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TWO-DIMENSIONAL HEAT CONDUCTION IN THE LAMINATE WITH THE FUNCTIONALLY GRADED PROPERTIES

Alina RADZIKOWSKA*, Artur WIROWSKI*
Department of Structural Mechanics,
Technical University of Łódź,
Al. Politechniki 6, 90-924 Łódź, Poland

The object of the contribution is the classical Fourier heat conduction in the laminate with functionally graded properties. The laminate is made of two conductors, non-periodically distributed as laminas along one direction. A macrostructure of the laminate is assumed to be functionally graded along this direction. In order to analyse heat conduction, the tolerance averaging technique is used. The approach is based on the book edited by Cz. Woźniak, Michalak and Jędrysiak [10] and by Cz. Woźniak et al. [6]. The aim of this paper is to apply the tolerance model equations of heat conduction for laminate with functionally graded properties to analyse two-dimensional stationary heat transfer. The equations of the tolerance model are solved by the finite difference method.

Keywords: heat conduction, functionally graded laminates, tolerance modelling.

1. INTRODUCTION

The main object under consideration is the laminate made of two conductors. These conductors are distributed non-periodically along the direction normal to the laminas. Every lamina has the thickness λ (λ is constant). It consists of two sublaminas (the thickness of each sublamina changes in every lamina). From the macroscopic point of view averaged (macroscopic) properties of this laminate are continuously varied along one direction, cf. Fig. 1a. However, the microstructure of the laminate can be defined by some distribution functions, which determine the thicknesses of sublaminas, cf. Fig. 1b.

* Corresponding author. E-mail: alina.radzikowska@p.lodz.pl, wirowski@p.lodz.pl

The laminates of this kind can be treated as made of the functionally graded materials (FGM), cf. Suresh and Mortensen [9].

There are many modelling techniques we can use to research thermo-mechanical problems of the functionally graded materials. One of them is the asymptotic homogenization. As an alternative approach, for FG-type materials the higher-order theory was proposed by Aboudi, Pindera and Arnold [1] and then with its reformulation by Bansal and Pindera [2]. In this paper we focus on the tolerance averaging technique (*the tolerance modelling*), extended on FG-type materials in books edited by Cz. Woźniak, Michalak and Jędrysiak [10] and by Cz. Woźniak et al. [6]. This technique was adopted to analyse various problems of FG-type materials and structures in series of papers, e.g. for heat conduction in transversally graded laminates (*TG laminates*) by Jędrysiak and Radzikowska [3, 4, 5] and in longitudinally graded composites by Michalak and M. Woźniak [7], Michalak, Cz. Woźniak and M. Woźniak [8]. Some additional examples of applications of this technique for composites can be found in the books [6, 10].

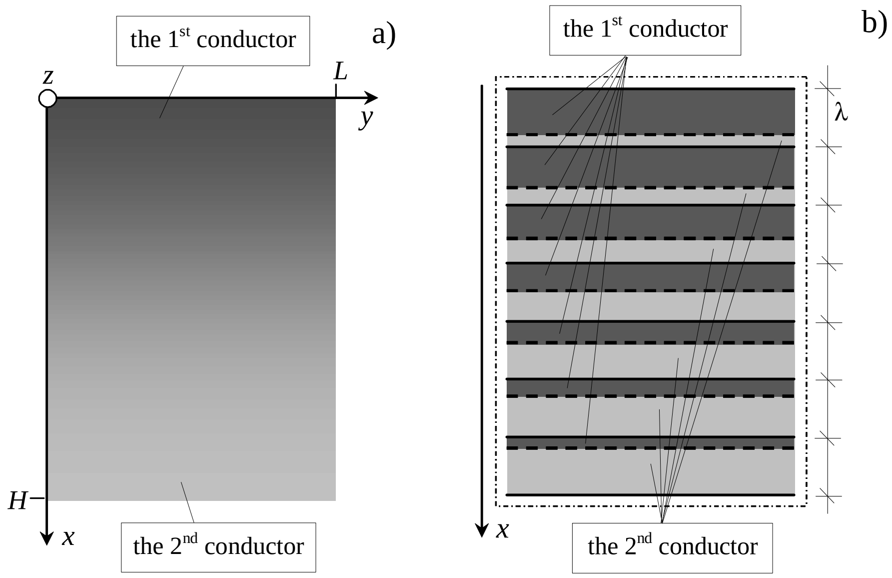

Fig. 1. A cross section of a laminate: a) the macroscopic point of view, b) the microscopic point of view

The main aim of this contribution is to apply the tolerance model equations of heat conduction for laminates with functionally graded properties to analyse stationary heat conduction.

