RHEOLOGICAL REDISTRIBUTION OF STRESSES IN MULTI-LAYERED BEAMS

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The paper concerns problems of rheology in wooden structures reinforced with polyester-glass composite bars glued to them by means of epoxy resin. A theoretical model of the behaviour of a multi-layered beam is presented. The component materials of this beam are described with equations for linear viscoelastic media (five- and six-parameter rheological models). The Bernoulli's plane sections hypothesis has been assumed and this assumption includes the statement that the particular layers are perfectly composed with no allowances for creep. Equations for stresses have been derived. To verify the elaborated theoretical model, a program of experimental tests has been prepared. The performed calculations have revealedthat the theoretical model is consistent with the experimental data. The redistribution of stresses occurs in the beam, also when global loads are constant in time.

Keywords: multi-layered structures, redistribution of stresses, rheology, viscoelasticity

1. INTRODUCTION

This paper concerns the analysis of stresses in viscoelastic multi-layered beams. The rheological characteristics of a beam made from a single material are manifested by deformations that increase with time (creep) or decreasing of stresses (relaxation of stresses). In the multi-layered beams a redistribution of stresses may occur due to different rheological properties of the component materials, which leads to changes in the location of the neutral axis, strains and displacements in time. The layers of the considered multi-layered beam are bonded together by means of an epoxy resin adhesive that exhibits pronounced rheological properties, which further enhances the viscous response of the beam as a whole.

2. THEORETICAL MODEL OF A MULTI-LAYERED BEAM

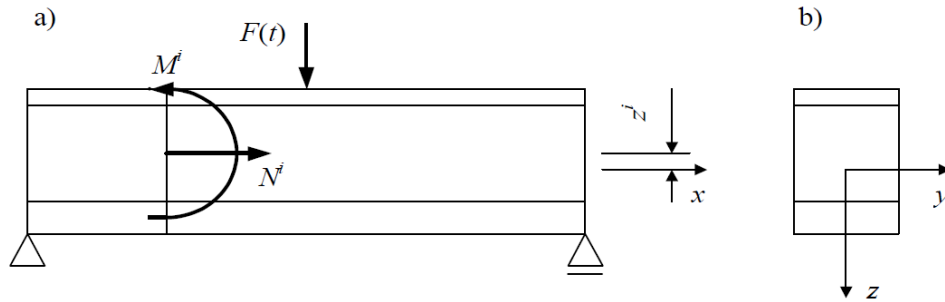A typical multilayer beam is illustrated in Fig. 1.

Fig. 1. A multi-layered beam: a) internal forces in ith layer,
b) a cross-section of the beam

The following assumptions have been made [1, 2, 3, 5, and 6]:
1. A cross section of a beam is symmetrical with respect to the vertical axis z.
2. A beam is composed of layers made of linear viscoelastic materials exhibiting the same rheological properties in tension and compression.
3. Planar cross-sections remain planar before and after bending (the Bernoulli hypothesis).
4. The layers are perfectly joined without slip.
5. Temperature and moisture conditions are constant in time.

As a result of a load applied at any point in an ith layer of the beam, a strain ε^i is caused, defining by the formula:

$$\varepsilon^i = \mathrm{K} \cdot z^i, \tag{2.1}$$

where K – curvature.

After differentiation dependence (2.1) assumes this form:

$$d\varepsilon^i = d\mathrm{K} \cdot z^i. \tag{2.2}$$

It follows from the formulae of equilibrium of statics that the sum total of normal forces operating in all layers equals 0:

$$N = \sum_{i=1}^{n} N^i = 0. \tag{2.3}$$

The normal force N^i, in turn, is the resultant force of normal stresses in an ith layer whose field in the cross section equals A^i:

$$N^i = \int_{A^i} \sigma^i dA^i. \tag{2.4}$$

The rheological behaviour of material in each layer is described by means of an integral representation of linear viscoelastic medium [2]:

$$\sigma = E * d\varepsilon ,$$ (2.5)

where:

E – a relaxation function,
$d\varepsilon$ – an increment of strain,
$*$ – the convolution product.

