlined below.

The starting point of the modelling is the formulation of the action functional (averaged over the plate thickness) in the form:

$$
\mathrm{A}(w(\cdot)) = \int_\Pi\int_{t_0}^{t_1}\Lambda(\mathbf{y},\partial_{\alpha\beta}w(\mathbf{y},t),\partial_\alpha w(\mathbf{y},t),\dot{w}(\mathbf{y},t),w(\mathbf{y},t))dtd\mathbf{y}, \tag{5.1}
$$

where the lagrangean Λ is given by:

$$
\Lambda = \tfrac{1}{2}(\mu\dot{w}\dot{w} - D_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w\partial_{\gamma\delta}w). \tag{5.2}
$$

The Euler-Lagrange equation

$$
\frac{\partial}{\partial t}\frac{\partial\Lambda}{\partial\dot{w}} - \frac{\partial\Lambda}{\partial w} - \partial_{\alpha\beta}\frac{\partial\Lambda}{\partial\,\partial_{\alpha\beta}w} = 0, \tag{5.3}
$$

written for lagrangean (5.2) leads to the fundamental equation of the plate theory (2.2), describing free vibrations for transversally graded plates.

Using the tolerance modelling to action functional (5.1) in the first step we substitute micro-macro decomposition (4.1) to action functional (5.1). In the next step, applying averaging operator (3.1) to the action functional we obtain the tolerance averaging of functional $\mathrm{A}(w(\cdot))$ in the form

$$
\mathrm{A}_h(W(\cdot),V^A(\cdot)) = \int_\Pi\int_{t_0}^{t_1}<\Lambda_h>(\mathbf{y},\partial_{\alpha\beta}W,\dot{W},\dot{V}^A,W,V^A)dtd\mathbf{y}, \tag{5.4}
$$

with the averaged form $<\Lambda_h>$ of lagrangean (5.2):

$$
\begin{aligned}
<\Lambda_h> = -\tfrac{1}{2}\{(&<D_{\alpha\beta\gamma\delta}>\partial_{\alpha\beta}W + 2<D_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^B>V^B)\partial_{\gamma\delta}W -\\
&-<\mu>\dot{W}\dot{W} + <D_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^A\partial_{\gamma\delta}h^B>V^AV^B - <\mu h^Ah^B>\dot{V}^A\dot{V}^B\}.
\end{aligned} \tag{5.5}
$$

The principle stationary action used to A_h leads to the following system of Euler-Lagrange equations:

$$\begin{aligned}&\frac{\partial<\Lambda_h>}{\partial\dot{W}}-\frac{\partial<\Lambda_h>}{\partial W}-\partial_{\alpha\beta}\frac{\partial<\Lambda_h>}{\partial(\partial_{\alpha\beta}W)}=0,\\&\frac{\partial<\Lambda_h>}{\partial\dot{V}^A}-\frac{\partial<\Lambda_h>}{\partial V^A}=0.\end{aligned}\tag{5.6}$$

Coefficients of equations (5.6) are slowly-varying functions in **x**.

6. MODEL EQUATIONS

Combining formula (5.5) with equations (5.6) the following system of equations for $W(\cdot,t)$ and $V^A(\cdot,t)$ is obtained:

$$\begin{aligned}&\partial_{\alpha\beta}(<B_{\alpha\beta\gamma\delta}>(\mathbf{x})\partial_{\gamma\delta}W+<B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}h^B>(\mathbf{x})V^B)+<\mu>(\mathbf{x})\ddot{W}=0,\\&<B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}h^A>(\mathbf{x})\partial_{\alpha\beta}W+<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^A\partial_{\gamma\delta}h^B>(\mathbf{x})V^B+\underline{<\mu h^A h^B>(\mathbf{x})}\ddot{V}^B=0.\end{aligned}\tag{6.1}$$

Equations $(6.1)_2$ involve the underlined terms with the microstructure parameter λ. The characteristic feature of equations (6.1) is that coefficients are slowly-varying functions in **x**. These equations and micro-macro decomposition (4.1) constitute *the tolerance model of thin transversally graded plates*, which describes the effect of the microstructure size on free vibrations of these plates. For a rectangular plate with midplane $\Pi=(0,L_1)\times(0,L_2)$ boundary conditions have to be formulated only for *the macrodeflection W* (on the edges $x_1=0$, L_1 and $x_2=0$, L_2), but not for *the fluctuation amplitudes* V^A, $A=1,\ldots,N$. These functions are slowly-varying in **x**.