2. TOLERANCE MODELLING

2.1. Modelling foundations

Let subscripts i, j related to the coordinate system Oxy, run over 1, 2. Introduce denotations: ∂_1 as a derivative of x; ∂_2 as a derivative of y. Let H be the laminate thickness along the x-axis, and L be the length dimension along the y axis. We assume that the laminate under consideration occupies the region $\Omega\times\Xi$ on the plane Oxy, where $\Omega\equiv(0,H)$, $\Xi\equiv(0,L)$. This laminate is made of two conductors distributed in m laminas with the constant thickness λ. Properties of these conductors are described by: specific heats c', c'' and heat conduction tensors with the components k'_{ij}, k''_{ij} i,j=1,2. It is assumed that the thickness λ satisfies the condition $\lambda<<H$ and is called *the microstructure parameter*. Every n^{th} lamina consists of two homogeneous sublaminas with thicknesses: λ'_n and $\lambda''_n=\lambda-\lambda'_n$, which are not constant, cf. Fig. 1b. We can introduce the material volume fractions in the n^{th} lamina defined as $\nu'_n\equiv\lambda'_n/\lambda$, $\nu''_n\equiv\lambda''_n/\lambda$. Sequence $\{\nu'_n\}$, n=1,…m, is monotone and satisfies the condition $|\nu'_{n+1}-\nu'_n|<<1$, for n=1,…,m-1. Because $\nu'_n+\nu''_n=1$ sequence $\{\nu''_n\}$ satisfies similar conditions. Sequences $\{\nu'_n\}$, $\{\nu''_n\}$, n=1,…m, can be approximated by continuous functions $\nu'(\cdot)$, $\nu''(\cdot)$, describing the gradation of material properties along the x axis. The functions $\nu'(\cdot)$, $\nu''(\cdot)$ can be called *the fraction ratios* of materials. Let us also define the non-homogeneity ratio by $\nu(\cdot)\equiv[\nu'(\cdot)\nu''(\cdot)]^{1/2}$. Moreover, these functions: $\nu'(\cdot)$, $\nu''(\cdot)$, $\nu(\cdot)$, are assumed to be slowly-varying, cf. the book edited by Cz. Woźniak, Michalak and Jędrysiak [10].

Let T denote the unknown temperature field. Moreover, the heat conduction problem in the TG laminate will be analysed in the framework of the Fourier's model, i.e. described by the following equation (without heat sources):

$$\partial_i\left(k_{ij}\partial_j T\right)-c\dot{T}=0. \tag{2.1}$$

For the TG laminate all coefficients in equation (2.1), i.e. $k_{ij}=k_{ij}(x)$, $c=c(x)$, are highly-oscillating, tolerance-periodic, non-continuous functions in x. Using the tolerance modelling, equation (2.1) can be replaced by the differential equations with continuous, smooth, slowly-varying coefficients, cf. the book edited by Cz. Woźniak, Michalak and Jędrysiak [10] and by Cz. Woźniak et al. [6].

2.2. Concepts and assumptions

The tolerance modelling concepts, the modelling assumptions and the modelling procedure can be found in books [6, 10]. Below, only few of them are reminded.

For an arbitrary integrable function f (which can also depend on y), defined in $\overline{\Omega}$, the averaging operator is given by:

$$< f > (x) = \lambda^{-1} \int_{x-\lambda/2}^{x+\lambda/2} f(\xi) d\xi \tag{2.2}$$

for $x \in [\lambda/2, H-\lambda/2]$. It can be shown that for the tolerance-periodic function f of x, its averaged value calculated from (2.2) is a slowly-varying function in x (cf. the books [6, 10]).

The fundamental modelling assumption is mentioned below.

