Appl. Math. and Comp. Sci., 1993, vol.3, No.2, 313–328

ANALYTICAL STUDY OF THE KALMAN FILTER FOR STATIONARY DYNAMIC SYSTEMS

PETRO BIDYUK*, VLADIMIR PODLADCHIKOV*, IRYNA PODLADCHIKOVA*

This paper is devoted to development of analytical approach to the Kalman filter study. This approach is based upon making use of Riccati equation solution for a class of dynamic systems considered, as well as Kalman and Wiener filters transition matrices. An example of a moving object state estimation shows advantages of the method proposed, the main of which are substantial reduction of computational expenses on modelling of estimation algorithm and a possibility of analytical study of estimation results. A case of correlated measurement noise is also considered. Optimal and suboptimal filters are applied to this case, and expressions for filtering errors are derived. The results obtained can be used to determine a possibility of using suboptimal instead of optimal filters, and to reduce computational expenses.

1. Introduction

The Kalman filter is nicely fit for computer realization, and is widely used to solve practical problems of real time data processing in different fields of engineering. However, recursive nature of the filter and difficulties with nonlinear Riccati equation solution make it necessary to utilize widely numerical techniques to perform qualitative analysis of corresponding algorithms, sensitivity analysis, etc.

The quality of state and parameter estimation for a dynamic system is determined by its dynamic properties as well as quality and statistics of measurements. If the speed of measurement data processing is very high (sampling period is short), then an assumption about statistical independence of measurable coordinates (states) may result in additional estimation errors. The reasons for this are the following:

- there exists correlation between measurements,
- the true data characterizing dynamic system behavior (e.g. in radar tracking system) are usually less optimistic than supposed.

On the other hand, optimal filtering of correlated noise puts ahead higher requirements to the computer for realization of corresponding algorithms. It follows from the above that optimal filtering of correlated noise or utilization of suboptimal

* Department of Applied Mathematics, Kiev Polytechnic Institute, 252056, Kiev, Ukraine

algorithm, based on the assumption of statistical independence of measurement errors requires *a priori* analysis and comparative study of their estimating properties. Usually analysis of the filter characteristics and development of engineering approaches to synthesis of specific filters is fulfilled by numerical techniques (Anderson and Moor, 1979; Rivkin *et al.*, 1976). The reason for this is that the Kalman filter, and solution of Riccati equation, which gives covariance matrix for errors of filtering, have an explicit recursive form. Such an approach imposes restriction on qualitative study of filter algorithm, and prevents from finding a solution to the problem of influence of a set of different conditions on the quality of estimation.

The practical aspects of changing initial conditions to improve quality of estimates, and to reduce transition period of an optimal filter were considered by Ljung and Kailath (1977). They considered specific conditions, and results of research were restricted by these conditions. Divergence analysis of an optimal filter is considered by Bidyuk (1985) who used integral equations of optimal filter that could be found in (Roitenberg, 1978). An influence study of of the finite word length of digital computer on the quality of optimal estimates was considered there. Possible reasons of estimation process divergence were established. A method of equivalent observation was used by Spencer (1981) to estimate sensitivity of errors to statistical parameters of measurement noise. Easily calculated partial derivatives of final error covariances with respect to parameter variances were found. Availability of the partial derivatives allows for an efficient ranking of error contributions to determine which error variances should be parametrically varied. The method is restricted to the white noise case. Discrete Kalman filter stability study by making use of integral representation of the filter was performed by Korbicz *et al.*, (1988). Here, transition matrix of the filter was used to analyse the influence of initial conditions on the quality of estimates including singular and non–singular initial conditions. The problem of reliable state estimation for linear dynamic systems is considered by Basin and Orlov (1992). The authors develop a theory for ellipsoidal estimation of state vector using continuous and discrete–time measurements. But this study touches only upon estimation with respect to small variations in measurements and skips general case. No special attention is devoted to asymptotic behavior of filter algorithm. In (Farber, 1992) analysis of filtering errors is performed from the point of view of finite word length in computer used for filter implementation. The author derived useful expressions for error estimation and gave recommendations for computer word length selection. The problem of linear filtering of stationary random process is considered by Golubev *et al.*, (1992).

The purpose of this paper is to determine analytical representation for the principal characteristics of the Kalman filter, that allow for a study of potential possibilities of filtering algorithms for a class of dynamic systems such as moving objects or targets. Application of the proposed analytical approach leads to substantial shortage of the computer simulation time during the process of filter design, and sometimes permits to exclude simulation at all. Here we also develop an analytical representation for variances of actual optimal and suboptimal errors of filtering for the case of correlated measurement noise. These analytical expressions are used to analyse possibilities of suboptimal algorithm application without substantial loss of estimation quality.

2. Kalman Filter Transition Matrix and Covariance Matrices for Estimation Error

2.1. Analytical Representation of the Kalman Filter

Consider a linear stationary dynamic system described by equations

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{G}\boldsymbol{w}(t) \tag{1}$$

$$\boldsymbol{z}(t) = \boldsymbol{H}\boldsymbol{x}(t) + \boldsymbol{v}(t) \tag{2}$$

where $\boldsymbol{x}$ is n–vector of dynamic system states, $\boldsymbol{F}$ - $(n \times n)$ matrix of system dynamics, $\boldsymbol{G}$ - $(n \times m)$ matrix of inputs, $\boldsymbol{H}$ - $(p \times n)$ matrix of observation, $\boldsymbol{w}(t)$ and $\boldsymbol{v}(t)$ are white noise random processes with zero mean and covariance matrices $\boldsymbol{Q}$, and $\boldsymbol{R}$, respectively.