It can be shown that $<\mu h^A h^B>\in O(\lambda^4)$. Thus, after neglecting the term with λ in equation $(6.1)_2$ the algebraic equations for the fluctuation amplitudes V^A are obtained:

$$V^A=-(<B_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^A\partial_{\gamma\delta}h^B>)^{-1}<B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}h^B>\partial_{\alpha\beta}W.\tag{6.2}$$

Substituting right-hand side of (6.2) into $(6.1)_1$ and denoting

$$B^h_{\alpha\beta\gamma\delta}\equiv<B_{\alpha\beta\gamma\delta}>-<B_{\alpha\beta\eta\kappa}\partial_{\eta\kappa}h^A>(<B_{\phi\mu\nu\pi}\partial_{\phi\mu}h^A\partial_{\nu\pi}h^B>)^{-1}<B_{\gamma\delta\chi\varepsilon}\partial_{\chi\varepsilon}h^B>$$

we arrive at the following equation for $W(\cdot,t)$:

$$\partial_{\alpha\beta}(B^h_{\alpha\beta\gamma\delta}(\mathbf{x})\partial_{\gamma\delta}W)+<\mu>(\mathbf{x})\ddot{W}=0.\tag{6.3}$$

Equation (6.3) together with micro-macro decomposition (4.1) represent *the asymptotic model of thin transversally graded plates*, which can be obtained in the framework of the formal asymptotic modelling procedure, cf. the book [9]. Equation (6.3) neglects the effect of the microstructure size on free vibrations of these plates.

7. APPLICATIONS – FREE VIBRATIONS OF A PLATE BAND

7.1. Introduction

Let us consider free vibrations of a thin plate band with span L along the x_1-axis, The plate band has a functionally graded material structure along its span. The material properties of the plate are assumed to be independent of the x_2-coordinate. A fragment of this plate is shown in Fig. 2.

The problem under consideration is treated as independent of the x_2-coordinate. Let us denote $x=x_1$, $z=x_3$, $x\in[0,L]$, $z\in[-d/2,d/2]$, where d is a constant plate thickness, and $\partial\equiv\partial_1$. It means that in this problem we assume the basic cell $\Omega\equiv[-\lambda/2,\lambda/2]$, in the interval $\Lambda\equiv[0,L]$, and λ, $\lambda<<L$, is the length of this cell. It is also assumed that $\lambda<< L$. A cell with a centre at $x\in[0,L]$ is denoted by $\Omega(x)\equiv[x-\lambda/2,x+\lambda/2]$.

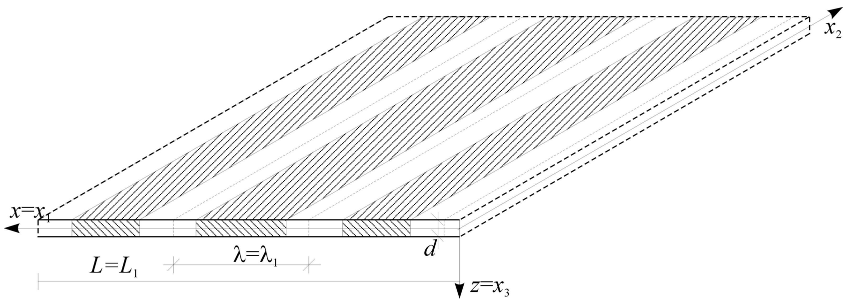

Fig. 2. A fragment of a thin transversally graded plate band

It is assumed that the plate band is made of two component elastic isotropic materials, characterised by: Young's moduli E', E'', Poisson's ratios ν', ν'' and mass densities ρ', ρ'', respectively. These materials are perfectly bonded across interfaces. It is assumed that $E(\cdot)$, $\rho(\cdot)$ are tolerance-periodic functions in x, $E(\cdot),\rho(\cdot)\in TP^0_\delta(\Lambda,\Omega)$, but Poisson's ratio $\nu\equiv\nu'=\nu''$ is constant. Hence, under condition $E'\neq E''$ and/or $\rho'\neq\rho''$ the material structure of the plate band can be treated as functionally graded along the x-axis.