The assumption is *the micro-macro decomposition*, in the following form:

$$T(x,y) = \vartheta(x,y) + \varphi(x)\psi(x,y), \tag{2.3}$$

where $\vartheta(\cdot y)$, $\psi(\cdot y)$ are slowly-varying functions. The basic unknown is function $\vartheta(\cdot,y)$ called *the macrotemperature*; an additional basic unknown is *the fluctuation amplitude* $\psi(\cdot,y)$; $\varphi(\cdot)$ is the known *fluctuation shape function*. The function $\varphi(\cdot)$ is assumed to be continuous, linear across every sublamina thickness and of an order $O(\lambda)$; it can be given by:

$$\varphi(x) = \begin{cases} \lambda\sqrt{3}\,\dfrac{\nu(\bar{x})}{\nu'(\bar{x})}[2x + \nu''(\bar{x})] & \text{for} \quad x \in (-\lambda/2,\, -\lambda/2 + \nu'(\bar{x})) \\ \\ \lambda\sqrt{3}\,\dfrac{\nu(\bar{x})}{\nu''(\bar{x})}[-2x + \nu'(\bar{x})] & \text{for} \quad x \in (\lambda/2 - \nu''(\bar{x}),\, \lambda/2) \end{cases} \tag{2.4}$$

where $\bar{x}$ is a centre of a cell with a length dimension λ and it's independent from x, cf. the book [10].

2.3. Tolerance model equations

Using the tolerance modelling and denoting the smooth functional coefficients: $K \equiv < k_{11} >,\ K_{22} \equiv < k_{22} >,\ \tilde{K} \equiv < k_{11}\partial_1\varphi >,\ \breve{K} \equiv < k_{11}\partial_1\varphi\partial_1\varphi >$, we obtain the averaged heat conduction equations in the form (cf. [3, 4, 5]):

$$\begin{aligned} &\partial_1\left(K\,\partial_1\vartheta\right) + K_{22}\,\partial_{22}\vartheta + \partial_1\left(\tilde{K}\,\psi\right) = 0\,, \\ &\tilde{K}\,\partial_1\vartheta + \breve{K}\,\psi - \lambda^2\nu^2 K_{22}\,\partial_{22}\psi = 0\,. \end{aligned} \tag{2.5}$$

Equations (2.5) have the coefficients being slowly-varying functions in x, in contrast to equation (2.1), which has functional, non-continuous, highly oscillating coefficients. Some terms depend explicitly on the microstructure parameter λ.

Equations (2.5) together with the micro-macro decomposition (2.3) constitute *the tolerance model of heat conduction for transversally graded laminates*. These equations take into account the effect of the microstructure size on heat transfer for these composites. It can be observed that for the TG laminate under consideration boundary conditions for the macrotemperature ϑ have to be formulated on the edges x=0, H and y=0, L, but for the fluctuation amplitude ψ only on the edges y=0, L.

2.4. Applications: stationary heat conduction

Denote by $k'_{11} = k'_{22} = k'$, $k''_{11} = k''_{22} = k''$ heat conduction coefficients in sublaminas and introducing notations:

$K(x) \equiv \nu'(x)k' + \nu''(x)k''$, $\tilde{K}(x) \equiv 2\sqrt{3}\nu(x)(k' - k'')$, $\breve{K}(x) \equiv 12(\nu'(x)k'' + \nu''(x)k')$,

equations (2.5) take the form:

$$\begin{aligned} &\partial_1(K\,\partial_1\vartheta) + K\,\partial_{22}\vartheta + \partial_1(\tilde{K}\,\psi) = 0\,, \\ &\tilde{K}\,\partial_1\vartheta + \breve{K}\,\psi - \lambda^2\,\nu^2\,K\,\partial_{22}\psi = 0\,. \end{aligned} \tag{2.6}$$

Equations (2.6) together with the boundary conditions were changed to the system of differential equations. A special numerical program based on the finite difference method was written by Mister Artur Wirowski. Using this program, equations (2.6) were solved.

3. RESULTS

In this section some effects of conductor properties c.f. heat conduction coefficients on the laminate temperature are presented.

