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APPROXIMATE CONTROLLABILITY OF SECOND ORDER DYNAMICAL SYSTEMS

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In the paper linear, infinite–dimensional second–order dynamical systems defined in a separable Hilbert spaces are considered. Using the spectral theory for linear, unbounded operators, necessary and sufficient conditions for various types of approximate controllability are formulated and proved. As an illustrative example approximate controllability of flexible mechanical dynamical system described by linear partial differential equations is investigated. Some additional remarks and comments on approximate controllability for different types of second–order abstract dynamical systems are also given. Approximate controllability conditions presented in this paper extend to the case of second–order dynamical systems the results given in some previous papers.

1. Introduction

In recent years modern control theory of linear dynamical system has been the subject of considerable interest of many research scientists. It has been motivated on the one hand by the wide range of applications of linear models in various areas of science and engineering and, on the other hand, by the difficult and stimulating theoretical problems posed by such systems.

This paper is intended to provide information about one of the fundamental concept in mathematical control theory which is controllability. Roughly speaking, controllability generally means, that it is possible to transform the dynamical system from an arbitrary initial state to an arbitrary final state using the set of so called admissible controls.

The present paper is devoted to a study of so called approximate controllability of linear second–order infinite–dimensional dynamical systems. We shall consider approximate controllability in an arbitrary time interval $[0, T]$ with unconstrained controls, and approximate controllability with nonnegative controls. Transforming second–order abstract differential equation into the set of linear first–order equations, we shall formulate and prove necessary and sufficient conditions for various types of approximate controllability. These conditions will be derived by using the methods of functional analysis, especially the theory of linear unbounded normal operators in separable Hilbert spaces. As illustrative example we shall consider

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the mechanical flexible dynamical system described by linear partial differential equation.

The results presented in this paper extend to the case of second–order infinite–dimensional dynamical systems, controllability conditions given in the book (Klamka, 1991a) and in the papers (Klamka, 1991b; Sakawa, 1975; 1983; 1985) and (Triggiani, 1975; 1976; 1978). The similar dynamical systems have been recently investigated also in the papers (Chen and Russell, 1982; Chen and Triggiani, 1989; 1990a; 1990b; Datko, 1988; Hishikira, 1989; Huang, 1988), where the conditions for existence of solutions are presented and proved using the spectral theory of linear operator.

2. System Description

Let us consider dynamical control system described by the following abstract second–order differential equation in Hilbert space X

$$\ddot{x}(t) + 2rA^{1/2}\dot{x}(t) + Ax(t) = \sum_{j=1}^{j=m} b_j u_j(t) = Bu(t), \quad t \geq 0 \tag{1}$$

where $x(t) \in X$ is the separable Hilbert space with inner product $< \cdot, \cdot >_X$, $u_j(t) : [0, \infty) \to R$, for $j = 1, 2, ..., m$ are Hölder continuous control function, $u(t) = [u_1(t), u_2(t), ..., u_j(t), ..., u_m(t)]^T$, T denotes the transposition, $b_j \in X$, for $j = 1, 2, ..., m$, $B = [b_1, b_2, ..., b_j, ..., b_m]$, $B : R^m \to X$ is the linear bounded operator, and $0 < r < 1$ is the damping coefficient.
$A : X \supset D(A) \to X$ is the linear, generally unbounded operator which has the following properties:

i) A is self–adjoint and positive–definite operator with dense domain $D(A)$ in X. Moreover, it is assumed that operator A has only pure discrete point spectrum consisting entirely with isolated real positive eigenvalues

$0 < s_2 < s_2 < ... < s_i < ...$ and $\lim_{i\to\infty} s_i = +\infty$

Each of eigenvalues is of finite multiplicity $n_i < \infty$, $i = 1, 2, ...$ equal to the dimensionality of the corresponding eigenmanifold . Hence, the resolvent $R(s, A)$ of the operator A is a compact operator as an operator on X for all s in the resolvent set $\varphi(A)$.