Let us assume that the properties of the plate band are described by the following functions:

$$\rho(\cdot,z)=\begin{cases}\rho', & \text{for} \quad z\in((1-\gamma(x))\lambda/2,(1+\gamma(x))\lambda/2),\\ \rho'', & \text{for} \quad z\in[0,(1-\gamma(x))\lambda/2]\cup[(1+\gamma(x))\lambda/2,\lambda],\end{cases} \tag{7.1}$$

$$E(\cdot,z)=\begin{cases}E', & \text{for} \quad z\in((1-\gamma(x))\lambda/2,(1+\gamma(x))\lambda/2),\\ E'', & \text{for} \quad z\in[0,(1-\gamma(x))\lambda/2]\cup[(1+\gamma(x))\lambda/2,\lambda],\end{cases} \tag{7.2}$$

where $\gamma(x)$ is a distribution function of material properties, cf. Fig. 3.

We assume only one fluctuation shape function, i.e. $A=N=1$. Hence, we denote $h \equiv h^1$, $V \equiv V^1$. Micro-macro decomposition (4.1) of the field $w(x,t)$ takes the form:

$$w(x,t) = W(x,t) + h(x)V(x,t),$$

where $W(\cdot,t), V(\cdot,t) \in SV_\delta^2(\Lambda,\Omega)$ for every $t \in (t_0,t_1)$, $h(\cdot) \in FS_\delta^2(\Omega,\Omega)$.

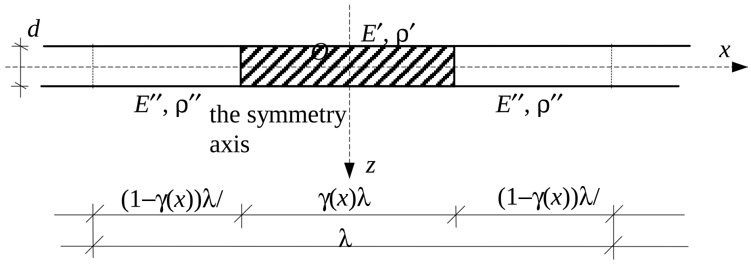

Fig. 3. A cell of the transversally graded plate band

For the cell shown in Fig. 3 the periodic approximation of the fluctuation shape function $h(x)$ can be assumed as

$$\tilde{h}(x,z) = \lambda^2[\cos(2\pi z/\lambda) + c(x)], \quad z \in \Omega(x), \quad x \in \Lambda,$$

where $c(x)$ is determined by $<\tilde{\mu}\tilde{h}>=0$. The parameter $c(x)$ is a slowly-varying function in x and has the form:

$$c = c(x) = \frac{\sin[\pi\tilde{\gamma}(x)](\rho' - \rho'')}{\pi\{\rho'\tilde{\gamma}(x) + \rho''[1 - \tilde{\gamma}(x)]\}},$$

where $\tilde{\gamma}(x)$ is the periodic approximation of the distribution function of material properties $\gamma(x)$. In calculations of derivatives $\partial\tilde{h}, \partial\partial\tilde{h}$ the parameter $c(x)$ is treated as constant. Hence:

$$\partial h(z) = -2\pi\lambda \sin(2\pi z/\lambda), \qquad \partial\partial h(z) = -4\pi^2 \cos(2\pi z/\lambda).$$