Let the layer thickness H be coupled with the microstructure parameter λ by the relation H=$m\lambda$ (m is the number of laminas). The fraction ratios of materials are defined as the exponential functions:

$$\nu'(x) = \frac{1 - \exp(2x/H)}{1 - \exp(2)}, \qquad \nu''(x) = 1 - \nu'(x)$$

For the macrotemperature ϑ we assume the following boundary conditions:

a) as a parabolical functions - Fig. 2a and 3a,

$$\vartheta(x,0) = \vartheta(x,L) = 100\left(\frac{x}{H}\right)^2 - 100\frac{x}{H} + 25\,, \quad \vartheta(0,y) = \vartheta(H,y) = 100\left(\frac{y}{L}\right)^2 - 100\frac{y}{L} + 25\,;$$

b) as a constant and a parabolical functions - Fig. 2b and 3b,

$$\vartheta(x,0) = \vartheta(x,L) = 25\,, \quad \vartheta(0,y) = \vartheta(H,y) = 100\left(\frac{y}{L}\right)^2 - 100\frac{y}{L} + 25\,.$$

The boundary conditions are shown as a schema in Fig. 4a,c. For the fluctuation amplitude ψ the boundary conditions we assume as equal zero.

Some calculation results are shown in Fig. 2-3. These plots are made for L=H, m=20, thus the ratio λ/H=0.05. In a numerical program the division for H it is 4000, for L – 30. In Fig. 2 there are presented the temperature distributions for the ratio k''/k'=10, but Fig. 3 shows plots of the temperature for the ratio k''/k'=0.1.

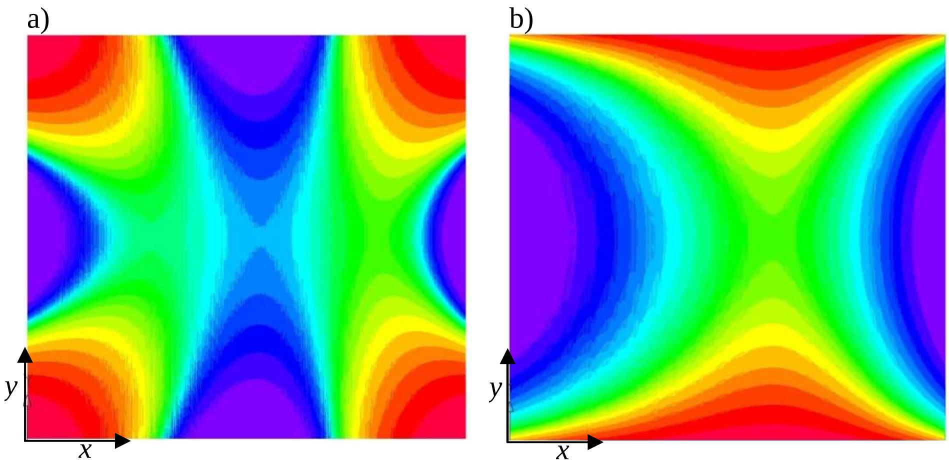

Fig. 2. Temperature distributions for the ratio k''/k'=10: a) for boundary conditions shown as a schema in Fig. 4a; b) for boundary conditions shown as a schema in Fig. 4c.