Continuous optimal filter equation for system (1), (2) is as follows (Anderson and Moor, 1979)

$$\frac{\mathrm{d}\hat{\boldsymbol{x}}(t)}{\mathrm{d}t} = \boldsymbol{F}\hat{\boldsymbol{x}}(t) + \boldsymbol{K}(t)[\boldsymbol{z}(t) - \boldsymbol{H}\hat{\boldsymbol{x}}(t)]$$

where $\hat{\boldsymbol{x}}$ is an optimal estimate of state vector, $\boldsymbol{K}(t) = \boldsymbol{P}(t)\boldsymbol{H}^T\boldsymbol{R}^{-1}$ is optimal filter gain, $\boldsymbol{P}(t)$ denotes covariance matrix of estimate errors, and is described by a differential Riccati equation

$$\frac{\mathrm{d}\hat{\boldsymbol{P}}(t)}{\mathrm{d}t} = \boldsymbol{F}\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{F}^T - \boldsymbol{P}(t)\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{P}(t) + \boldsymbol{G}\boldsymbol{Q}\boldsymbol{G}^T, \tag{3}$$

$$\boldsymbol{P}(0) = \boldsymbol{P}_0$$

Most often the solution of equation (3) is determined be making use of numerical techniques (Andreyev, 1976). But for some class of models it is possible to find an explicit solution of Riccati equation. Using results given by Andreyev (1976) analytical solution of Riccati equation for stationary systems can be found as follows

$$\begin{aligned}\boldsymbol{P}(t) &= \boldsymbol{P}_a + \{\exp(-\boldsymbol{A}^T t)(\boldsymbol{P}_0 - \boldsymbol{P}_a)^{-1}\exp(-\boldsymbol{A}t) \\ &+ \int_0^t [\exp(\boldsymbol{A}^T(\tau - t))\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}\exp(\boldsymbol{A}(\tau - t))\mathrm{d}\tau\}^{-1}\end{aligned} \tag{4}$$

where $\boldsymbol{A} = \boldsymbol{F} - \boldsymbol{P}_a\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}$ is a matrix of dynamics for the Wiener filter designed for system (1), (2); $\boldsymbol{P}_a$ is a solution of algebraic Riccati equation.

2.2. Transition Matrix for Optimal Filter

Explicit form of transition matrix for optimal filter can be found for the following free dynamic system

$$\frac{\mathrm{d}\boldsymbol{x}^*(t)}{\mathrm{d}t} = \boldsymbol{F}\boldsymbol{x}^*(t) \tag{5}$$

$$\boldsymbol{z}(t) = \boldsymbol{H}\boldsymbol{x}^*(t) + \boldsymbol{v}(t) \tag{6}$$

If $\boldsymbol{A} = \boldsymbol{F} - \boldsymbol{P}_a\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}$ is a transition matrix for the Wiener filter constructed for system (1), and covariance matrix $\boldsymbol{P}_0^* = \boldsymbol{P}_0 - \boldsymbol{P}_a$, where $\boldsymbol{P}_0$ is a covariance matrix for initial estimate of state $\boldsymbol{x}_0$, and $\boldsymbol{P}_a$ is a steady–state solution for equation (3), then transition matrices of optimal Kalman filter for system (1), (2), and for a free dynamic system (5), (6) are identical.

It is shown by Roytenberg (1978) that transition matrix of the Kalman filter for the free dynamic system (5), (6) is as follows:

$$\boldsymbol{\Psi}^*(t,\tau) = \boldsymbol{P}^*(t)\exp(\boldsymbol{A}^T(\tau - t))(\boldsymbol{P}^*(\tau))^{-1}$$

The Kalman filter transition matrix $\boldsymbol{\Psi}(t,\tau)$ for dynamic system (1), (2) satisfies the following differential equation

$$\frac{\mathrm{d}\boldsymbol{\Psi}(t,\tau)}{\mathrm{d}t} = [\boldsymbol{F} - \boldsymbol{P}(t)\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}]\boldsymbol{\Psi}(t,\tau)$$

Transition matrix $\boldsymbol{\Psi}^*(t,\tau)$ also satisfies equation

$$\frac{\mathrm{d}\boldsymbol{\Psi}^*(t,\tau)}{\mathrm{d}t} = [\boldsymbol{A} - \boldsymbol{P}^*\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}]\boldsymbol{\Psi}^*(t,\tau)$$

It follows from the equality $[\boldsymbol{F} - \boldsymbol{P}^*\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}] = [\boldsymbol{A} - \boldsymbol{P}^*\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}]$ that $\boldsymbol{\Psi}(t,\tau) = \boldsymbol{\Psi}^*(t,\tau)$. Thus, we have

$$\boldsymbol{\Psi}(t,\tau) = [\boldsymbol{P}(t) - \boldsymbol{P}_a]\exp[(\boldsymbol{F} - \boldsymbol{P}_a\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^T(\tau - t)][\boldsymbol{P}(\tau) - \boldsymbol{P}_a]^{-1} \tag{7}$$