Using definitions of tolerance-periodic functions (2.1) for the plate band under consideration we have:

$$\mu(x) \equiv d\rho(x), \qquad B(x) \equiv \tfrac{d^3}{12(1-\nu^2)} E(x).$$

Introducing denotations:

$$\begin{aligned} &\hat{B} \equiv < B >, && B_{11} \equiv < B\partial\partial h >, && B^{11} = < B\partial\partial h\partial\partial h >, \\ &\hat{\mu} = < \mu >, && \mu^{11} = < \mu hh >, && \end{aligned} \tag{7.3}$$

tolerance model equations (6.1) take the form:

$$\begin{aligned}&\partial\partial(\hat{B}\partial\partial W + B_{11}V) + \hat{\mu}\ddot{W} = 0,\\&B_{11}\partial\partial W + B^{11}V + \mu^{11}\ddot{V} = 0.\end{aligned} \tag{7.4}$$

Equations (7.4) describe free vibrations of the transversally graded plate band under consideration in the framework of the tolerance model.

Moreover, using denotations (7.3) equation (6.3) has the form

$$\partial\partial[(\hat{B} - (B_{11})^2 / B^{11})\partial\partial W] + \hat{\mu}\ddot{W} = 0, \tag{7.5}$$

which describes free vibrations of this plate band within the asymptotic model. Coefficients of equations (7.4) and (7.5) are slowly-varying functions in x.

7.2. Example – the finite differences method applied to the asymptotic model equation

Our example is restricted to analyse free vibrations of the transversally graded plate band using only the asymptotic model described by equation (7.5). This equation has slowly-varying functional coefficients. Hence, the problem of free vibrations will be analysed using the finite differences method.

Let us assume the solution to equation (7.5) in the form

$$W(x,t) = U(x)\cos(\omega t), \tag{7.6}$$

where ω is a free vibrations frequency, $U(\cdot)$ is so called eigenvalue function for the macrodeflection W, satisfying boundary conditions for x=0, L. Substituting (7.6) into equation (7.5) we have

$$\partial\partial[(\hat{B} - (B_{11})^2 / B^{11})\partial\partial U] - \omega^2\hat{\mu}U = 0. \tag{7.7}$$

Let us consider two cases of the boundary conditions for the plate band:

1) the simply supported plate band

$$\begin{aligned}&U(x=0)=0, &&\partial\partial U(x=0)=0,\\&U(x=L)=0, &&\partial\partial U(x=L)=0;\end{aligned} \tag{7.8}$$

2) the clamped plate band

$$\begin{aligned}&U(x=0)=0, &&\partial U(x=0)=0,\\&U(x=L)=0, &&\partial U(x=L)=0.\end{aligned} \tag{7.9}$$

Introducing denotations:

$$\begin{aligned}&A^{(0)} = \hat{B}, && A^{(1)} = \partial\hat{B}, && A^{(2)} = \partial\partial\hat{B}, &&\\&D^{(0)} = B_{11}, && D^{(1)} = \partial B_{11}, && D^{(2)} = \partial\partial B_{11}, && C^{(0)} = \hat{\mu},\\&F^{(0)} = B^{11}, && F^{(1)} = \partial B^{11}, && F^{(2)} = \partial\partial B^{11}, &&\end{aligned}$$

equation (7.7) can be written in the form

$$
\begin{aligned}
&[A^{(0)}-\frac{(D^{(0)})^2}{F^{(0)}}]\partial\partial\partial\partial U+2[A^{(1)}-2\frac{D^{(0)}}{F^{(0)}}D^{(1)}+\frac{(D^{(0)})^2}{(F^{(0)})^2}F^{(1)}]\partial\partial\partial U+\\
&+[A^{(2)}-2\frac{D^{(0)}D^{(2)}+(D^{(1)})^2}{F^{(0)}}+\frac{(D^{(0)})^2F^{(2)}+4D^{(0)}D^{(1)}F^{(1)}}{(F^{(0)})^2}-\\
&\qquad-2\frac{(D^{(0)})^2}{(F^{(0)})^3}(F^{(1)})^2]\partial\partial U-\omega^2C^{(0)}U=0.
\end{aligned}
\tag{7.10}
$$