ii) There exists a corresponding complete orthonormal set $\{x_{ik}\}, x_{ik} \in X$ $i = 1, 2, ...,$ $k = 1, 2, ..., n_i$ of eigenvectors of the operator A. Hence, for every $x \in X$ we have the following unique expansion

$$x = \sum_{i=1}^{i=\infty} \sum_{k=1}^{k=n_i} < x, x_{ik} >_X x_{ik}$$

iii) Operator A has the spectral representation

$$Ax = \sum_{i=1}^{i=\infty} s_i \sum_{k=1}^{k=n_i} < x, x_{ik} >_X x_{ik} \quad \text{for} \quad x \in D(A)$$

where s_i, $i = 1, 2, \dots$ are eigenvalues of the operator A.

iv) Operator A is the infinitesimal generator of the analytic semigroup $S(t) : X \to X$, $t > 0$, of bounded linear operators. This semigroup is explicitly given by the following formula

$$S(t)x = \sum_{i=1}^{i=\infty} \exp(s_i t) \sum_{k=1}^{k=n_i} < x, x_{ik} >_X x_{ik}, \qquad \text{for } t \geq 0 \text{ and } x \in X$$

v) For the operator A fractional power $A^{1/2}$ can be defined as follows

$$A^{1/2}x = \sum_{i=1}^{i=\infty} \sqrt{s_i} \sum_{k=1}^{k=n_i} < x, x_{ik} >_X x_{ik}, \text{ for } x \in D(A^{1/2}) \subset X$$

In a quite similar way we can define the arbitrary fractional power $A^\alpha, \alpha \in (0,1)$ of the operator A.

It should be stressed, that generally even for differentiating operator A, the operator $A^{1/2}$ may have quite different nature. It depends mainly of on form of the operator A and imposed boundary conditions.

vi) Since the spectrum of the operator A is bounded away from zero, then as a consequence $D(A^{1/2}) \supset D(A)$. Therefore operator $A^{1/2}$ is linear, self–adjoint and positive–definite with dense domain in the space X. Similarly, A^{-1} and $A^{-1/2}$ are both linear, nonnegative, bounded, self–adjoint operators on X.

All the fundamental properties of the operators A and $A^{1/2}$ listed above will be extensively used in the next sections of the paper. Several other behaviors of these operators can be found for example in the papers (Chen and Russell, 1982; Chen and Triggiani,1989; 1990a; 1990b) and (Kunimatsu and Ito, 1988).

Let the initial conditions for the equation (1) be given

$$x(0) \in X \quad \text{and} \quad \dot{x}(0) \in X \tag{2}$$

It is well known (see e.g. Chen and Russell, 1982; Datko, 1988; Huang, 1988; Hishikira, 1989; Chen and Triggiani, 1989; 1990a; 1990b), that the equation (1) with the initial conditions (2) has for each $t_1 > 0$ an unique, so called mild solution $x(t) : [0, t_1] \to X$, which satisfies the following conditions

$$x(t) \in C^{(2)}(\ [0,t_1], X\ )$$

$$x(t) \in D(A) \quad \text{and} \quad \dot{x}(t) \in D(A) \quad \text{for} \quad t \in (0,t_1]$$

The abstract differential equation (1) is a mathematical model of many distributed parameter dynamical systems described by various type of partial differential equations. The damping term in the equation reflects the dissipation of energy which is empirically observed in nature. The equations of the type (1) describe the vibrations in mechanically flexible systems which have applications to attitude control of flexible spacecraft, active tendon control of structures and a control of manipulator with flexible arm (Sakawa and Matsushita, 1975; Sakawa, 1984; 1985).