Taking into consideration that the transition matrix can be represented as a product of two matrices, i.e. $\boldsymbol{\Psi}(t,\tau) = \boldsymbol{S}(t)\boldsymbol{S}^{-1}(\tau)$, we find that

$$\boldsymbol{S}(t) = [\boldsymbol{P}(t) - \boldsymbol{P}_a]\exp\{[-(\boldsymbol{F} - \boldsymbol{P}_a\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})]^T t\}$$

2.3. Sensitivity Analysis of the Kalman Filter

If an optimal filter processes data with poorly defined statistics, there usually appear extra errors of estimation, which are studied by making use of numerical techniques. Using transition matrix of the Kalman filter an explicit expression for covariance matrices of extra filtering errors is derived for the case when state noise model does not fit exactly the process noise. Suppose dynamic system is influenced by two noise processes $\xi_1(t)$, and $\xi_2(t)$, then its model can be written as follows

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{G}\boldsymbol{\xi}_1(t) + \boldsymbol{G}\boldsymbol{\xi}_2(t) \tag{8}$$

$$\boldsymbol{z}(t) = \boldsymbol{H}\boldsymbol{x}(t) + \boldsymbol{v}(t),$$

where $\mathrm{cov}\{\boldsymbol{\xi}_1(t)\} = \boldsymbol{Q}_1$, $\mathrm{cov}\{\boldsymbol{\xi}_2(t)\} = \boldsymbol{Q}_2$, $\mathrm{cov}\{\boldsymbol{v}(t)\} = \boldsymbol{R}$

Let us not take into consideration during the process of filter design the other noise component $\boldsymbol{\xi}_2(t)$ i.e. the process model looks as follows

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{G}\boldsymbol{\xi}_1(t)$$

$$\boldsymbol{z}(t) = \boldsymbol{H}\boldsymbol{x}(t) + \boldsymbol{v}(t)$$

Optimal filter for this model is described by equations

$$\frac{\mathrm{d}\widehat{\boldsymbol{x}}(t)}{\mathrm{d}t} = (\boldsymbol{F} - \boldsymbol{K}(t)\boldsymbol{H})\boldsymbol{x}(t) + \boldsymbol{K}(t)\boldsymbol{z}(t) \tag{9}$$

$$\boldsymbol{K}(t) = \boldsymbol{P}_p(t)\boldsymbol{H}^T\boldsymbol{R}^{-1}.$$

Here $\boldsymbol{P}_p(t)$ denotes a covariance matrix that corresponds to an approximate model (without $\boldsymbol{\xi}_2(t)$), and satisfies the following Riccati equation

$$\frac{\mathrm{d}\boldsymbol{P}_p(t)}{\mathrm{d}t} = \boldsymbol{F}\boldsymbol{P}_p(t) + \boldsymbol{P}_p(t)\boldsymbol{F}^T + \boldsymbol{P}_p(t)\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{P}_p(t) + \boldsymbol{G}\boldsymbol{Q}_1\boldsymbol{G}^T$$

Denoting actual estimation error as $\widetilde{\boldsymbol{x}} = \boldsymbol{x} - \widehat{\boldsymbol{x}}$, and subtracting expression (9) from expression (8) we get

$$\frac{\mathrm{d}\widetilde{\boldsymbol{x}}(t)}{\mathrm{d}t} = (\boldsymbol{F} - \boldsymbol{K}(t)\boldsymbol{H}^T)\widetilde{\boldsymbol{x}}(t) + \boldsymbol{K}(t)\boldsymbol{v}(t) + \boldsymbol{\xi}_1(t) + \boldsymbol{\xi}_2(t) \tag{10}$$

A solution for the differential equation (10) can be found in the form

$$\widetilde{\boldsymbol{x}}(t) = \psi(t,0)\widetilde{\boldsymbol{x}}_0 + \int_{t_0}^{t} \psi(t,\tau)(\boldsymbol{G}\boldsymbol{\xi}_1(\tau) + \boldsymbol{G}\boldsymbol{\xi}_2(\tau) - \boldsymbol{K}(\tau)\boldsymbol{v}(\tau))\mathrm{d}\tau$$

Now the covariance matrix for the actual error of estimation is described by equation

$$\boldsymbol{P}_\phi(t) = \mathrm{cov}\{\widetilde{\boldsymbol{x}}(t)\} =$$

$$= \psi(t,0)\boldsymbol{P}_p\psi^T(t,0) + \int_0^t \psi(t,\tau)[\boldsymbol{K}(t)\boldsymbol{R}\boldsymbol{K}^T(\tau) + \boldsymbol{Q}_1 + \boldsymbol{Q}_2]\psi^T(t,\tau)\mathrm{d}\tau$$

In the same way the covariance matrix can be found for the case of an approximate system model

$$\boldsymbol{P}_p(t) = \psi(t,0)\boldsymbol{P}_0\psi^T(t,0) + \int_o^t \psi(t,\tau)[\boldsymbol{K}(\tau)\boldsymbol{R}\boldsymbol{K}^T(\tau) + \boldsymbol{Q}_1]\psi^T(t,\tau)\mathrm{d}\tau$$

Thus, additional filtering error generated during the filter run is defined by equation

$$\Delta P(t) = P_{\phi}(t) - P_{p}(t) = \int_{0}^{t} \psi(t,\tau) Q_2 \psi^T(t,\tau) \mathrm{d}\tau \tag{11}$$

as the state noise term $\xi_2(t)$ was omitted.