Replacing representation (2.4) with (2.5), we obtain:

$$N^i = \int_{A^i} E^i * d\varepsilon^i dA^i ,$$ (2.6)

and taking into account representation (2.2):

$$N^i = \int_{A^i} E^i * d\mathrm{K} \cdot z^i dA^i .$$ (2.7)

Therefore, the equilibrium of normal forces (2.3) assumes the form:

$$N^i = \sum_{i=1}^{n} \int_{A^i} E^i * d\mathrm{K} \cdot z^i dA^i = 0 ,$$ (2.8)

and after transformation:

$$N = \sum_{i=1}^{n} E^i * d\mathrm{K} \cdot \int_{A^i} z^i dA^i = 0 .$$ (2.9)

The integral in representation (2.9) corresponds to the static moment of cross-sectional area A^i of i*th* layer with respect to the neutral axis of the beam:

$$S^i = \int_{A^i} z^i dA^i ,$$ (2.10)

and so representation (2.9) can be presented in the following form:

$$N = \sum_{i=1}^{n} E^i * d\mathrm{K} \cdot S^i = 0 .$$ (2.11)

It follows from equation (2.11) that:

$$\sum_{i=1}^{n} E^i \cdot S^i = 0 .$$ (2.12)

Equation (2.12) makes possible calculating the time changeable distance from the neutral axis.

The bending moment in an i*th* layer can be calculated from the representation:

$$M^i = \int_{A^i} \sigma^i \cdot z^i dA^i \,,\qquad (2.13)$$

and taking into account representation (2.2) and the integral representation of constitutive equations of linear viscoelastic behaviour (2.5):

$$M^i = \int_{A^i} E^i * dK \cdot \left(z^i\right)^2 dA^i \,.\qquad (2.14)$$

The above equation can be transformed into:

$$M^i = E^i * dK \cdot \int_{A^i} \left(z^i\right)^2 dA^i \,,\qquad (2.15)$$

where:

$$I^i = \int_{A^i} \left(z^i\right)^2 dA^i \,,\qquad (2.16)$$

corresponds to the moment of inertia of cross-sectional area A^i of ith layer with respect to the neutral axis of beam.

Having considered (2.16), equation (2.15) assumes the following form:

$$M^i = E^i * dK \cdot I^i \,.\qquad (2.17)$$

The total bending moment in cross-section will be equal to the sum total of partial bending moment in each layer:

$$M = \sum_{i=1}^{n} M^i = \sum_{i=1}^{n} E^i * dK \cdot I^i \,.\qquad (2.18)$$

The convolution product can be identified on the basis of equation (2.17):

$$E^i * dK = \frac{M^i}{I^i} \,.\qquad (2.19)$$

The same product can be established from representations (2.2) and (2.5):

$$E^i * dK = \frac{\sigma^i}{z^i} \,.\qquad (2.20)$$

After comparing the right sides of equations (2.19) and (2.20), the equation representing the normal stresses in an i*th* layer is obtained:

$$\sigma^i = \frac{M^i \cdot z^i}{I^i} \,. \tag{2.21}$$

This equation has a form which is analogical with the classical dependence on the distribution of normal stresses in a homogenous bending beam. However, the moment M^i remains unknown. It can be identified by means of the condition of strain deformation. Representations (2.17) and (2.18) lead to the following dependences:

$$d\mathrm{K} = M^i * \left(E^i \cdot I^i\right)^{-1}, \tag{2.22}$$

$$d\mathrm{K} = M * \left(\sum_{i=1}^{n} E^i \cdot I^i\right)^{-1}. \tag{2.23}$$

Comparing the right sides of the above equations, it is possible to derive the dependence describing the bending moment in an i*th* layer:

$$M^i = E^i \cdot I^i * \left(\sum_{i=1}^{n} E^i \cdot I^i\right)^{-1} * M \,. \tag{2.24}$$

Integral equation (2.23) makes possible calculating curvature K, and to be more specific, the discrete set of curvature values in subsequent points in time t. Using curvature values, deflection can be identified by means of the equation:

$$\mathrm{K} = \frac{d^2 u}{dx^2} \,. \tag{2.25}$$

Integration of the above equation enables calculating beam deflections u at any time t.

Unfortunately, the convolution product, appearing in the above equations, is a source of significant difficulties in a precise calculation. Therefore, to calculate curvature K, an approximated calculation was used, where the convolution product (2.18) is approximated by the sum with a variable summation limit. Dependence (2.20) was transformed similarly. Both equations have the following form:

$$M(t_m) = \sum_{j=0}^{m} \left\{ \left[\sum_{i=1}^{n} I^i(t_m) E^i(t_m - t_j) \right] \Delta \mathrm{K}(t_j) \right\} \tag{2.26}$$

$$\sigma^i(t_m) = \sum_{j=0}^{m} E^i(t_m - t_j) \cdot \Delta \mathrm{K}(t_j) \cdot z^i(t_m) \tag{2.27}$$

Curvature increments in subsequent moments were calculated from formula (2.26), and on their basis, by means of formula (2.27), normal stresses in each layer were specified.