In the literature there are many different mathematical models of dynamical systems with inherent damping. For example in the papers (Chen and Triggiani, 1989; 1990a; 1990b) is considered abstract differential equation of the form

$$\ddot{x}(t) + 2rA^{\alpha}\dot{x}(t) + Ax(t) = 0 \tag{3}$$

for damping coefficient $r > 0$ and fractional power $\alpha \in (0,1)$. The case $\alpha = 1$ is studied in the papers (Sakawa, 1984; 1985), while the case $\alpha = 0$ is mentioned in the papers (Chen and Triggiani, 1989) and (Huang, 1988). In (Kunimatsu and Ito, 1988) the abstract differential equation with linear combination of the terms $A^{1/2}$ and A is investigated. Moreover, in the paper (Triggiani,1975) the special case $r = 0$ is detailed studied for more general class of the operator A.

It should be also stressed that in the papers (Datko, 1988; Hishikira, 1989; Lasiecka and Triggiani, 1991; Nambu, 1984; 1985; Narukawa, 1982; 1984) similar second–order abstract differential equations are formulated and studied in detail.

In the sequel, for comparisons we shall consider instead of the second–order equation (1) also the first–order differential equation

$$\dot{x}(t) + Ax(t) = Bu(t) \tag{4}$$

Dynamical systems of the form (4) have been analysed in many publication (see e.g. Son, 1990; Triggiani, 1975; 1976; 1978).

3. First Order Equation

The main purpose of this section is to transform the second–order abstract differential equation (1) to first–order one by using the procedure proposed in the papers (Sakawa and Matsushita, 1975) and (Chen, 1982; Sakawa, 1984; Huang, 1988; Chen and Triggiani, 1990b).

First of all let us convert the equation (1) into equivalent first–order differential equation in Hilbert space $H = X \times X$

$$\dot{y}(t) = \widetilde{A}_r y(t) + \widetilde{B}u(t) \tag{5}$$

where

$$y = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \in H = X \times X \quad \text{and} \quad y^1 = x, \; y^2 = \dot{x}$$

$$\widetilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix} : R^m \to H$$

$$\widetilde{A}_r = \begin{bmatrix} 0 & I \\ -A & -2rA^{1/2} \end{bmatrix} \qquad D(\widetilde{A}_r) = D(A) \times D(A^{1/2})$$

The unbounded linear operator $\widetilde{A}_r$ is similar to a normal operator. In order to explain it, let us introduce the nonsingular transformation represented by the linear invertible operator $F : D(A^{1/2}) \times D(A^{1/2}) \to H$ which is densely defined in the space $H : \overline{D(A^{1/2}) \times D(A^{1/2})} = H$

$$F = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ e^{j\psi} A^{1/2} & -e^{-j\psi} A^{1/2} \end{bmatrix} \tag{6}$$

where $\psi = \arctan\left(\frac{r}{\sqrt{1-r^2}}\right) = \arctan(r/p)$, $p = 1/\sqrt{1-r^2}$. Since by assumption $0 < r < 1$, then $\psi \in (\pi/2, \pi)$. The inverse transformation F^{-1} exists and it is a bounded linear operator $F^{-1} : H \to H$ given the following formula

$$F^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} pe^{j(\psi-\pi/2)} I & pe^{-j\pi/2} A^{-1/2} \\ pe^{j(-\psi+\pi/2)} I & pe^{j\pi/2} A^{-1/2} \end{bmatrix} \tag{7}$$

Letting

$$z = \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} = Fy = F \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \tag{8}$$

we transform the equation (5) into equivalent differential equation

$$\dot{z}(t) = A_r z(t) + B_r u(t) \tag{9}$$

where

$$A_r = F^{-1} \widetilde{A}_r F + \begin{bmatrix} A_r^+ & 0 \\ 0 & A_r^- \end{bmatrix} \tag{10}$$

$$B_r = \begin{bmatrix} pe^{-j\pi/2} A^{-1/2} B \\ pe^{j\pi/2} A^{-1/2} B \end{bmatrix} \tag{11}$$

and A_r^+, A_r^- are linear unbounded normal operators given by the following equalities

$$A_r^+ = e^{j\psi} A^{1/2}, \quad A_r^- = e^{-j\psi} A^{1/2} \tag{12}$$

Let us collect the fundamental spectral properties of the operator A_r

i) Spectrum $\sigma(A_r)$ of the operator A_r consists entirely of the isolated eigenvalues λ_i^+, λ_i^-, $i = 1, 2, ...,$ where (Chen and Russell, 1982; Sakawa, 1985; Huang, 1988; Kunimatsu and Ito, 1988; Chen and Triggiani, 1990b)