In the finite differences method interval $\Lambda=[0,L]$ is divided on n intervals. Derivatives of function $g=g(x)$, $x\in\Lambda$, with respect to x can be written as differences:

$$
\begin{aligned}
&g(x)=g_i, \qquad \partial g(x)=\tfrac{1}{2\Delta x}(g_{i+1}-g_{i-1}),\\
&\partial\partial g(x)=\tfrac{1}{(\Delta x)^2}(g_{i+1}-2g_i+g_{i-1}),
\end{aligned}
\tag{7.11}
$$

where Δx is an increment of argument x, $i=0,1,\dots n$. Hence, using formulas (7.11) equation (7.10) takes the form of system of differences equations:

$$
\begin{aligned}
&[A^{(0)}{}_i-\frac{(D^{(0)}{}_i)^2}{F^{(0)}{}_i}]\tfrac{U_{i+2}-4(U_{i+1}+U_{i-1})+6U_i+U_{i-2}}{(\Delta x)^4}+\\
&+[A^{(1)}{}_i-2\frac{D^{(0)}{}_i}{F^{(0)}{}_i}B^{(1)}{}_i+\frac{(D^{(0)}{}_i)^2}{(F^{(0)}{}_i)^2}F^{(1)}{}_i]\tfrac{U_{i+3}-3(U_{i+1}-U_{i-1}-U_{i+2}+U_{i-2})-U_{i-3}}{(\Delta x)^3}+\\
&+[A^{(2)}{}_i-2\frac{D^{(0)}{}_iD^{(2)}{}_i+(D^{(1)}{}_i)^2}{F^{(0)}{}_i}+\frac{(D^{(0)}{}_i)^2F^{(2)}{}_i+4D^{(0)}{}_iD^{(1)}{}_iF^{(1)}{}_i}{(F^{(0)}{}_i)^2}-\\
&\qquad-2\frac{(D^{(0)}{}_i)^2}{(F^{(0)}{}_i)^3}(F^{(1)}{}_i)^2]\tfrac{U_{i+1}-2U_i+U_{i-1}}{(\Delta x)^2}-\omega^2C^{(0)}{}_iU_i=0.
\end{aligned}
\tag{7.12}
$$

Formula (7.12) is a system of algebraic linear equations for unknowns U_i. Using boundary conditions (7.8) and (7.9) we have:

1) for the simply supported plate band

$$
\begin{aligned}
&U_0=0, \qquad \tfrac{U_1-2U_0+U_{-1}}{(\Delta x)^2}=0,\\
&U_n=0, \qquad \tfrac{U_{n+1}-2U_n+U_{n-1}}{(\Delta x)^2}=0;
\end{aligned}
\tag{7.13}
$$

2) for the clamped plate band

$$
\begin{aligned}
&U_0=0, \qquad \tfrac{U_1-U_{-1}}{2\Delta x}=0,\\
&U_n=0, \qquad \tfrac{U_{n+1}-U_{n-1}}{2\Delta x}=0.
\end{aligned}
\tag{7.14}
$$

Equations (7.12) with formulas (7.13) or (7.14) make it possible to find values U_i, $i=1,\dots n-1$.

7.3. Results

Let us consider three distribution functions of material properties $\gamma(x)$, assuming their periodic approximations in the form:

– the 1st case (ϕ=1)

$$\tilde{\gamma}(x) = \sin^2(\pi x / L)\,; \tag{7.15}$$

– the 2nd case (ϕ=2)

$$\tilde{\gamma}(x) = \cos^2(\pi x / L)\,; \tag{7.16}$$

– the 3rd case (ϕ=3)

$$\tilde{\gamma}(x) = (2x / L - 1)^2\,. \tag{7.17}$$

Let a nondimensional frequency parameter be given by:

$$\Omega^2 \equiv \frac{12(1-\nu^2)\rho'}{E'} L^2 \omega^2, \tag{7.18}$$

with the free vibrations frequency ω determined by the finite differences method from equation (7.10).