2.4. Example of Estimation of Moving Object States

Dynamic System Model. Using equation (1) characteristics of the Kalman filter that is used for estimation of position and velocity of a moving object is determined. Only position is measured, and velocity is disturbed by random, non–correlated in time, acceleration.

Dynamic system matrices are defined as follows

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Phi(\tau, t) = e^{F(\tau - t)} \simeq \begin{bmatrix} 1 & \tau - t \\ 0 & 1 \end{bmatrix}$$

$$G = [0\ 1]^T \quad H = [1\ 0], \quad Q = \sigma_a^2, \quad R = \sigma_r^2$$

Riccati Equation. A matrix Riccati equation for the model under consideration can be replaced by the following system of first order equations

$$\begin{aligned} &2p_{12a} - p_{11a}^2/\sigma_r^2 = 0 \\ &p_{22a} - p_{11a}p_{12a}/\sigma_r^2 = 0 \\ &-p_{11a}^2/\sigma_r^2 + \sigma_a^2 = 0 \end{aligned} \tag{12}$$

where p_{11a}, p_{12a}, p_{22a} are elements of matrix P_a,

$$P_a = \begin{bmatrix} p_{11a} & p_{12a} \\ p_{21a} & p_{22a} \end{bmatrix}$$

Solving equations (12), we get the following formula

$$P_a = \sigma_r^2 \begin{bmatrix} (2\kappa)^{1/2} & \kappa \\ \kappa & \kappa(2\kappa)^{1/2} \end{bmatrix}$$

where $\kappa = \sigma_a/\sigma_r$

Matrix of Dynamics and Transition Matrix for the Wiener Filter. Matrix of dynamics for the Wiener filter is defined in this case as follows

$$A = F - P_a H^T R^{-1} = \begin{bmatrix} -(2\kappa)^{1/2} & 1 \\ -\kappa & 0 \end{bmatrix}$$

Hamilton–Cayley theorem allows us to find the Wiener filter transition matrix

$$\exp(\boldsymbol{A}t) = \begin{bmatrix} -(2\kappa)^{1/2}\alpha(t) + \beta(t) & \alpha(t) \\ -\kappa\alpha(t) & \beta(t) \end{bmatrix} \tag{13}$$

where

$$\alpha(t) = \left(\sin((\kappa/2)^{1/2}t)/(\kappa/2)^{1/2}\right)\exp(-(\kappa/2)^{1/2}t)$$
$$\beta(t) = \exp(-(\kappa/2)^{1/2}t)\left(\cos((\kappa/2)^{1/2}t) + \sin((\kappa/2)^{1/2}t)\right)$$

Estimation Error Covariances. Suppose that *a priori* information characterizing the initial system state is not available, i.e. $\boldsymbol{P}_0^{-1} = 0$. Substituting matrix (13) into equation (4), and taking integral we get

$$(\boldsymbol{P}(t) - \boldsymbol{P}_a)^{-1} =$$
$$= \frac{1}{2}\begin{bmatrix} \left[(2 + \sin((2\kappa)^{1/2}t) - \cos((2\kappa)^{1/2}t))\exp((2\kappa)^{1/2}t)\right](2\kappa)^{-1/2} & \left[\exp((2\kappa)^{1/2}t)\left(\cos((2\kappa)^{1/2}t) - 1\right)\right]\kappa^{-1} \\ \left[\exp((2\kappa)^{1/2}t)\left(\cos((2\kappa)^{1/2}t) - 1\right)\right]\kappa^{-1} & \left[(2 - \sin((2\kappa)^{1/2}t) - \cos((2\kappa)^{1/2}t))\left(\exp((2\kappa)^{1/2}t) - 1\right)\right](2\kappa)^{-1/2} \end{bmatrix} \tag{14}$$

Calculating inverse for the matrix determines an explicit expression for the covariances of filtering errors in transition mode of the filter as an explicit function of time and variances of state and measurement noise

$$\boldsymbol{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \tag{15}$$

where

$$p_{11}(t) = (\sigma_r^2(2\kappa)^{1/2}/D)(\exp(2(2\kappa)^{1/2}t) - 2\exp((2\kappa)^{1/2}t)\sin((2\kappa)^{1/2}t) + 1)$$
$$p_{12}(t) = p_{21}(t) = (\sigma_r^2\kappa/D)(\exp(2(2\kappa)^{1/2}t) - 2\exp((2\kappa)^{1/2}t)\cos((2\kappa)^{1/2}t) + 1)$$
$$p_{22}(t) = (\sigma_r^2\kappa(2\kappa)^{1/2}/D)(\exp(2(2\kappa)^{1/2}t) + 2\exp((2\kappa)^{1/2}t)\sin((2\kappa)^{1/2}t) - 1)$$
$$D = \exp(2(2\kappa)^{1/2}t) - 2\exp((2\kappa)^{1/2}t)(2 - \cos((2\kappa)^{1/2}t) + 1$$