$$\lambda_i^+ = e^{j\psi}\sqrt{s_i}, \quad \text{and} \quad \lambda_i^- = e^{-j\psi}\sqrt{s_i}, \quad \text{for} \quad i = 1, 2, ... \tag{13}$$

ii) Taking into account the definitions of the operators A_r^+ and A_r^- we obtain the following equalities

$$A_r \begin{bmatrix} x_{ik} \\ 0 \end{bmatrix} = \begin{bmatrix} A_r^+ x_{ik} \\ 0 \end{bmatrix} = \lambda_i^+ \begin{bmatrix} x_{ik} \\ 0 \end{bmatrix} \text{ for } i = 1, 2, ..., \ k = 1, 2, ..., n_i \tag{14}$$

$$A_r \begin{bmatrix} 0 \\ x_{ik} \end{bmatrix} = \begin{bmatrix} 0 \\ A_r^- x_{ik} \end{bmatrix} = \lambda_i^- \begin{bmatrix} 0 \\ x_{ik} \end{bmatrix} \text{ for } i = 1, 2, ..., \ k = 1, 2, ..., n_i \tag{15}$$

Therefore $[x_{ik}, 0]^T \in H$ and $[0, x_{ik}]^T \in H$, for $k = 1, 2, ..., n_i$ are eigenvectors of the operator A_r corresponding to the eigenvalues λ_i^+ and λ_i^-, $i = 1, 2, ...,$ respectively.

iii) The set of eigenvectors of the operator A_r, $\{[x_{ik}, 0]^T, [0, x_{ik}]^T \ i = 1, 2, ..., k = 1, 2, ..., n_i\}$ is a complete orthonormal system in Hilbert space $H = X \times X$.

iv) The operator A_r is the infinitesimal generator of an analytic semigroup $S_r(t) : H \to H$, for $t > 0$, represented by the formula

$$S_r(t)z = S_r(t) \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} = \sum_{i=1}^{i=\infty} \left(\exp(t\lambda_i^+) \sum_{k=1}^{k=n_i} < z^1, x_{ik} >_X \begin{bmatrix} x_{ik} \\ 0 \end{bmatrix} \right) +$$

$$+ \sum_{i=1}^{i=\infty} \left(\exp(t\lambda_i^-) \sum_{k=1}^{k=n_i} < z^2, x_{ik} >_X \begin{bmatrix} 0 \\ x_{ik} \end{bmatrix} \right), \quad \text{for } z \in H \tag{16}$$

All above statements are a natural generalization to the infinite-dimensional separable Hilbert spaces of finite-dimensional facts from the space R^n.

It should be pointed out, that although $n_i < \infty$ for all $i = 1, 2...$ this does not ensure in general that $\sup_i n_i < \infty$ (see e.g. Triggiani, 1975; 1976). If $n_i = 1$ for all $i = 1, 2, ...$ (i.e. operator A has only single eigenvalues s_i, $i = 1, 2, ...$), then we simply write $\{x_i, \ i = 1, 2, ...\}$ instead of $\{x_{i1}, \ i = 1, 2, ...\}$.

In this paper operator A_r is considered in the Hilbert space $H = X \times X$. It is possible to do similar consideration for the case, when the operator A_r is defined in the so called *energy space* $E = D(A^{1/2}) \times X$ (Chen and Russell, 1982; Chen and Triggiani, 1989; 1990a; 1990b). In order to do that, it is enough to take the transformation $w^1 = A^{1/2}z^1$ and $w^2 = z^2$. Hence, the eigenvalues of the operator A_r will be the same as before, but the corresponding eigenfunctions will be a little different (see e.g. Chen and Russell, 1982; for details). Since in the sequel we shall consider only approximate controllability of dynamical system (1), hence it is more suitable and more natural to consider the operator A_r in the Hilbert space H, as in the paper (Kunimatsu and Ito, 1988).