Calculational results are shown as plots in Figs. 4-9. Curves presented in Figs. 4-6 are made for the simply supported plate band, but in Figs. 7-9 – for the clamped plate band. Figs. 4 and 7 show results for the distribution function of material properties $\gamma(x)$ given by (7.15) (ϕ=1), Figs. 5 and 8 – for the function $\gamma(x)$ given by (7.16) (ϕ=2), Figs. 6 and 9 – for the function $\gamma(x)$ given by (7.17) (ϕ=3). Plots of the frequency parameters versus ratio ρ''/ρ' are presented in Figs. 4a, 5a, 6a, 7a, 8a, 9a (for E''/E'=0.25, 0.5, 0.75, 1), but in Figs. 4b, 5b, 6b, 7b, 8b, 9b there are shown plots of the frequency parameters versus ratio E''/E' (for ρ''/ρ'=0.08, 0.51, 0.74, 1). All these results are calculated for the Poisson's ratio ν=0.3 and the ratios: λ/L=0.1, d/λ=0.01.

a) b)

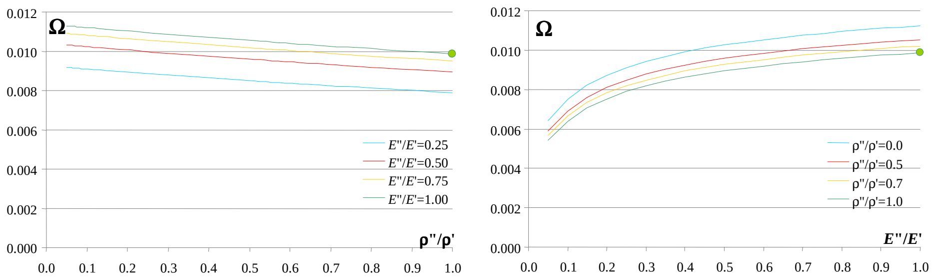

Fig. 4. Plots of the lower frequency parameters Ω for the simply supported plate band and function $\gamma(x)$ denoted by ϕ=1 versus: a) the ratio of mass densities ρ''/ρ', b) the ratio of Young's moduli E''/E' (ν=0.3, ratio λ/L=0.1, ratio d/λ=0.01)

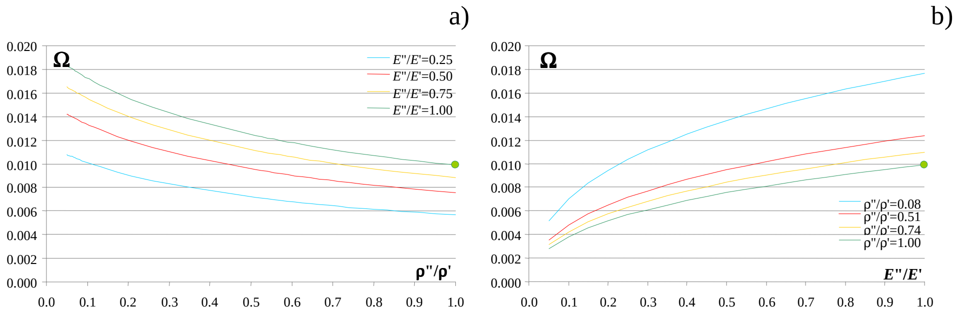

Fig. 5. Plots of the lower frequency parameters Ω for the simply supported plate band and function $\gamma(x)$ denoted by ϕ=2 versus: a) the ratio of mass densities ρ''/ρ', b) the ratio of Young's moduli E''/E' (ν=0.3, ratio λ/L=0.1, ratio d/λ=0.01)