For the special case of elimination of the state noise, and with $\kappa \longrightarrow 0$, we find the following result

$$\boldsymbol{P}(t) = \begin{bmatrix} 4/t & 6/t^2 \\ 6/t^2 & 12/t^3 \end{bmatrix}$$

Optimal Kalman Filter Transition Matrix. Now determine matrix $\boldsymbol{S}^{-1}(\tau)$ for the system model under consideration. Substituting expression (14) for $(\boldsymbol{P}(\tau) - \boldsymbol{P}_a)^{-1}$ into equation (7), and $\exp(\boldsymbol{A}) = \exp(\boldsymbol{F} - \boldsymbol{P}_a\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})$, after appropriate transformations we get

$$S^{-1}(\tau) = \begin{bmatrix} b_{11}(\tau) & b_{12}(\tau) \\ b_{21}(\tau) & b_{22}(\tau) \end{bmatrix}$$

where

$$b_{11} = (2\kappa)^{-1/2}(\cos((\kappa/2)^{1/2}t)\mathrm{sh}((\kappa/2)^{1/2}t) + \sin((\kappa/2)^{1/2}t)\mathrm{ch}((\kappa/2)^{1/2}t)$$

$$b_{12} = -\left[(\sin((\kappa/2)^{1/2}t))\kappa^{-1}\right]\mathrm{sh}((\kappa/2)^{1/2}t)$$

$$b_{21} = \left[(\sin((\kappa/2)^{1/2}t))\kappa^{-1}\right]\mathrm{sh}((\kappa/2)^{1/2}t)$$

$$b_{22} = (\kappa(2\kappa)^{-1/2})(\cos((\kappa/2)^{1/2}t)\mathrm{sh}((\kappa/2)^{1/2}t) - \sin((\kappa/2)^{1/2}t)\mathrm{ch}((\kappa/2)^{1/2}t))$$

Now explicit form for the optimal filter transition matrix is as follows

$$\psi(t,\tau) = (d(t))^{-1}\begin{bmatrix} b_{22}(t) & -b_{12}(t) \\ -b_{21}(t) & b_{11}(t) \end{bmatrix}\begin{bmatrix} b_{11}(\tau) & b_{12}(\tau) \\ b_{21}(\tau) & b_{22}(\tau) \end{bmatrix}, \qquad (16)$$

where

$$d(t) = (8\kappa^2)^{-1}(\mathrm{ch}(2\kappa)^{1/2}t) + \cos((2\kappa)^{1/2}t) - 2)$$

Kalman Filter Sensitivity to State Noise Model Errors. Define additional error of filtering for the class of models (1), (2). Substituting expression (16) for $\psi(t,\tau)$ into formula (11), and taking integral we have

$$\Delta P(t) = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}$$

where

$$\begin{aligned}\Delta_{11} = \sigma_{a2}^2(4((2\kappa)^{1/2})^{11}D_1^2)^{-1}\{\gamma - \lambda + \gamma_1 \times \\ \times \left[4((2\kappa)^{1/2}t)(\sin((2\kappa)^{1/2}t) - 1) + 2\cos((2\kappa)^{1/2}t) - 2\sin((2\kappa)^{1/2}t) - 4\right] + \\ +\lambda\left[-4(2\kappa)^{1/2}t(\sin((2\kappa)^{1/2}t) + 1) - 2\sin((2\kappa)^{1/2}t) - 2\cos((2\kappa)^{1/2}t) + 4\right] + \\ + 8\sin((2\kappa)^{1/2}t) - 4\sin((2\kappa)^{1/2}t)\cos((2\kappa)^{1/2}t) + 8(2\kappa)^{1/2}t\cos((2\kappa)^{1/2}t)\}\end{aligned}$$

$$\begin{aligned}\Delta_{12} = \Delta_{21} = \sigma_{a2}^2(8((2\kappa)^{1/2})^{10}D_1^2)^{-1}\{2\gamma + 2\lambda + \gamma_1 \times \\ \times\left[-4(2\kappa)^{1/2}t(-\sin((2\kappa)^{1/2}t) - \cos((2\kappa)^{1/2}t + 1) - 8\right] + \\ +\lambda_1\left[4(2\kappa)^{1/2}t(\sin((2\kappa)^{1/2}t) - \cos((2\kappa)^{1/2}t) + 1) - 8\right] + \\ + 16\cos((2\kappa)^{1/2}t) - 4 + 8\sin^2((2\kappa)^{1/2}t) - 8(2\kappa)^{1/2}t\sin((2\kappa)^{1/2}t)\}\end{aligned}$$