The more detailed analysis of various properties of the operator A_r is presented in many papers, even for more general case than those given by the equalities (10) and (11), (see e.g. Chen and Russell, 1982; Huang, 1988; Kunimatsu and Ito, 1988; Hishikira, 1989; Chen and Triggiani, 1989; 1990a; 1990b; Lasiecka and Triggiani, 1991). In the publications (Chen and Triggiani, 1989; 1990a; 1990b) the operator A_r with the term A^{α}, $\alpha = (0,1)$ instead of $A^{1/2}$ is carefully investigated. Especially, analyticity and uniform stability of corresponding semigroups of linear bounded operators are considered in detail. Similar considerations are also given in the paper (Huang, 1988) where is stated, that $\alpha = 1$ and $\alpha = 1/2$ are the break for the stability and analytic property of the corresponding semigroup, respectively.

The paper (Kunimatsu and Ito, 1988) contains the detailed analysis of the spectral properties of the operator A_r in the case when instead of the operator $A^{1/2}$ we have linear combination of the operators A and $A^{1/2}$. In this case the form of the spectrum $\sigma(A_r)$ is more complicated as before. Some additional considerations about the spectral properties of the operator A_r can be found also in the papers (Sakawa, 1983; 1984; 1985).

4. Basic Definitions

For infinite-dimensional dynamical systems of the form (1) we may define many different notions of controllability. In the sequel we shall concentrate on so called approximate controllability.

In order to do it, first of all let us introduce the concept of so called attainable set for the dynamical system (1). The attainable set for the dynamical system (1) defined at time $t > 0$ and from zero initial conditions is given by the following formula

$$K_t = \left\{ \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \in H = X \times X : x = x(t,0,u), \quad \dot{x} = \dot{x}(t,0,u), \quad u \in U \right\} \tag{17}$$

where $x(t,0,u)$ is the unique mild solution of the abstract differential equation (1) with zero initial conditions and admissible controls $u \in U$, U is the set of admissible controls.

Similarly the attainable set for dynamical system (1) is defined as follows

$$K_{\infty} = \bigcup_{t>0} K_t \tag{18}$$

Taking into account the notions of the attainable sets given by formulae (17) and (18) we may define the concepts of approximate controllability for the dynamical system (1).

Definition 1. Dynamical system (1) is said to be approximately controllable in the time interval $[0,T]$ in the set of admissible controls U if

$$\overline{K}_T = H = X \times X \tag{19}$$

Definition 2. Dynamical system (1) is said to be approximately controllable in the set of admissible controls U if

$$\overline{K}_\infty = H = X \times X \tag{20}$$

In some cases approximate controllability in the time interval $[0,T]$ is equivalent to approximate controllability (see Triggiani, 1975; 1976; 1987; for details), but generally approximate controllability in $[0,T]$ is essentially stronger notion than approximate controllability. However, it should be stressed, that in the case when the appropriate semigroup of operators is analytic, then these two notitions of controllability are equivalent (Triggiani, 1978).

5. Approximate Controllability

In this section we shall formulate and prove necessary and sufficient conditions for approximate controllability. We shall also consider the case, when the control functions are constrained and take their values from the nonnegative convex cone.

For the abbreviation let us introduce the following notation

$$B_i = \begin{bmatrix} <b_1, x_{i1}>_X, & <b_2, x_{i1}>_X, & \dots, & <b_m, x_{i1}>_X \\ <b_1, x_{i2}>_X, & <b_2, x_{i2}>_X, & \dots, & <b_m, x_{i2}>_X \\ \dots\dots & \dots\dots & \dots, & \dots\dots \\ <b_1, x_{in_i}>_X, & <b_2, x_{in_i}>_X, & \dots, & <b_m, x_{in_i}>_X \end{bmatrix} \tag{21}$$

where B_i are $(n_i \times m)$–dimensional constant matrices for $i = 1, 2, \dots$.