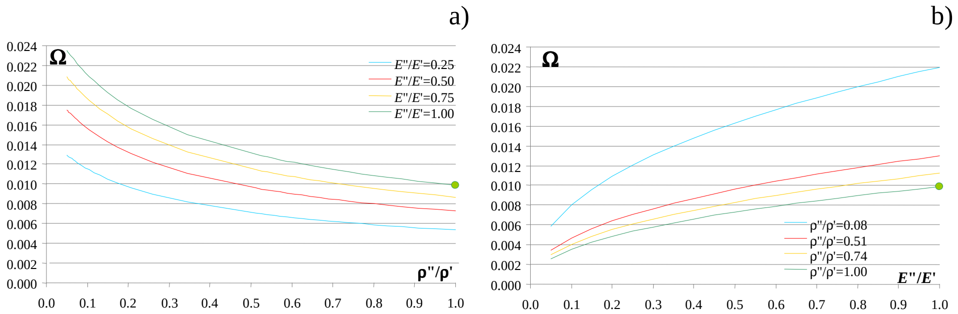

Fig. 6. Plots of the lower frequency parameters Ω for the simply supported plate band and function $\gamma(x)$ denoted by ϕ=3 versus: a) the ratio of mass densities ρ''/ρ', b) the ratio of Young's moduli E''/E' (ν=0.3, ratio λ/L=0.1, ratio d/λ=0.01)

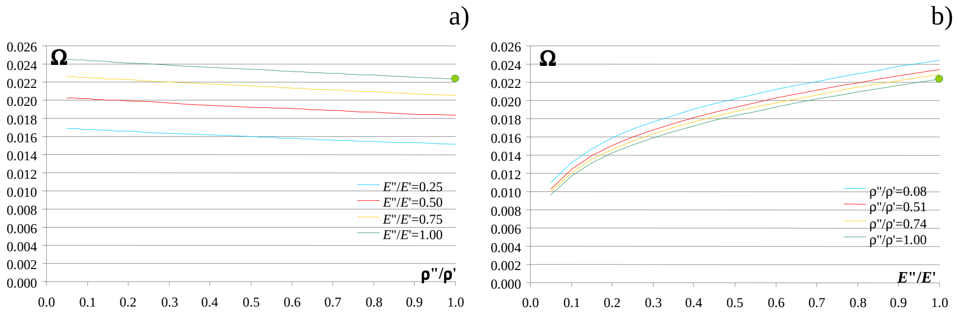

Fig. 7. Plots of the lower frequency parameters Ω for the clamped plate band and function $\gamma(x)$ denoted by ϕ=1 versus: a) the ratio of mass densities ρ''/ρ', b) the ratio of Young's moduli E''/E' (ν=0.3, ratio λ/L=0.1, ratio d/λ=0.01)

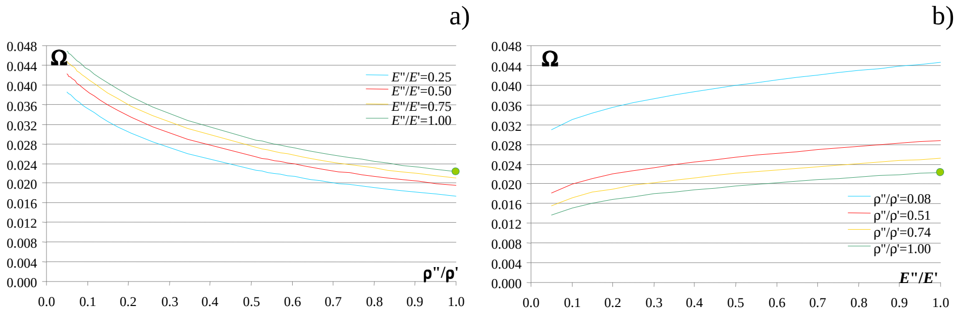

Fig. 8. Plots of the lower frequency parameters Ω for the clamped plate band and function $\gamma(x)$ denoted by $\phi=2$ versus: a) the ratio of mass densities ρ''/ρ', b) the ratio of Young's moduli E''/E' ($\nu=0.3$, ratio $\lambda/L=0.1$, ratio $d/\lambda=0.01$)