It needs to be stressed that at the starting point $t = 0$ the relaxation function $E^i(t)$ of the material in ith layer it assumes the value of modulus of elasticity E_0^i of the material, that is:

$$E^i(t = 0) = E_0^i. \qquad (2.28)$$

In a homogenous linear viscoelastic beam, during creep, an increment in the value of deflection will be observed, whereas the stresses are constant in time. In the case of a multi-layered and multi-material arrangement, the aforementioned phenomenon may only occur when the relaxation functions in the particular layers of material are similar, that is:

$$E^i(t) = k^i \cdot E(t), \qquad (2.29)$$

where k^i is the factor of proportionality specified for each layer, and $E(t)$ is the basic relaxation function. In any other case the redistribution of stresses will occur between the constituent layers of the intersection.

Using the above dependencies to calculate stresses and deflections in a viscoelastic multilayered bar is possible after specifying the relaxation function $E^i(t)$ of rheological model of the material in each layer. The values of modulus of elasticity E_j^i in the equations and coefficients of elasticity η_j^i are to be identified experimentally in creep tests or relaxation tests. Due to significant difficulties in carrying out relaxation tests, the former solution is usually applied. The method of determining values of parameters E_j^i and η_j^i is discussed in [5].

3. LABORATORY TESTS

Long-term experimental investigations of layered beams were conducted at the laboratory in the Institute of Building Engineering at the University of Zielona Góra. The multi-layered beams of natural dimensions made of wood, polyester glass composite bars and epoxy adhesive, were examined in the four-point bending test (Fig. 2).

With the aim of comparison, additional tests were carried out on wooden beams of homogenous cross-section with the identical geometrical characteristics as the layered beam's cross-section. The cross-sections of the beams and their geometrical characteristics are displayed in Fig. 3, wherein the symbols denote:

A^D, A^K, A^L, A^P – areas of cross-section,

$E_0^D, E_0^K, E_0^L, E_0^P$ – elastic moduli,

I^L, I^Z – the moment of inertia of cross-sectional area,

W^L, W_g^Z, W_d^Z – section moduli,

D, K, L, P, Z – index of materials: wood, adhesive, homogeneous beam,
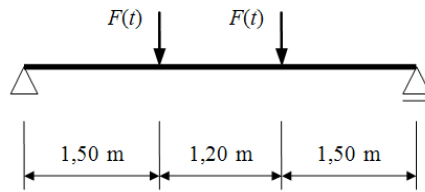composite bar, reinforced beam.

Fig. 2. General view of LS and ZS beams

Up to the point of stress value being equal appr. 30% of failure load, the wood may be treated as linear viscoelastic medium [5]. Therefore, in experiments the equal stress 30% $\overline{F}_n^L$ was adopted, where $\overline{F}_n^L$ is the average force destroying the homogenous beam, marked earlier in tests on short-term loads. The duration of the load equalled 100 days. Then the beams were relieved for 35 days, whereupon they were loaded again. They were relieved and loaded again. The whole multi-stage loading program is presented in Fig. 4. Six beams of each type (LS and ZS) were subjected to the test and the mean values of the results were calculated.
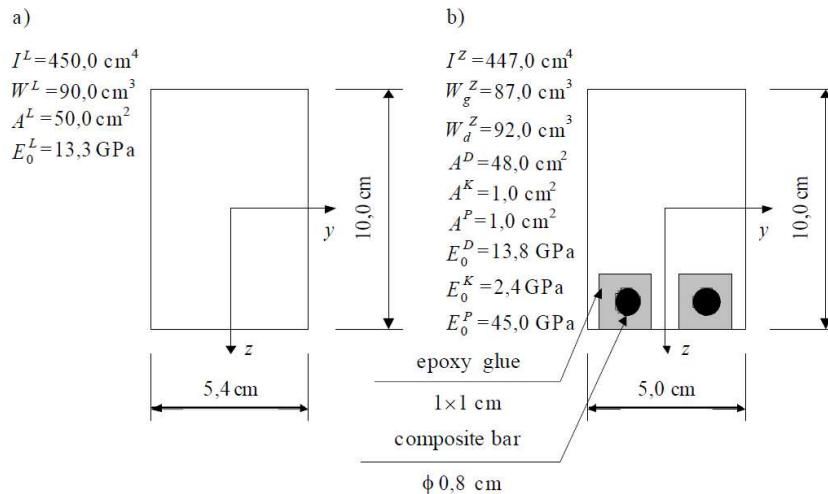

