LOCAL OPTIMAL CONTROL SYNTHESIS IN THE SYSTEMS WITH DELAYED ACTION

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This study is devoted to the problems of optimal control synthesis in stochastic delayed action systems on the basis of the local quality criteria, characterizing the current precision of stabilization of the controlled motion.

1. Introduction

To a number of important factors, defining the quality of different object control and efficiency of their functioning, a delay in the channel of transfer and information treatment is related.

Many investigations have been concerned with the problems of analysis and synthesis of systems with delayed action, valuable results have been derived, for example, in studies (Chernous'ko and Kolmanovsky 1978; Górecki, 1974; Janushevsky, 1978). Relatively few works are devoted to the problems of optimal control of systems with delayed action. In particular many problems of optimal control for the uncertainty conditions are not resolved.

The application for the case of electronic computer continuous dynamic objects control creates the peculiarities of the control realization, to one of which time quantization is related. In this case the control remains constant between the moments of the control signal formation that must be taken into account in the control synthesis of such systems.

This study is devoted to the problems of optimal control synthesis in stochastic delayed action systems on the basis of the local quality criteria, characterizing the current precision of stabilization of the controlled motion.

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2. Problem Statement

Let the motion of the controlled object be defined by the vector equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{A}_1(t)\boldsymbol{x}(t-h_1) + \boldsymbol{B}(t)\boldsymbol{u}(t) + \boldsymbol{C}(t)\boldsymbol{w}(t) \qquad (1)$$

with the initial conditions

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad \boldsymbol{x}(\tau) = \varphi(\tau), \quad \tau \in [t_0 - h_1, t_0).$$

The equation of the measurement has the form

$$\boldsymbol{y}(t) = \boldsymbol{H}(t)\boldsymbol{x}(t) + \boldsymbol{v}(t) \qquad (2)$$

Here $\boldsymbol{x}(t)$ is a state vector, $\boldsymbol{u}(t)$ is a control vector, $\boldsymbol{w}(t)$ is a vector of the stochastic disturbance actions, $\boldsymbol{y}(t)$ is a measurement vector, $\boldsymbol{v}(t)$ is a vector of noise, h_1 is a delay constant, $\boldsymbol{A}(t)$, $\boldsymbol{A}_1(t)$, $\boldsymbol{B}(t)$, $\boldsymbol{C}(t)$ and $\boldsymbol{H}(t)$ are matrices of the corresponding dimensions. Suppose the stochastic initial states $\boldsymbol{x}_0$ and $\varphi(\tau)$, disturbance action $\boldsymbol{w}(t)$ and measurement noise $v(t)$ are statistically known, and defined as follows

$$\begin{aligned}
&E[\boldsymbol{x}_0] = \overline{\boldsymbol{x}}_0, \quad E[\varphi(\tau)] = \overline{\varphi}(\tau),\\
&E[\boldsymbol{x}(t_0+s)\boldsymbol{x}^T(t_0)] = \boldsymbol{X}_1(t_0+s),\\
&E[\boldsymbol{x}(t_0+r)\boldsymbol{x}^T(t_0+s)] = \boldsymbol{X}_2(t_0+r, t_0+s), \quad r \in [-h_1, 0], \quad s \in [-h_1, 0],\\
&E[\boldsymbol{w}(t)] = 0, \quad E[\boldsymbol{w}(t)\boldsymbol{w}^T(\tau)] = \boldsymbol{Q}(t)\delta(t-\tau),\\
&E[\boldsymbol{v}(t)] = 0, \quad E[\boldsymbol{v}(t)\boldsymbol{v}^T(\tau)] = \boldsymbol{R}(t)\delta(t-\tau),\\
&E[\varphi(\tau)\boldsymbol{w}^T(t)] = 0, \quad E[\varphi(\tau)\boldsymbol{v}^T(t) = 0,
\end{aligned}$$

where $\overline{\boldsymbol{x}}_0$, $\varphi(\tau)$, $\boldsymbol{X}_1(\cdot)$, $\boldsymbol{X}_2(\cdot)$, $\boldsymbol{Q}(t)$, $\boldsymbol{R}(t)$ are the known vectors and matrices, $\boldsymbol{Q}(t)$ is a symmetrical non-negative definite, $\boldsymbol{R}(t)$ is a symmetrical definite-positive matrix.