Using the general results concerning approximate controllability which are given in the papers (Triggiani, 1975; 1976; 1978) we can formulate necessary and sufficient conditions for approximate controllability of dynamical system (1). First, we shall consider the case, when there is no any contraints on the controls.

Theorem 1. *Dynamical system (1) is approximately controllable in an arbitrary time interval* $[0,T]$ *if and only if the dynamical system (4) is approximately controllable.*

Proof. First of all, let us observe, that since operator A is the infinitesimal generator of analytic semigroup $S(t)$, then approximate controllability of dynamical system (4) is equivalent to approximate controllability in an arbitrary time interval $[0,T]$, (Triggiani, 1976). Similarly since the operator A_r is the infinitestimal generator of analytic semigroup $S_r(t)$, (Chen and Triggiani, 1989; 1990b), then the same statement is true for dynamical system (9). Next, let us observe, that the nonsigular transformations of the dynamical system (1) into dynamical system (9) are invertible and densely defined, then approximate controllability of dynamical system (1) is equivalent to approximate controllability of dynamical system (9). The same statement is of course true for approximate controllability in an arbitrary time interval $[0,T]$. Therefore instead the dynamical sytem (1) we may consider equivalently approximate controllability in an arbitrary time interval of the dynamical system (9).

Since the operators A_r^+ and A_r^- are both normal operators, then taking into account the equality (10) we can easily conclude that also the operator A_r is normal. Therefore, in order to check approximate controllability of dynamical system (9) we may use the necessary and sufficient conditions given by Triggiani (1976). Hence, the dynamical system (9) is approximately controllable in an arbitrary time interval if and only if

$$\text{rank } B_{ri}^+ = n_i, \quad \text{for all} \quad i = 1, 2, ... \tag{22}$$

and

$$\text{rank } B_{ri}^- = n_i, \quad \text{for all} \quad i = 1, 2, ... \tag{23}$$

where

$$B_{ri}^+ = pe^{-j\pi/2}(s_i)^{-1/2}B_i, \quad \text{for } i = 1, 2, ... \tag{24}$$

$$B_{ri}^- = pe^{j\pi/2}(s_i)^{-1/2}B_i, \quad \text{for } i = 1, 2, ... \tag{25}$$

The above statements immediately follows from the equality (11) and from the fact, that the operator A is self–adjoint, which implies that also the operator $A^{-1/2}$ is self–adjoint. Moreover, taking into account the equalities (22)–(25) it is easy to verify that dynamical system (9) is approximately controllable in an arbitrary time interval $[0,T]$ if and only if

$$\text{rank } B_i = n_i, \quad \text{for all} \quad i = 1, 2, ... \tag{26}$$

The last statement is an immediate consequence of the observation that for all $i = 1, 2, ...$ we have $pe^{-j\pi/2}(s_i)^{-1/2} \neq 0$ and similarly, $pe^{j\pi/2}(s_i)^{-1/2} \neq 0$.

On the other hand the formula (26) is the necessary and sufficient condition for approximate controllability of the first–order dynamical system (4), (see Triggiani, 1976; for details).

Therefore we have proved, that approximate controllability in an arbitrary time interval of second–order dynamical system (1) is equivalent to approximate controllability of first–order dynamical system (4). Hence, our theorem follows.

Corollary 1. *If the operator A has only single eigenvalues ($n_i = 1$, for $i = 1, 2, ...$), then the dynamical system (1) is approximately controllable in an arbitrary time interval $[0,T]$ if and only if*

$$< b_1, x_i >_X^2 + < b_2, x_i >_X^2 + ... + < b_m, x_i >_X^2 \neq 0, \quad \text{for all} \quad i = 1, 2, ... \tag{27}$$

Proof. For the case $n_i = 1$, $i = 1, 2, \ldots$ we have

$$B_i = [< b_1, x_i >_X, < b_2, x_i >_X, \ldots, < b_m, x_i >_X] \quad \text{for} \quad i = 1, 2, \ldots \tag{28}$$

where for simplicity: $x_i = x_{i1}$, $i = 1, 2, \ldots$. Hence, condition (26) is equivalent to inequalities (27). Therefore our corollary follows.