Fig. 3. Cross-sections of the tested beams: a) homogeneous wooden cross-section, LS,
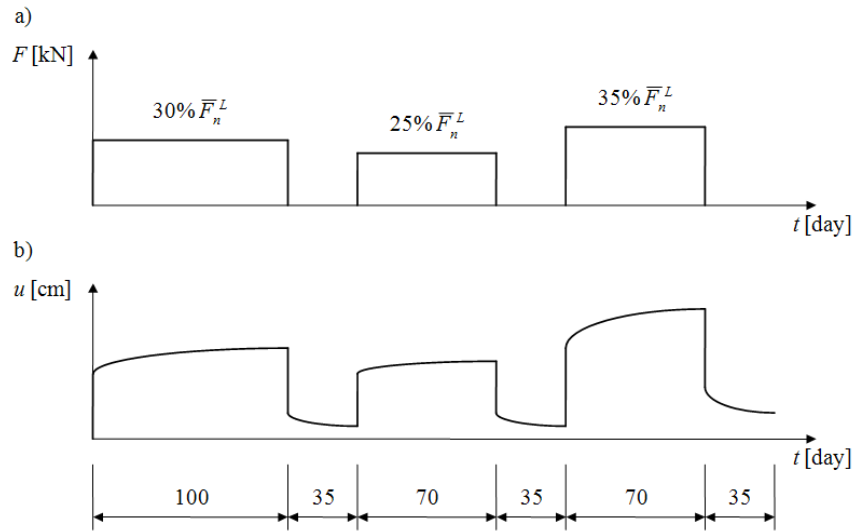b) cross-section with embedded composite reinforcement, ZS

a)

b)

Fig. 4. The multi-step programme of loads of LS and ZS beams:
a) load change, b) hypothetical deflection change

4. RHEOLOGICAL MODELS OF COMPONENT MATERIALS

The rheological properties of the polyester glass and the epoxy adhesive are described with the five-parameter model shown in Fig. 5a, whereas the six-parameter model shown in Fig. 5b is used for the wood.
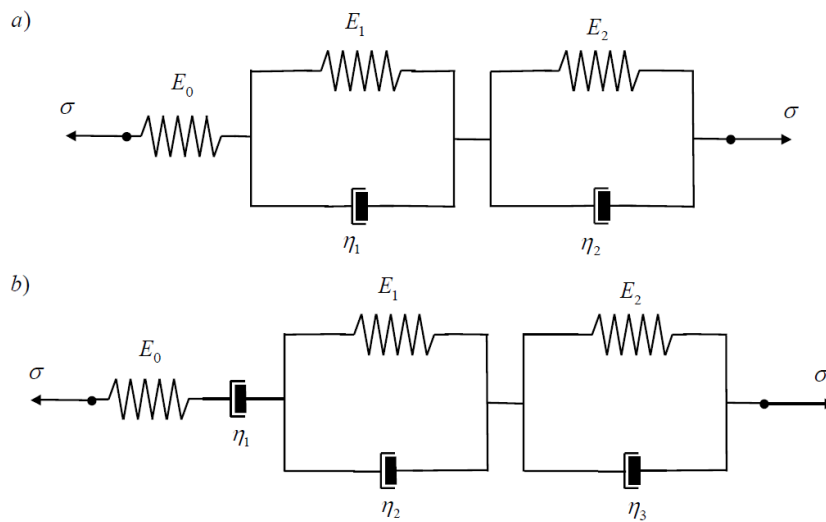
a)

b)

Fig. 5. Rheological models: a) five-parameter model, b) six-parameter model

The material parameters E and η of these models have been determined from additional separate tests conducted on samples of these materials and evaluated by the method of least squares. The mathematical formulae for the relaxation function of each of the applied models were derived from the constitutive relations of linear viscoelasticity in differentia form by making use of the Laplace transformation [4, 5, and 7].

5. REDISTRIBUTIONS OF STRESSES

The pertinent wooden beams in glued-in composite reinforcement are statically indetermined arrangements. Deformations in each material are limited owing to the condition of agreement of deformations. Furthermore, each material displays different rheological features, and so a redistribution of stresses occurs in the bar, even when the global loads are constant in time.

The results of the calculations performed by means of equations (2.26) and (2.27) in the form of stress values in selected days are presented in Table 1. Stresses in the lower (σ_{tt}^{Dd}) and upper (σ_{tt}^{Dg}) edge of the wooden part of the reinforced cross-section and in the glue (σ_{tt}^{K}) and the reinforced bar (σ_{tt}^{P}) were correlated. In order to compare, extreme stresses in the beams LS (σ_{tt}^{L}, at the lower and upper edge of the beam cross-section) are presented.