The quality of the controlled motion will be evaluated with the help of the local functional

$$\begin{aligned}
J(t) \;=\; & E\{\boldsymbol{x}^T(t)\boldsymbol{G}_1(t)\boldsymbol{x}(t) + \int_{-h_1}^{0} \boldsymbol{x}^T(t)\boldsymbol{G}_2(t,s)\boldsymbol{x}(t+s)\,ds +\\
&+ \int_{-h_1}^{0} \boldsymbol{x}^T(t+r)\boldsymbol{G}_3(t,r)\boldsymbol{x}(t)\,dr +\\
&+ \int_{-h_1}^{0}\int_{-h_1}^{0} \boldsymbol{x}^T(t+r)\boldsymbol{G}_4(t,r,s)\boldsymbol{x}(t+s)\,dr\,ds +\\
&+ \int_{t_0}^{t} \boldsymbol{u}^T(\tau)\boldsymbol{G}_0(\tau)\boldsymbol{u}(\tau)\,d\tau\}, \qquad (3)
\end{aligned}$$

characterizing the current precision of the stabilization with due regard for the control cost.

It is supposed that matrix functions $G_i(\cdot)$ $(i = 0, 1..., 4)$ have the following properties

$$G_0(t) = G_0^T(t), \quad G_1(t) = G_1^T(t), \quad G_2(t,s) = G_3^T(t,s),$$

$$G_3(t,r) = G_2^T(t,r), \quad G_4(t,r,s) = G_4^T(t,s,r).$$

Besides matrices $G_0(t)$, $G_1(t)$ will be taken as definitely positive, and $G_2(t,s)$, $G_3(t,r)$ and $G_4(t,r,s)$ are such that at each t moment $J(t) > 0$ is satisfied on the trajectory of the system (1) – (2). The task is to find the control

$$u(t) = u\{y(\tau), \quad t_0 - h_1 \leq \tau \leq t\},$$

which minimizes the functional (3).

To define the control $u(t)$ the local optimality condition is used (Degtyarev and Syrazetdinov, 1986): $\min_{U_0(t)} \frac{dJ(t)}{dt}$ where $\frac{dJ(t)}{dt}$ is evaluated according to th motion equations.

It is shown by Degtyarev and Terent'yev (1989) that for the considered system with delayed action having noncomplete and nonprecise state measurement just as for the system without delay the problem is divided into two: the problem of the linear optimal observer and the determined problem of the optimal regulator synthesis, and the control is defined by the expression

$$u(t) = -G_0^{-1}(t)\left[B^T(t)G_1(t)\hat{x}(t) + B^T(t)\int_{-h_1}^{0} G_3(t,s)\hat{x}(t+s)\,ds\right], \quad (4)$$

where $\hat{x}(t)$ is a local-optimal estimate of the state vector.

3. Problem Solution

Let us consider now the solution of the problem of the parametric synthesis for the case of local-optimal piecewise constant control in the systems with delayed action.

Let the measurement has a delay as well

$$y(t) = H(t)x(t - h_2) + v(t).$$

The control action is formed at the discrete moment of the time $t = lT$:

$$u(lT) = \sum_{i=0}^{k} f_i(lT) y(lT - \tau_i),$$

$$\tau_0 = 0, \quad \tau_{i+1} > \tau_i, \quad \tau_k < T \tag{5}$$

and remains unchanged at the interval between the moments of the control formation $u(t) = u(lT)$ with $lT \le t < (l+1)T$.

The value of τ_i in (5) may be defined, for example, as $\tau_i = iT_2$ $(kT_2 < T)$, where T_2 is a time period with which the measurement is performed.