Corollary 2. *If the operator A has only single eigenvalues and moreover, $m = 1$ (scalar control), then the dynamical system (1) is approximately controllable in an arbitrary time interval $[0, T]$ if and only if*

$$< b_1, x_i >_X \neq 0, \quad \text{for all} \quad i = 1, 2, \ldots \tag{29}$$

Proof. Corollary 2 immediately follows from equality (26) and the corollary 1.

Now, we shall concentrate on approximate controllability with constrained controls. We shall asume, that the controls $u_j(t) \geq 0$, for $j = 1, 2, \ldots, m$.

Theorem 2. *Dynamical system (1) is approximately controllability with nonegative controls if and only if it is approximately controllable in an arbitrary time interval $[0, T]$ without any constraints on control.*

Proof. Since operator A has only complex eigenvalues then from the papers (Klamka, 1991b) and (Son, 1990) follows that dynamical systems (9) or equivalently (1) are approximately controllable with nonnegative controls if and only if the condition (25) holds. Therefore by theorem 1 dynamical system (1) is approximately controllable with nonnegative controls if and only if it is approximately controllable in an arbitrary time interval $[0, T]$ without any constraints on the controls. Hence, our theorem follows.

6. Illustrative Example

In this section we shall present example of dynamical system (1) with inherent damping. We shall consider the flexible slender beam of the lenght L supported on two ends. Such mechanical structure with internal viscous damping of the Voigt type can be described by the following linear partial differential equation (Kunimatsu and Ito, 1988)

$$w_{tt}(t, q) - 2r w_{tqq}(t, q) + w_{qqqq}(t, q) = b(q) u(t) \tag{30}$$

defined for $q \in [0, L]$ and $t > 0$, with initial conditions

$$w(0, q) = w_0(q), \quad \text{and} \quad w_t(0, q) = w_1(q), \quad \text{for} \quad q \in [0, L] \tag{31}$$

and boundary conditions

$$w(t, 0) = w(t, L) = w_{qq}(t, 0) = w_{qq}(t, L) = 0, \quad \text{for} \quad t \geq 0 \tag{32}$$

In the equation (30), $r \in (0,1)$ is a damping coefficient and function $b(q) \in X = L^2([0,L],R)$. Moreover, it is assumed that the control $u(t) \geq 0$, for $t \geq 0$.

The function $w(t,q)$ denotes the displacement from the reference state at time $t \geq 0$ and in the position $q \in [0,L]$. The boundary conditions (32) correspond to hinged ends of the flexible beam.

Linear partial differential equation (30) can be represented as a linear abstract differential equation of the form (1) defined in a separable Hilbert space $X = L^2([0,L],R)$. In order to do that, let us define the linear unbounded operator $A : X \supset D(A) \to X$ as follows (Kunimatsu, 1988)

$$Aw = Aw(t,q) = \frac{\partial^4 w(t,q)}{\partial x^4} = w_{qqqq}(t,q) \tag{33}$$

The domain $D(A)$ of the operator A is dense in Hilbert space $X = L^2([0,L],R)$ and is given explicitly by the following equality

$$D(A) = \{w \in X : w \in H^4[0,L],\ w(t,0) = w(t,L) = w_{qq}(t,0) = w_{qq}(t,L) = 0\} \tag{34}$$

where the term $H^4[0,L]$ denotes the Sobolev space of order four (Kunimatsu, 1988).