Table 1. Normal stresses in beams ZS and LS

Phase	Day	Wood – ZS beams		Glue	Composite	LS beams
		σ_{tt}^{Dg} [MPa]	σ_{tt}^{Dd} [MPa]	σ_{tt}^{K} [MPa]	σ_{tt}^{P} [MPa]	σ_{tt}^{L} [MPa]
load	0	-15,86	14,94	2,89	47,15	15,42
	100	-15,48	14,09	2,00	57,71	15,42
unload	100	0,38	-0,85	-0,89	10,56	0,0
	135	0,20	-0,46	-0,23	5,33	0,0
load	135	-13,02	11,99	2,19	44,63	12,85
	205	-12,82	11,56	1,52	50,24	12,85
unload	205	0,39	-0,89	-0,89	10,95	0,0
	240	0,24	-0,55	-0,31	6,40	0,0
load	240	-18,25	16,87	3,06	61,37	17,99
	310	-17,95	16,18	2,17	70,10	17,99
unload	310	0,54	-1,24	-1,21	15,12	0,0
	345	0,33	-0,78	-0,39	8,97	0,0

The results of the calculations in the form of function of relative stress change in time t (quotient of normal stresses in time t and time $t=0$, $\left(\sigma_{tt}^{i}/\sigma_{0t}^{i}\right)\cdot 100\%$) are shown on Fig. 6-11.
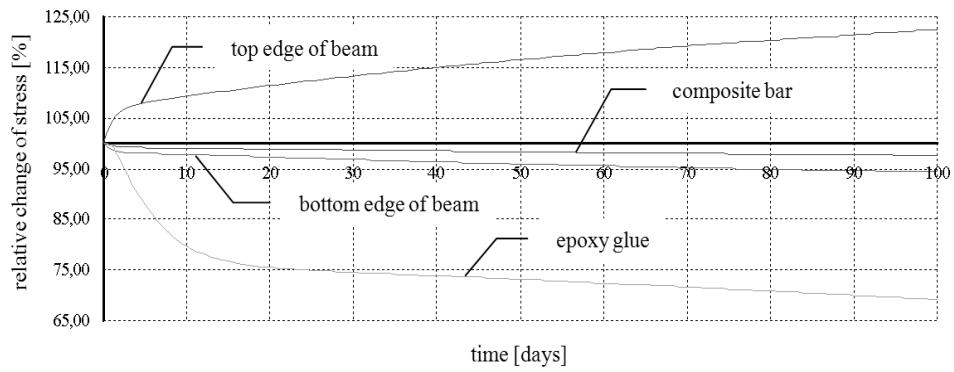
Fig. 6. Relative changes of normal stresses in ZS beams during the period
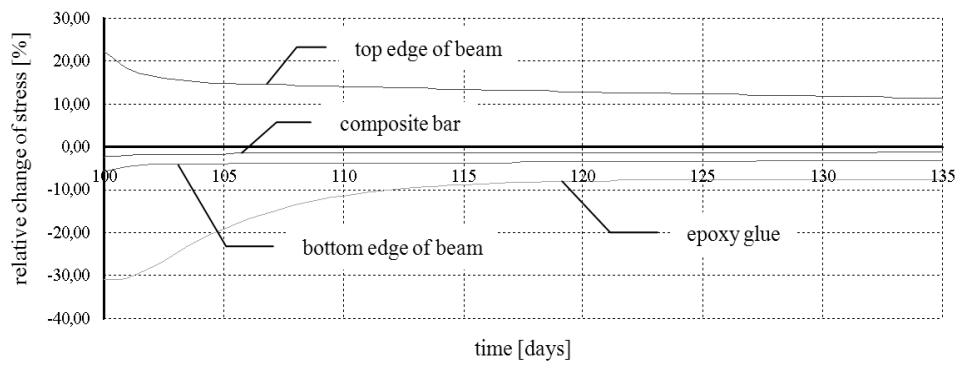0 – 100 days

Fig. 7. Relative changes of normal stresses in ZS beams during the period
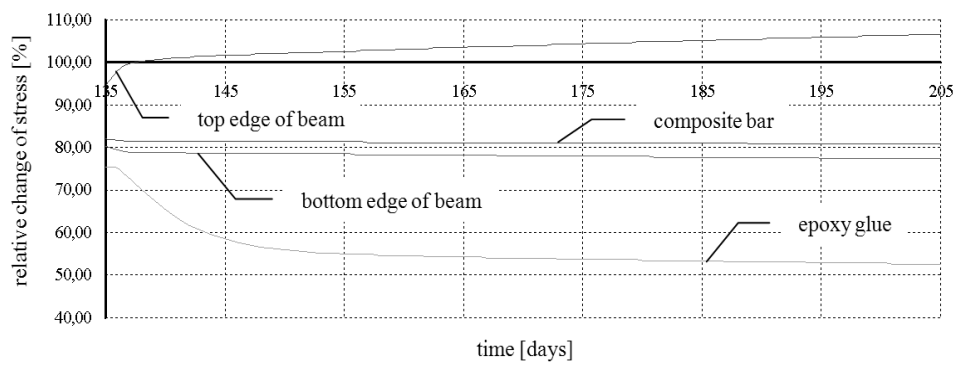100 – 135 days