$$\begin{aligned}\Delta_{22} = \sigma_{a2}^2(8((2\kappa)^{1/2})^{9}D_1^2)^{-1}\{3\gamma - 3\lambda + \gamma_1 \times \\ \times\left[6\sin((2\kappa)^{1/2}t) + 6\cos((2\kappa)^{1/2}t) - 12 - 4(2\kappa)^{1/2}t(1 - \cos((2\kappa)^{1/2}t))\right] + \\ +\lambda_1\left[6\sin((2\kappa)^{1/2}t) - 6\cos((2\kappa)^{1/2}t) + 12 - 4(2\kappa)^{1/2}t(1 - \cos((2\kappa)^{1/2}t))\right] +\end{aligned}$$

$$+8(2\kappa)^{1/2}t(1-\cos((2\kappa)^{1/2}t))-24\sin((2\kappa)^{1/2}t)+$$
$$+12\sin((2\kappa)^{1/2}t)\cos((2\kappa)^{1/2}t)\}$$

$$D_1^2=(8\kappa^2)^{-1}(\gamma_1+\lambda_1-4+2\cos((2\kappa)^{1/2}t))$$
$$\kappa=\sigma_{a1}/\sigma_r,\quad \gamma=\exp(2(2\kappa)^{1/2}t),\quad \gamma_1=\exp((2\kappa)^{1/2}t)$$
$$\lambda=\exp(-2(2\kappa)^{1/2}t),\quad \lambda_1=\exp(-(2\kappa)^{1/2}t)$$

If t is going to infinity, steady–state estimation errors caused by modeling errors of the state noise have the following covariances

$$\lim_{t\to\infty}\Delta\boldsymbol{P}(t)=\sigma_{a2}^2\begin{bmatrix}(2\kappa(2\kappa)^{1/2})^{-1} & (2\kappa)^{-1}\\ (2\kappa)^{-1} & 3(2(2\kappa)^{1/2})^{-1}\end{bmatrix}$$

Thus, additional error of a position signal estimation is bounded by the value $\sigma_{a2}^2(2\kappa(2\kappa)^{1/2})^{-1}$, and for the velocity signal error we have the following limit: $3\sigma_{a2}^2(2(2\kappa)^{1/2})^{-1}$.

If σ_{a1} tends to zero we will find an explicit expression for additional errors of estimation for the special case when noise term ξ_1 is not present

$$\lim_{\sigma_{a1}\to 0}\Delta\boldsymbol{P}(t)=\sigma_{a2}^2\begin{bmatrix}t^3/105 & 11t^2/210\\ 11t^2/210 & 13t/35\end{bmatrix}$$

This expression shows that additional errors of estimation verge to infinity with $t\to\infty$, i.e. the Kalman filter estimates diverge. This is true for the case when the actual system is disturbed by the state noise, but the filter is built for a free dynamic system.

3. Error of Filtering for a Case of Correlated Measurement Noise

3.1. Mathematical Problem Statement

Let system dynamics be described by a differential equation as follows (free dynamic system)

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}=\boldsymbol{F}\boldsymbol{x}(t)\tag{17}$$

$$\boldsymbol{z}(t)=\boldsymbol{H}\boldsymbol{x}(t)+\boldsymbol{v}(t)\tag{18}$$

where $\boldsymbol{x}$ is a state vector, and $\boldsymbol{F}$ denotes a matrix of system dynamics.

Assuming that the components of the state vector are measured at the presence of additive noise, a linear measurement equation will be like (2). In this equation $\boldsymbol{v}(t)$ is a random noise process with known statistics

$$E\{v(t)\} = 0$$

$$E\{v(t)v^T(\tau)\} = R\exp[-\lambda(t-\tau)] = \sigma^2 I \exp[-\lambda(t-\tau)] = L(t,\tau) \quad (19)$$

where σ^2 is a variance of measurement error for the component of state vector, $\lambda = 1/\tau$ is an inverse value with respect to the correlation time, and I is a unity matrix.

Generalization of the optimal filtering problem for the errors correlated in time is based on the filter design that establishes a link between the vector of measurement errors $v(t)$, and the white noise vector $\eta(t)$

$$\frac{dv(t)}{dt} = -\lambda v(t) + \eta(t) \quad (20)$$

The optimal filtering algorithm for the case of a coloured measurement noise is represented by the modified Kalman filter with the following observation matrix (Sage and Melsa, 1971)

$$H^*(t) = \frac{dH(t)}{dt} + H(t)F(t) - \lambda H(t) \quad (21)$$

with measurements

$$z^*(t) = \frac{dz(t)}{dt} - \lambda z(t) \quad (22)$$

The filtering algorithm is defined by the equation (Rivkin *et al.*, 1976)

$$\frac{d\hat{x}(t)}{dt} = F(t)\hat{x}^*(t) + K^*(t)\nu(t) \quad (23)$$

where $\hat{x}^*(t)$ is an optimal estimate of state vector, and innovation $\nu(t)$ is defined by the equation

$$\nu(t) = \frac{dz(t)}{dt} - \lambda z(t) - H^*(t)\hat{x}^*(t) \quad (24)$$

The filter gain is computed using a known equation

$$K^*(t) = P^*(t)H^{*T}(t)R^{*-1}(t) \quad (25)$$

where $R^*(t) = \mathrm{cov}\{\eta(t)\} = 2\sigma^2\lambda I$. The covariance matrix of filtering errors (argument t is dropped to simplify writing) is given by

$$\frac{dP^*(t)}{dt} = FP^* + P^*F^T - P^*H^{*T}R^{*-1}H^*P^{*-1} \quad (26)$$

with the following initial conditions

$$P_0^* = \left[H^T(0)\,H(0)\right]^{-1}\sigma^2, \quad x_0^*(0) = \left[H^T(0)\,H(0)\right]^{-1}\left[H^T(0)z(0)\right] \quad (27)$$

The problem is to determine the covariance matrix of estimate error as an explicit function of time for the cases of optimal and suboptimal filtering, and then to use these expressions to develop recommendations on the possibilities of application of suboptimal algorithms.