Linear operator A defined by the equalities (33) and (34) is self–adjoint, positive–definite and has pure discrete point spectrum, consisting entirely with real positive isolated eigenvalues (Kunimatsu, 1988)

$$s_i = (\pi i/L)^4, \quad \text{for } i = 1,2,... \tag{35}$$

each of multiplicity $n_i = 1$, for $i = 1,2,....$

The corresponding complete orthonormal set of eigenfunctions in the space $X = L^2([0,L],R)$ has the following form (Kunimatsu, 1988)

$$x_i(q) = (2/L)^2 \sin(\pi i q/L), \quad \text{for } q \in [0,L] \tag{36}$$

Therefore combining the results given in corollary 2 and in theorem 2 one can easily to conclude, that the second–order dynamical system (30) is approximately controllable with nonnegative controls if and only if

$$< b, x_i >_X = \int_0^L b(q)(2/L)^{1/2} \sin(\pi i q/L) dq \neq 0, \quad \text{for all } i = 1,2,... \tag{37}$$

Therefore, for example if we take the function $b(q) = 1$, for $q \in [0,L]$, then since $< b, x_i >_X = 0$, for $i = 2n$, $n = 1,2,...$ the dynamical system (1) is not approximately controllable with nonnegative controls, and in fact by theorem 2 it is not approximately controllable in any time interval $[0,T]$ with arbitrary unconstrained controls.

However, if we take the function $b(q) = q$, for $q \in [0, L]$, then since $< b, x_i >_X \neq 0$, for all $i = 1, 2, ...$, the dynamical system (1) is approximately controllable with nonnegative controls.

Finally, it should be stressed, that in this case the operator $A^{1/2}$ is given by the following formula (Kunimatsu and Ito, 1988)

$$A^{1/2}w = A^{1/2}w(t,q) = -\frac{\partial^2 w(t,q)}{\partial t} = -w_{qq}(t,q) \tag{38}$$

and the domain $D(A^{1/2})$ is expressed as follows (Kunimatsu and Ito, 1988)

$$D(A^{1/2}) = \{w \in X : w \in H^2[0,L], w(t,0) = w(t,L) = 0\} \tag{39}$$

where the term $H^2[0,L]$ denotes the Sobolev space of order two (Kunimatsu and Ito, 1988). Since $D(A) \subset D(A^{1/2}) \subset X$, then of course the set $D(A^{1/2})$ is dense in X.

Similar mechanical dynamical systems which can be described by mathematical model (1) may be found, for example, in the papers (Sakawa, 1983; 1984; 1985), where flexible arm of the manipulator is considered in detail.

7. Concluding Remarks

In this final section let us collect some remarks and comments on controllability conditions given the section 5.

Remark 1. The case $r = 0$ has been considered in the paper (Triggiani, 1978) where the relationships between approximate controllability in $[0, T]$ and approximate controllability of the first and second–order abstract dynamical systems defined in separable Banach spaces have been analyzed. The present paper extends these results to the case $r \in (0, 1)$.

Remark 2. The case $r = 1$ has not yet been considered in the literature except the conditions for uniqueness and existence of solution (see e.g. Chen and Russell, 1982; Huang, 1988; Chen and Triggiani, 1989; 1990a; 1990b; for details). In this case the appropriate linear first–order abstract differential equation possesses operator A_r in Jordan canonical form with nontrivial two–dimensional Jordan blocks. Hence, the approximate controllability conditions are in this case more complicated and desire more refined technique.

Remark 3. In the case, when we consider operator A^α, $\alpha \in (0, 1)$ instead of the operator $A^{1/2}$, the similar technique as in the present paper can be used to derive necessary and sufficient conditions for approximate controllability of the second–order dynamical system (1). However, it should be pointed out, that in this case operator A_r, $r \in (0, 1)$ generally has real and complex eingenvalues, and hence the analysis of approximate controllability with nonnegative controls is not so easy as in the case $\alpha = 1/2$.

Remark 4. It is possible to extend the presented results for the case of semilinear second–order dynamical systems, when the nonlinear term is small enough. Conditions for existence of solution of first–order semilinear abstract differential equation can be found for example in the papers (Zhou, 1983; 1984).

Remark 5. Other approximate controllability conditions for linear abstract dynamical systems are presented by Lasiecka and Triggiani (1991), Nambu (1984; 1985) and Narukawa (1982; 1984) using the various methods of functional analysis.

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