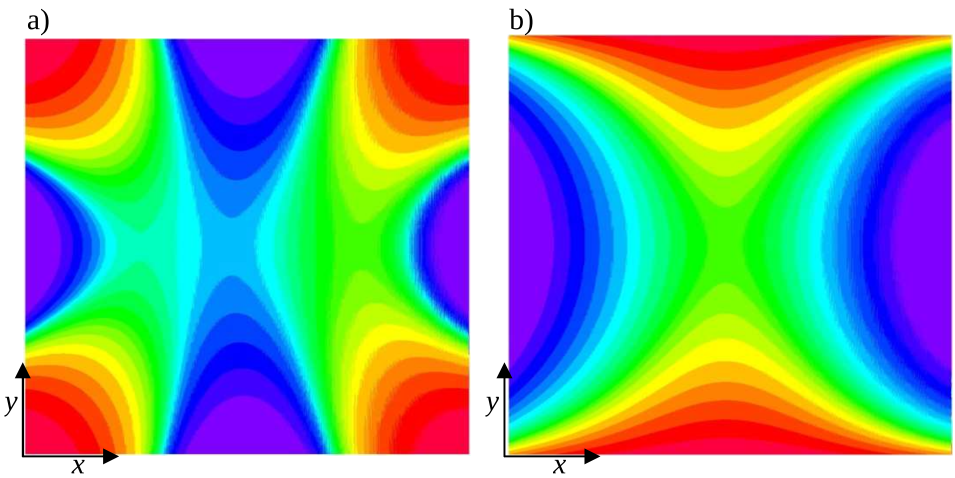

Fig. 3. Temperature distributions for the ratio k''/k'=0.1:
a) for boundary conditions shown as a schema in Fig. 4a;
b) for boundary conditions shown as a schema in Fig. 4c.

a) b) c)

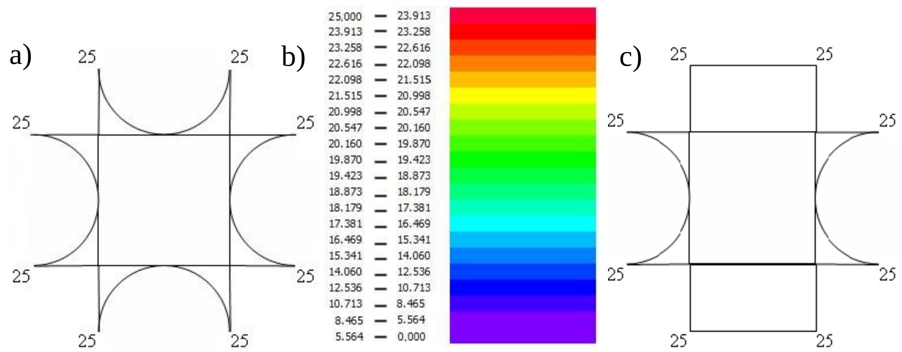

Fig. 4. Schema of the boundary conditions for the macrotemperature ϑ (a), (c) and the legend of the coloring plots (b).

4. REMARKS

Under the calculation results for the stationary heat conduction, we can observe:

- Distributions of the temperature:
  - depend on the differences between heat conduction coefficients k', k'',
  - are symmetric along the y axis,
  - are not symmetric along the x axis, because of the laminas distribution.

The tolerance averaging technique it seems to be an effective tool to research heat conduction in transversally graded laminates.

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DWUWYMIAROWE ZAGADNIENIE PRZEWODNICTWA CIEPŁA W LAMINACIE O FUNKCYJNEJ GRADACJI WŁASNOŚCI

S t r e s z c z e n i e

Przedmiotem rozważań pracy jest przewodnictwo ciepła w klasycznym, fourierowskim sformułowaniu, odniesione do laminatu z jednokierunkową funkcyjną gradacją własności. Rozpatrywany jest laminat złożony z wielu dwuskładnikowych warstw, którego właściwości termiczne zmieniają się funkcyjnie w kierunku prostopadłym do laminowania. W pracy zajęto się przypadkiem, gdy grubość warstw jest stała, a każda warstwa złożona jest z dwóch różnych, jednorodnych i izotropowych materiałów. W skali makro własności laminatu zmieniają się w sposób ciągły i gładki wzdłuż kierunku prostopadłego do laminowania. W mikroskali laminat ma budowę określoną przez jednorodne funkcje rozkładu materiałów, dobrane tak, aby sąsiednie warstwy „mało” się od siebie różniły. Celem pracy jest analiza stacjonarnego, dwuwymiarowego zagadnienia przewodnictwa ciepła z użyciem techniki tolerancyjnego uśredniania. Technika ta pozwala zastąpić równanie przewodnictwa ciepła o silnie oscylujących, nieciągłych współczynnikach funkcyjnych układem równań różniczkowych o współczynnikach funkcyjnych gładkich i wolnozmiennych. Otrzymany układ równań rozwiązano metodą różnic skończonych. Przedstawiono rozkłady temperatury w zależności od proporcji między współczynnikami przewodzenia ciepła, przy różnych warunach brzegowych.