3.2. Derivation of Covariance Matrices for Estimate Error for a Case of Coloured Measurement Noise

An integral representation of the Kalman filter will be used to analyse the quality of estimation. It is shown by Roitenberg (1979) that for continuous–time systems in the absence of state noise an optimal estimate of the system state vector can be found as an explicit function of time and model parameters.

As in the case of white measurement noise an optimal estimate of state vector for a system with coloured errors is determined by the equation

$$\hat{\boldsymbol{x}}^*(t) = \boldsymbol{P}^*(t) \int_0^t \boldsymbol{\phi}^T(\tau, t)[\boldsymbol{H}^*(\tau)]^T [\boldsymbol{R}^*(\tau)]^{-1} \boldsymbol{z}(\tau) \mathrm{d}\tau \tag{28}$$

where

$$\boldsymbol{P}^*(t) = \left\{ \int_0^t \boldsymbol{\phi}^T(\tau, t)[\boldsymbol{H}^*]^T [\boldsymbol{R}^*]^{-1} \boldsymbol{H}^* \boldsymbol{\phi}(\tau, t) \mathrm{d}\tau + \boldsymbol{\phi}^T(0, t)[\boldsymbol{P}_0^*]^{-1} \boldsymbol{\phi}(0, t) \right\}^{-1} \tag{29}$$

The suboptimal estimate of a state vector, with assumption that measurement noise is non–correlated, can be calculated using the equation

$$\hat{\boldsymbol{x}}(t) = \boldsymbol{P}(t) \int_0^t \boldsymbol{\phi}^T(\tau, t) \boldsymbol{H}^T(\tau) \boldsymbol{R}^{-1}(\tau) \boldsymbol{z}(\tau) \mathrm{d}\tau \tag{30}$$

where $\boldsymbol{P}(t)$ is defined by expression

$$\boldsymbol{P}(t) = \left[\int_0^t \boldsymbol{\phi}^T(\tau, t) \boldsymbol{H}^T \boldsymbol{R}^{-1} \boldsymbol{H} \boldsymbol{\phi}(\tau, t) \mathrm{d}\tau \right]^{-1} \tag{31}$$

Now constructing the relation

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{P}(t) \boldsymbol{P}^{-1}(t) \boldsymbol{x}(t) = \\ &= \boldsymbol{P}(t) \left[\int_0^t \boldsymbol{\phi}^T(\tau, t) \boldsymbol{H}^T \boldsymbol{R}^{-1} \boldsymbol{H} \boldsymbol{\phi}(\tau, t) \mathrm{d}\tau \right] \boldsymbol{x}(t) \end{aligned} \tag{32}$$

and using (19), (30), and the property of transition matrix that $\boldsymbol{\phi}(\tau, t) = \boldsymbol{\phi}(\tau, \sigma) \boldsymbol{\phi}(\sigma, t)$ the estimate error can be determined as follows

$$\hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t) = \boldsymbol{P}(t) \int_0^t \boldsymbol{\phi}^T(\tau, t) \boldsymbol{H}^T(\tau) \boldsymbol{R}^{-1}(\tau) \boldsymbol{v}(\tau) \mathrm{d}\tau \tag{33}$$

3.3. Illustrative Example

Let dynamic system matrices be defined as in section 2.4 and measurement noise statistics correspond to (19). Thus $\boldsymbol{H}^*(t) = [\lambda \;\; 1]^T$. Initial conditions correspond to (27). Such a model can be used to describe moving object tracking process, with state vector $x = \left[x \; \frac{\mathrm{d}x}{\mathrm{d}t}\right]^T$, where the first element is object position, and the other is its velocity.

Substituting components of matrices $\boldsymbol{\Phi}^*, \boldsymbol{H}^*, \boldsymbol{R}^*$, and $\boldsymbol{P}_0^*$ into equation (29) we have

$$[\boldsymbol{P}^*(t)]^{-1} = \frac{1}{2\sigma^2\lambda}\begin{bmatrix} \lambda^2 t^2 & (2\lambda t - \lambda^2 t^2)/2 \\ (2\lambda t - \lambda^2 t^2)/2 & (\lambda^2 t^3 - 3\lambda t^3 + 3t)/3 \end{bmatrix} +$$

$$+\frac{1}{\sigma^2}\begin{bmatrix} 1 & -t \\ -t & t^2 \end{bmatrix} = \frac{1}{2\sigma^2\lambda}\begin{bmatrix} \lambda^2 t^2 + 2\lambda & -(\lambda^2 t^2 + 2\lambda t)/2 \\ -(\lambda^2 t^2 + 2\lambda t)/2 & (\lambda^3 t^3 + 3\lambda t^2 + 3t)/3 \end{bmatrix} \quad (34)$$