The quality of the controlled motion will be evaluated by the local functional.

$$\begin{aligned} J(t) &= E[x^T(t)G_1(t)x(t) + \int_{-\bar{h}}^{0} x^T(t)G_2(t,s)x(t+s)\,ds + \\ &+ \int_{-\bar{h}}^{0} x(t+r)G_3(t,r)x(t)\,dr + \\ &+ \int_{-\bar{h}}^{0}\int_{-\bar{h}}^{0} x^T(t+r)G_4(t,r,s)x(t+r)\,dr\,ds + \\ &+ \int_{t_0}^{t} u^T(r)G_0(r)u(r)\,dr], \end{aligned} \tag{6}$$

where $\bar{h} = \max\{h_1, h_2, \tau_0, \tau_1, ..., \tau_k\}$.

We assume that the matrix functions $G_i(\cdot)$ $i = 0, 1...4$ possess the above given properties.

In the control law (5) the unknown parameters are the regulator matrix coefficients $f_i(lT)$, $i = \overline{0, k}$. Let us define these coefficients from the optimality condition by the criterion (6), i.e. we shall solve the problem of the parametric synthesis.

The unknown regulator coefficients will be defined at the moments of the control formation from the condition

$$\min_{f_i(lT),\, i=0,1,\dots,k} \left. \frac{dJ(t)}{dt} \right|_{t=lT}. \tag{7}$$

The following notations are used for a more compact representation

$$\boldsymbol{x}_p^T(lT) = [\boldsymbol{x}^T(lT-\tau_0)\ \boldsymbol{x}^T(lT-\tau_1)\ ...\ \boldsymbol{x}^T(lT-\tau_k)],$$

$$\boldsymbol{v}_p^T(lT) = [\boldsymbol{v}^T(lT-\tau_0)\ \boldsymbol{v}^T(lT-\tau_1)\ ...\ \boldsymbol{v}^T(lT-\tau_k)],$$

$$\boldsymbol{y}_p^T(lT) = [\boldsymbol{y}^T(lT-\tau_0)\ \boldsymbol{y}^T(lT-\tau_1)\ ...\ \boldsymbol{y}^T(lT-\tau_k)],$$

$$\boldsymbol{F}_p(lT) = [f_0(lT)\ f_1(lT)\ ...\ f_k(lT)],$$

$$\boldsymbol{H}_p(lT) = \operatorname{diag}\{\boldsymbol{H}(lT-\tau_0), \boldsymbol{H}(lT-\tau_1), ..., \boldsymbol{H}(lT-\tau_k)\},$$

$$\boldsymbol{P}_1(t) = E[\boldsymbol{x}(t)\boldsymbol{x}^T(t)], \qquad \boldsymbol{P}_2(t,r) = E[\boldsymbol{x}(t+r)\boldsymbol{x}^T(t)],$$

$$\boldsymbol{P}_3(t,s) = E[\boldsymbol{x}(t)\boldsymbol{x}^T(t+s)], \quad \boldsymbol{P}_4(t,r,s) = E[\boldsymbol{x}(t+r)\boldsymbol{x}^T(t+s)],$$

$$\boldsymbol{w}_2(t, lT-h_2-\tau_i) = E[\boldsymbol{x}(lT-h_2-\tau_i)\boldsymbol{x}^T(t)] = \boldsymbol{P}_2(t, t-(lT-h_2-\tau_i)),$$

$$\begin{aligned} \boldsymbol{w}_4(lT-h_2-\tau_i, lT-h_2-\tau_j) &= E[\boldsymbol{x}(lT-h_2-\tau_i)\boldsymbol{x}^T(lT-h_2-\tau_j)] = \\ &= \boldsymbol{P}_4(lT, h_2+\tau_i, h_2+\tau_j), \end{aligned} \tag{8}$$

$$\boldsymbol{w}_{2p}^T(t, lT-h_2) = [w_2^T(t, lT-h_2-\tau_0)\ ...\ w_2^T(t, lT-h_2-\tau_k)],$$