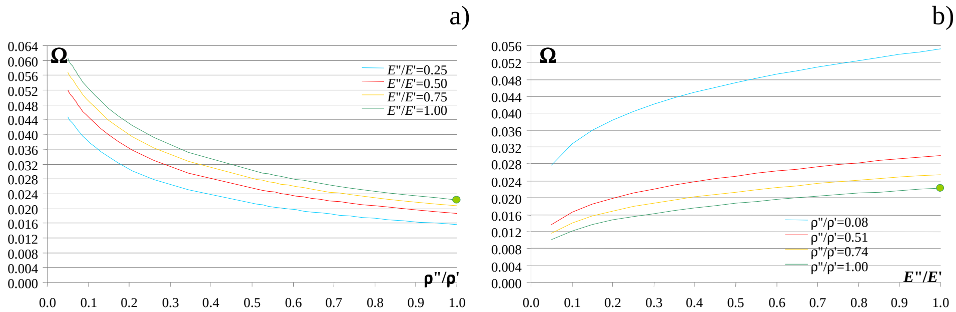

Fig. 9. Plots of the lower frequency parameters Ω for the clamped plate band and function $\gamma(x)$ denoted by $\phi=3$ versus: a) the ratio of mass densities ρ''/ρ', b) the ratio of Young's moduli E''/E' ($\nu=0.3$, ratio $\lambda/L=0.1$, ratio $d/\lambda=0.01$)

Analysing results presented in Figs. 4-9 the following remarks can be formulated:

1) values of the lower free vibrations frequencies depend on ratios E''/E' and ρ''/ρ', i.e.:
   a. they increase with the increasing of the ratio E''/E' (cf. Figs. 4b, 5b, 6b, 7b, 8b, 9b),
   b. they decrease with the increasing of the ratio ρ''/ρ' (cf. Figs. 4a, 5a, 6a, 7a, 8a, 9a);
2) values of the lower free vibrations frequencies depend on distribution functions of material properties $\gamma(x)$ under consideration, e.g.:
   a. the highest values of these frequencies are obtained for the 3rd case ($\phi=3$) given by (7.17), cf. Figs. 6, 9,
   b. the smallest values of these frequencies – for the 1st case ($\phi=1$) given by (7.15), cf. Figs. 4, 7;

3) differences between the values of the free vibrations frequencies for various ratios ρ''/ρ' depend on the distribution functions of material properties $\gamma(x)$, e.g. for the 1st case (ϕ=1) these differences are smaller than for the other cases (ϕ=2,3), cf. Figs. 4, 7.

8. FINAL REMARKS

Using the tolerance modelling to the known differential equation of Kirchhoff-type plates with a transversally graded macrostructure the tolerance model equations are derived. In the framework of this technique the governing differential equation with non-continuous, tolerance-periodic coefficients is replaced by the system of differential equations with slowly-varying coefficients. The characteristic feature of the obtained tolerance model equations is that they describe the effect of the microstructure size on the overall behaviour of transversally graded plates under consideration. On the other hand, the asymptotic model neglects this effect and describes these plates only on the macrolevel.

Our considerations in the presented example are restricted to analyse only lower free vibrations frequencies, which can be calculated within both the proposed models – tolerance and asymptotic. From this example it can be observed that these frequencies decrease with the increasing of the ratio of the mass densities ρ''/ρ' and increase with the increasing of the ratio of the Young's moduli E''/E'. These frequencies depend also on the distribution function of material properties $\gamma(x)$.

Other special problems of vibrations for the transversally graded plates and some comparisons with results obtained within other methods will be presented in forthcoming papers.

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DRGANIA SWOBODNE PŁYT CIENKICH O POPRZECZNEJ GRADACJI WŁASNOŚCI

S t r e s z c z e n i e

W pracy rozpatrywane są drgania cienkich płyt kompozytowych, charakteryzujących się „wolną” zmianą własności makroskopowych (uśrednionych). Płyty nazywane są płytami o poprzecznej gradacji własności. Zaproponowano pewne matematyczne modele takich płyt. Następnie, stosując te modele obliczono podstawowe częstości drgań swobodnych pasma płytowego, wykorzystując metodę różnic skończonych.