Fig. 8. Relative changes of normal stresses in ZS beams during the period
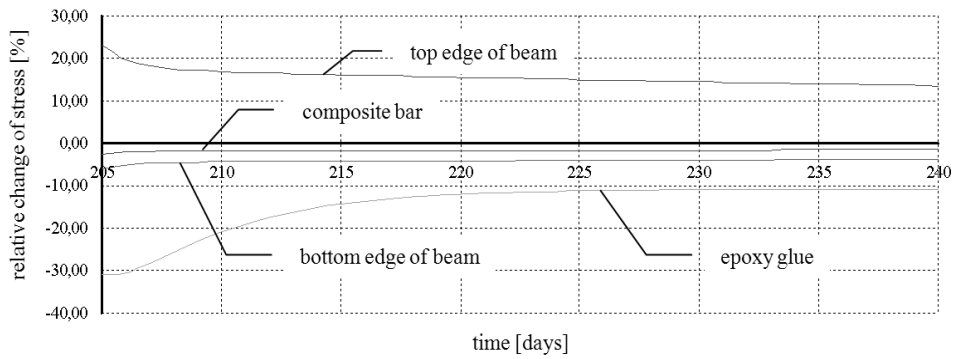135 – 205 days

Fig. 9. Relative changes of normal stresses in ZS beams during the period
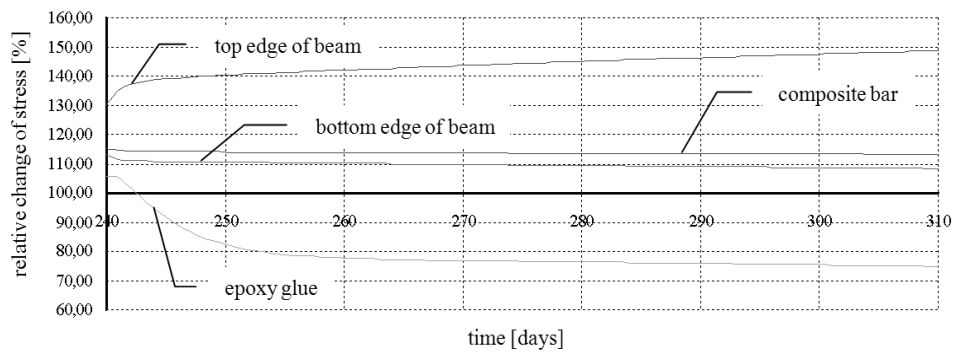205 – 240 days

Fig. 10. Relative changes of normal stresses in ZS beams during the period
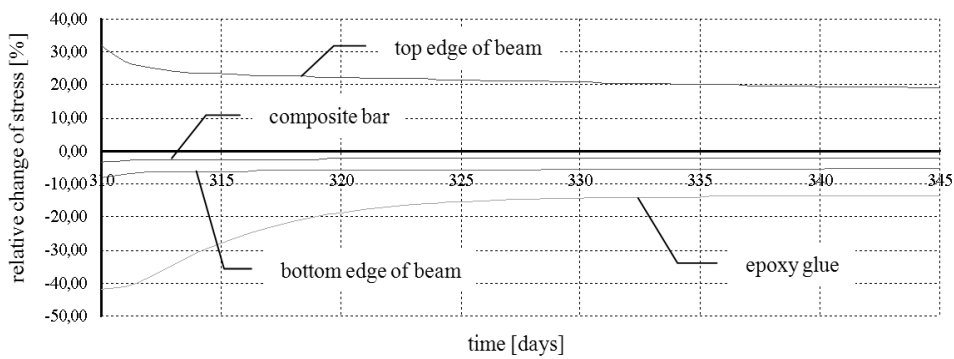240 – 310 days

Fig. 11. Relative changes of normal stresses in ZS beams during the period
310 – 345 days

An analysis of the diagrams enables making the conclusion that, in bent wooden elements with glued-in composite reinforcement, a significant redistribution of stresses in the intersection is observed. When the load is active, the composite bar is additionally loaded and in extreme cases the stresses rise by 50% with relation to stresses at time $t=0$. The stresses in the wood decrease slightly (appr. 10%), both in the compressed and the tensioned area. The stresses in the glued joint decrease significantly (appr. 30%). This results from its sensitivity to creep (in glue tests the rheological deflections rose by 50%).

During the periods of relieving the beams, owing to different rheological properties of the particular materials, a redistribution of stresses is also observed. In each part of cross-section a decrease in stress values is observed, though at a different velocity. The largest changes are observed in the composite bar and the glue.