$$\boldsymbol{w}_{4p}(lT-h_2, lT-h_2) = \\ = \begin{bmatrix} w_4(lT-h_2-\tau_0, lT-h_2-\tau_0) & ... & w_4(lT-h_2-\tau_0, lT-h_2-\tau_k) \\ ... & ... & ... \\ w_4(lT-h_2-\tau_k, lT-h_2-\tau_0) & ... & w_4(lT-h_2-\tau_k, lT-h_2-\tau_k) \end{bmatrix}$$

$$\boldsymbol{T}_p(t, lT) = E[\boldsymbol{y}(t)\boldsymbol{v}_p^T(lT)],$$

$$\boldsymbol{R}_p(lT) = \operatorname{diag}\{\boldsymbol{R}(lT-\tau_0), \boldsymbol{R}(lT-\tau_1), ..., \boldsymbol{R}(lT-\tau_k)\},$$

where $lT \le t < (l+1)T$.

Using the notations (8) the control law (5) may be presented in the form

$$\boldsymbol{u}(lT) = \boldsymbol{F}_p(lT)\boldsymbol{y}_p(lT). \tag{9}$$

With the help of the optimality condition (7) we shall gain an expression to define the regulator optimal matrix coefficients

$$\begin{aligned} \boldsymbol{F}_p(lT) &= -\boldsymbol{G}_0^{-1}(lT)\boldsymbol{B}^T(lT)[\boldsymbol{G}_1(lT)\boldsymbol{\psi}(lT, lT) + \\ &+ \int_{-\bar{h}}^{0} \boldsymbol{G}_3^T(lT, s)\boldsymbol{\psi}(lT+s, lT)\, ds]\boldsymbol{\xi}^{-1}(lT), \end{aligned} \tag{10}$$

where $\psi(lT, lT) = w_{2p}(lT)H_p^T(lT) + T_p(lT, lT)$,

$$\xi(lT) = H_p(lt)w_{4p}(lT - h_2, lT - h_2)H_p^T(lT) + R_p(lT). \tag{11}$$

$P_i(\cdot)$ $i = \overline{1,4}$ and $T_p(\cdot)$ included in (10) and (11) are defined from the equations solution

$$\begin{aligned}
\frac{dP_1(t)}{dt} &= A(t)P_1(t) + P_1(t)A^T(t) + A_1(t)P_2(t, -h_1) + \\
&+ P_2^T(t, -h_1)A_1^T(t) + B(t)F_p(lT)\psi^T(t, lT) + \\
&+ \psi(t, lT)F_p^T(lT)B^T(lT) + C(t)Q(t)C^T(t), \\
\frac{\partial P_2(t,r)}{\partial t} &= \frac{\partial P_2(t,r)}{\partial r} + P_2(t,r)A^T(t) + P_4(t,r,-h_1)A_1^T(t) + \\
&+ \psi(t + r, lT)F_p^T(lT)B(lT), \qquad (12) \\
\frac{\partial P_3(t,s)}{\partial t} &= \frac{\partial P_3(t,s)}{\partial s} + A(t)P_3(t,s) + A_1(t)P_4(t, -h_1, s) + \\
&+ B(t)F_p(lT)\psi^T(t + s, lT), \\
\frac{\partial P_4(t,r,s)}{\partial t} &= \frac{\partial P_4(t,r,s)}{\partial r} + \frac{\partial P_4(t,r,s)}{\partial s}, \\
\frac{dT_p(t,lT)}{dt} &= A(t)T_p(t, lT) + A_1(t)T_p(t - h_1, lT) + \\
&+ B(t)F_p(lT)[H_p(lT)T_p(lT - h_2, lT) + R_p(lT)],
\end{aligned}$$

with the corresponding initial and boundary conditions for $P_i(\cdot)$, and the initial condition

$$T_p(0,0) = 0. \tag{13}$$

It follows that the local-optimal piecewise constant control will be defined from the equations (9), (10) and contained with in them matrix functions $w_{2p}(\cdot)$, $w_{4p}(\cdot)$ and $T_p(\cdot)$ are defined from the equations (12) taking into account the definitions of these functions in (8).