6. VERIFICATION OF BERNOULLI HYPOTHESIS

In order to test Bernoulli hypothesis, results of measurements of deformations were used. More specifically, the mean values of deformations $\bar{\varepsilon}_{td}^{LS}$ and $\bar{\varepsilon}_{td}^{ZS}$ are calculated on the basis of 36 readings.

The degree of agreement of mean deformation values with Bernoulli hypothesis will be even larger when the deformation function is nearer a straight line, and the measure of this agreement corresponds to the value of linear correlation R between average measurement result and the point of measurement at the depth of the beam.

The results of calculations and measurements are presented in Fig. 12, 13, and 14. The results are given for time $t=0$, $t=310$ and $t=345$ days, that is the time of commencement, completion of the test and the last day of activity of the largest load. Unfortunately, due to the measurement by means of a mechanic tensometer and the way the reinforcement was located in ZS beams, only wood deformations were measured.

The analysis of correlation between localisation of deflection measuring point and mean values of deflections has revealed that, during all the period of time, strong linear interdependence was observed, both for homogenous and reinforced beams. In each case the coefficient value of linear correlation equalled R=0,999.

Diagrams of theoretical deformation function ε_{tt}^{LS} and ε_{tt}^{ZS} are presented in Fig. 12, 13 and 14. The experimental deformations display a large degree of agreement with the theoretical model during the period of load activity. More divergences are observed in no-load periods, during the last stage in particular.
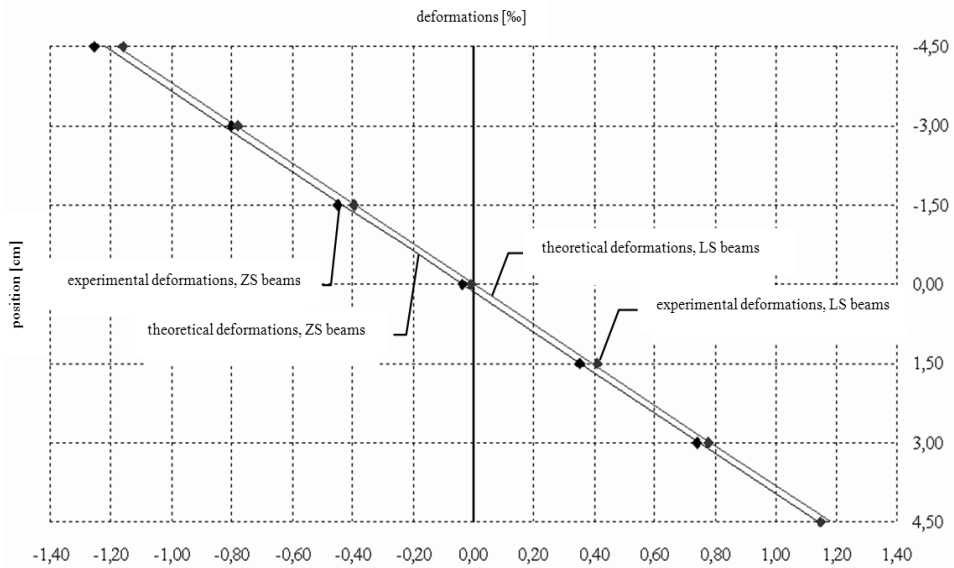
Fig. 12. Theoretical and experimental deformations of ZS and LS beams
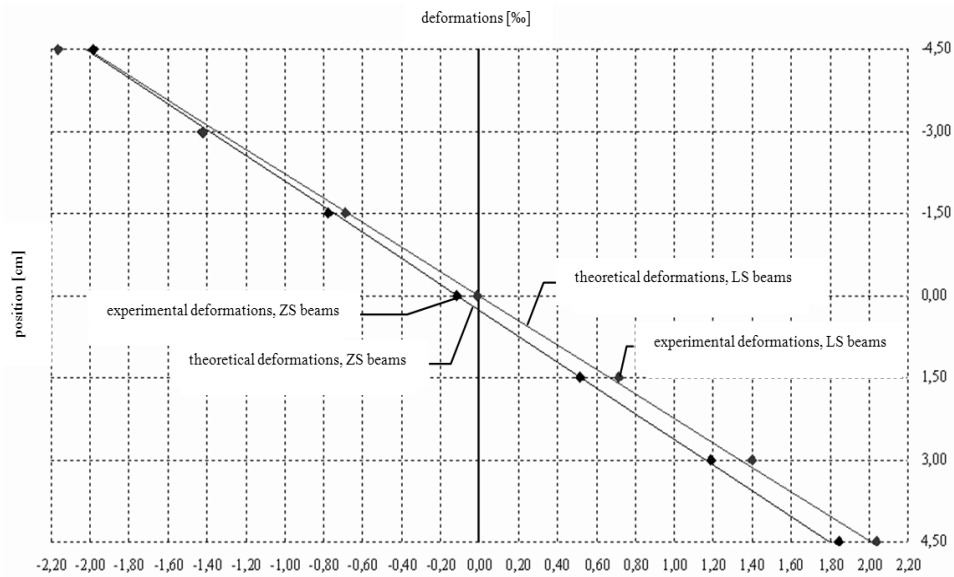at time t=0 days