It must be noted that in the case of delay absence in the system, i.e. $h_1 = h_2 = 0$ the evaluating equations are considerably simplified.

4. Discrete-Time System

Let us consider a case when the object model is reduced to the equivalent discrete system of the form

$$\boldsymbol{x}(k+1) = \sum_{j=0}^{N} \boldsymbol{A}_j \boldsymbol{x}(k-j) + \sum_{j=0}^{P} \boldsymbol{B}_j \boldsymbol{u}(k-j) + \boldsymbol{C}\boldsymbol{w}(k) \tag{14}$$

with the known initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}_o$, $\boldsymbol{x}(i) = \boldsymbol{x}_i$, $i = -N, ..., -1$.

The measurement equation has the following form

$$\boldsymbol{y}(k) = \boldsymbol{H}\boldsymbol{x}(k-r) + \boldsymbol{v}(k)$$

The control action is formed at the moments of time $t = kT$:

$$\boldsymbol{u}(k) = \sum_{i=0}^{s} a(k,i)\ y(k-i) + \sum_{l=1}^{s} b(k,l)\ u(k-l) \tag{15}$$

In contrast to the regulator of the form (5) the control values at the preceding moments of time are introduced into regulator (15) and so it has a more general form.

The quality of the controlled motion will be evaluated with the help of the local functional

$$\begin{aligned} J(k) &= E\{\boldsymbol{x}^T(k)\boldsymbol{G}_1(k)\boldsymbol{x}(k) + \sum_{j=0}^{k-1}[\boldsymbol{u}^T(i)\boldsymbol{G}_o(i)\boldsymbol{u}(i) + \boldsymbol{x}^T(i)\boldsymbol{G}_2(i)\boldsymbol{u}(i) + \\ &+ \boldsymbol{u}^T(i)\boldsymbol{G}_2^T(i)\boldsymbol{x}(i) + \boldsymbol{x}^T(i)\boldsymbol{G}_3(i)\boldsymbol{x}(i)]\}. \end{aligned} \tag{16}$$

Let us introduce the notations (by convention it is assumed $s \leq p < N$):

$\boldsymbol{x}_p^T(k) = [\boldsymbol{x}^T(k)\boldsymbol{x}^T(k-1)\ ...\ \boldsymbol{x}^T(k-N)\boldsymbol{u}^T(k-1)\boldsymbol{u}^T(k-2)\ ...\ \boldsymbol{u}^T(k-p)]$,

$\boldsymbol{y}_p(k) = [\boldsymbol{y}^T(k)\boldsymbol{y}^T(k-1)\ ...\ \boldsymbol{y}^T(k-N)\boldsymbol{u}^T(k-1)\boldsymbol{u}^T(k-2)\ ...\ \boldsymbol{u}^T(k-p)]$,

$\boldsymbol{v}_p^T(k) = [\boldsymbol{v}^T(k)\boldsymbol{v}^T(k-1)\ ...\ \boldsymbol{v}^T(k-N)\ 0]$,

$$
A_p = \left[\begin{array}{cccc|ccccc}
A_0 & A_1 & \cdots & A_N & B_1 & B_2 & \cdots & B_{p-1} & B_p \\
I & 0 & \cdots & 0 & & & & & \\
0 & I & \cdots & 0 & & & 0 & & \\
\cdots & \cdots & \cdots & \cdots & & & & & \\
0 & 0 & \cdots & I & & & & & \\
\hline
 & & & & 0 & 0 & \cdots & 0 & 0 \\
 & & & & I & 0 & \cdots & 0 & 0 \\
 & & 0 & & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & 0 & 0 & \cdots & I & 0
\end{array}\right] \qquad (17)
$$

$$
\begin{aligned}
&\boldsymbol{B}_p^T = [\boldsymbol{B}_o^T \; 0 \; I^T \; 0], \quad \boldsymbol{C}_p^T = [\boldsymbol{C}^T \; 0], \\
&\boldsymbol{H}_p = [0 \; \mathrm{diag}\{\boldsymbol{H},...,\boldsymbol{H},I,...,I\}], \\
&\boldsymbol{F}(k) = [a(k,0) \; a(k,1) \; ... \; a(k,s) \;\; 0 \;\; b(k,1) \; ... \; b(k,s) \;\; 0 \;], \\
&\boldsymbol{G}_{1p}(k) = \mathrm{diag}\{G_1(k),0\}, \\
&\boldsymbol{G}_{2p}(k) = \mathrm{diag}\{G_2(k),0\}, \\
&\boldsymbol{G}_{3p}(k) = \mathrm{diag}\{G_3(k),0\}.
\end{aligned}
$$

With the help of the notations (17) the law of the control formation may be written in the form

$$\boldsymbol{u}(k) = \boldsymbol{F}(k)\boldsymbol{y}_p(k). \qquad (18)$$

From the local optimality condition $\min_U \Delta J(k)$ the matrix of coefficients in the regulator is defined as

$$
\begin{aligned}
\boldsymbol{F}(k) \;&=\; -[\boldsymbol{B}_p^T\boldsymbol{G}_{1p}(k)\boldsymbol{B}_p + \boldsymbol{G}_2(k)]^{-1}[\boldsymbol{H}_p\boldsymbol{P}(k)\boldsymbol{A}_p\boldsymbol{G}_{1p}(k) + \\
&+\; \boldsymbol{T}^T(k)\boldsymbol{A}_p^T\boldsymbol{G}_{1p}(k) + \boldsymbol{H}_p\boldsymbol{P}(k)\boldsymbol{G}_{2p}(k) + \boldsymbol{T}(k)\boldsymbol{G}_{2p}(k)] \times \\
&\times\; [\boldsymbol{H}_p\boldsymbol{P}(k)\boldsymbol{H}_p^T + \boldsymbol{R}_p(k) + \boldsymbol{H}_p\boldsymbol{T}(k) + \boldsymbol{T}^T(k)\boldsymbol{H}_p^T]^{-1}. \qquad (19)
\end{aligned}
$$

The matrix functions $\boldsymbol{P}(k)$ and $\boldsymbol{T}(k)$ satisfy the recurrente expressions

$$
\begin{aligned}
\boldsymbol{P}(i+1) \;&=\; \boldsymbol{A}_{op}(i)\boldsymbol{P}(i)\boldsymbol{A}_{op}^T(i) + \boldsymbol{B}_{op}(i)\boldsymbol{R}_p(i)\boldsymbol{B}_{op}^T(i) + \\
&+\; \boldsymbol{C}_p\boldsymbol{Q}(i)\boldsymbol{C}_p^T + \boldsymbol{A}_{op}(i)\boldsymbol{T}(i)\boldsymbol{B}_{op}^T(i) + \boldsymbol{B}_{op}(i)\boldsymbol{T}^T(i)\boldsymbol{A}_{op}^T(i), \\
\boldsymbol{P}(0) \;&=\; E[\boldsymbol{x}_{po}\boldsymbol{x}_{po}^T], \qquad (20)
\end{aligned}
$$

$$T(i+1) = [A_{op}(i)T(i) + B_{op}(i)R_p(i)]I_1, \quad T(0) = 0,$$

where

$$A_{op}(i) = A_p + B_p(i)F(i)H_p, \quad B_{op}(i) = B_p(i)F(i),$$

$$I_1 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The optimal control is formed in accordance with (18), (19), where $P(k)$ and $T(k)$ are defined from (20) at each step of the discrete time.

Suppose $N = P = r = 0$ and we shall get a case for a system without delay for which the control synthesis algorithm holds true.

5. Conclusion

The local optimal control synthesis algorithms in stochastic systems with delayed action were derived in this study. The cases of continuous dynamic objects control with the help of regulators having the continuously changing control signals and with the help of regulators having control signal with time quantization are considered, besides the case of control synthesis of the discrete systems have been studied.

The proposed algorithm of the discrete control system synthesis has been used for the control design, stabilizing the elastic object. The obtained results have confirmed the algorithm efficiency and showed sufficiently high precision of the controlled object stabilization.

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