Fig. 13. Theoretical and experimental deformations of ZS and LS beams
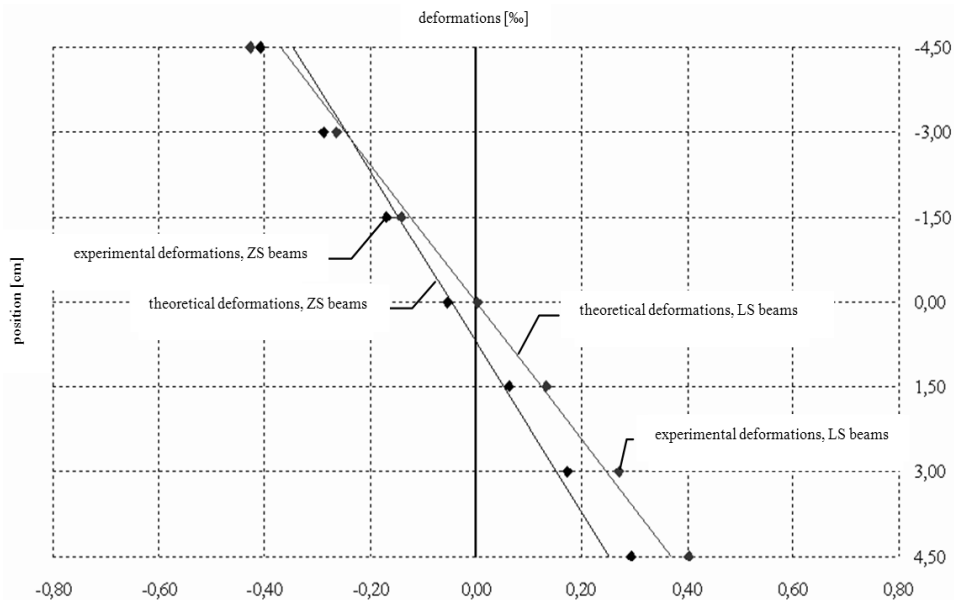at time t=310 days

Fig. 14. Theoretical and experimental deformations of ZS and LS beams
at time *t*=345 days

7. CONCLUSIONS

The experimental tests presented in the article and the theoretical elaboration
of their results enable drawing the following conclusions:

- The adopted theoretical model of reinforced beam displays good agreement
  with empirical data and can be used to calculate a broad range of settings,
  such as beams with glued-in reinforcement or the typical composite
  structures.

- A significant redistribution of stresses in the cross-section occurs in the bent
  wooden elements with glued-in composite reinforcement. The composite bar
  is further loaded when the load is active, in extreme cases the stresses rise
  by 50% with relation to stresses at time *t*=0. Stresses in the wood decrease
  slightly (appr. 10%), both in the compression and tensioned area. In turn,
  stresses in the glue joint decrease significantly, which results from its
  sensitivity to creep (in tests of glue samples the increment of rheological
  deformations equalled 50% of elastic ones). During no-load periods, a
  decrease in deformation values is observed at each place of intersection,
  though at a different pace. The largest changes are observed in the
  composite bar and the glue.

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REOLOGICZNA REDYSTRUBUCJA NAPRĘŻEŃ W BELKACH
WIELOWARSTWOWYCH

S t r e s z c z e n i e

W referacie przedstawiono problemy reologii konstrukcji wielowarstwowych na przykładzie belek drewnianych wzmacnianych wklejonymi prętami kompozytowymi (trzy warstwy: drewno, spoina klejowa o dużej grubości, pręt kompozytowy). Zaprezentowano teoretyczny model pracy tego typu belki zbudowany w oparciu o równania ośrodków liniowo lepkosprężystych, hipotezę płaskich przekrojów oraz założenie idealnego zespolenia elementów składowych. Wykorzystując rachunek operatorowy wyprowadzono równania ugięć i naprężeń. W celu weryfikacji modelu teoretycznego przeprowadzono badania doświadczalne na belkach w skali naturalnej, litych i wzmocnionych. Wykonane obliczenia wykazały dobrą zgodność modelu teoretycznego z danymi doświadczalnymi. Stwierdzono istotną redystrybucję naprężeń w obrębie przekroju poprzecznego.