Finding an inverse for matrix (34) we can write the optimal estimate error covariance matrix in the form

$$\boldsymbol{P}^*(t) = \frac{24\lambda t}{\lambda^4 t^4 + 8\lambda^3 t^3 + 24\lambda^2 t^2 + 24\lambda t}\begin{bmatrix} p_{11}^* & p_{12}^* \\ p_{21}^* & p_{22}^* \end{bmatrix}, \quad (35)$$

where $p_{11}^* = (\lambda^2 t^2 + 3\lambda t + 3)/3$; $p_{12}^* = p_{21}^* = (\lambda^2 t^2 + 2\lambda t)/2t$; $p_{22}^* = (\lambda^2 t^2 + 2\lambda t)/t^2$

In fact, p_{11}^* is a variance of position estimate error, p_{22}^* is a variance of velocity estimate error, and p_{12}^* is a correlation moment for the position and velocity estimate errors.

Let the measurement model used by the optimal filter not contain information about correlation of measurement errors. In such a case the covariance matrix of actual estimate errors $\boldsymbol{P}_a$ will have distinctions with respect to the nominal one. By making use of equation (31), and matrices $\boldsymbol{\Phi}$, $\boldsymbol{H}$, $\boldsymbol{R}$, defined previously, covariance matrix $\boldsymbol{P}(t)$ can be found

$$\boldsymbol{P}(t) = \sigma^2\begin{bmatrix} 4/t & 6/t^2 \\ 6/t^2 & 12/t^3 \end{bmatrix} \quad (36)$$

Actual covariance of estimate error $\boldsymbol{P}_a(t)$ for considered suboptimal case can be found using the following integral equation (Brown, 1983)

$$\boldsymbol{P}_a(t) = \boldsymbol{P}(t)\left\{\int_0^t \int_0^t \boldsymbol{\Phi}^T(\tau,t)\boldsymbol{H}^T(\tau)\boldsymbol{R}^{-1}(\tau)\boldsymbol{L}(\tau,\sigma)\boldsymbol{R}^{-1}(\sigma)\boldsymbol{H}(\sigma) \times \right.$$

$$\left. \times \quad \boldsymbol{\Phi}(\sigma,t)\mathrm{d}\tau\,\mathrm{d}\sigma \right\}\boldsymbol{P}(t) \quad (37)$$

Substitution of $\boldsymbol{P}(t)$, defined by (36), and $\boldsymbol{\Phi}$, $\boldsymbol{R}$, $\boldsymbol{H}$, $\boldsymbol{L}(\tau, t)$ into equation (37) yields the actual covariance

$$\boldsymbol{P}_a(t) = \begin{bmatrix} p_{a11} & p_{a12} \\ p_{a21} & p_{a22} \end{bmatrix}, \tag{38}$$

where

$$p_{a11} = 4\left[18 - 5\lambda^2 t^2 - 2\exp(-\lambda t)(9 + 9\lambda t + 2\lambda^2 t^2) + 2\lambda^3 t^3\right]$$

$$p_{a12} = p_{a21} = 12\left[12 - 3\lambda^2 t^2 - 3\exp(-\lambda t)(2 + \lambda t^2) + \lambda^3 t^3\right] t^{-1}$$

$$p_{a22} = 24\left[12 - 3\lambda^2 t^2 - 3\exp(-\lambda t)(2 + \lambda t)^2 + \lambda^3 t^3\right] t^{-2}$$

3.4. Comparative Study of Optimal and Suboptimal Filters

Figures 1 and 2 illustrate time history of normalized variances of position and velocity estimates for optimal and suboptimal filtering found by making use of (37) and (31) for the values of correlation interval $\tau = 0.1s$ and $\tau = 0.5s$. Figure 2 a has different time scale for the time interval $t > 4s$.

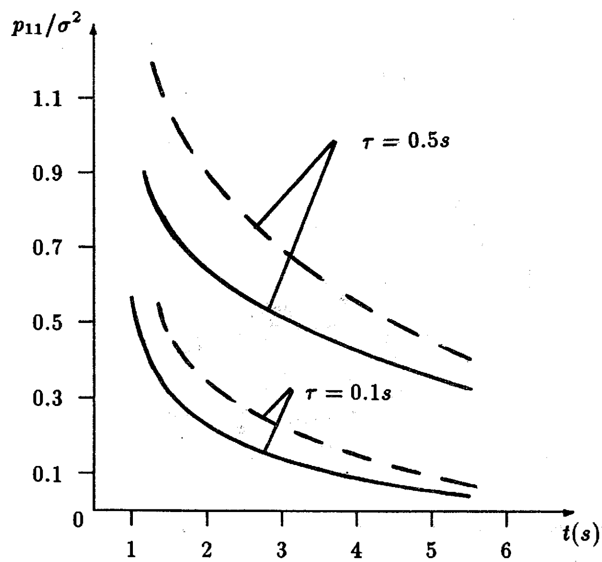

Fig. 1. Variance of filtering error versus observation time: ——— optimal filter, – – – suboptimal filter (position estimation).

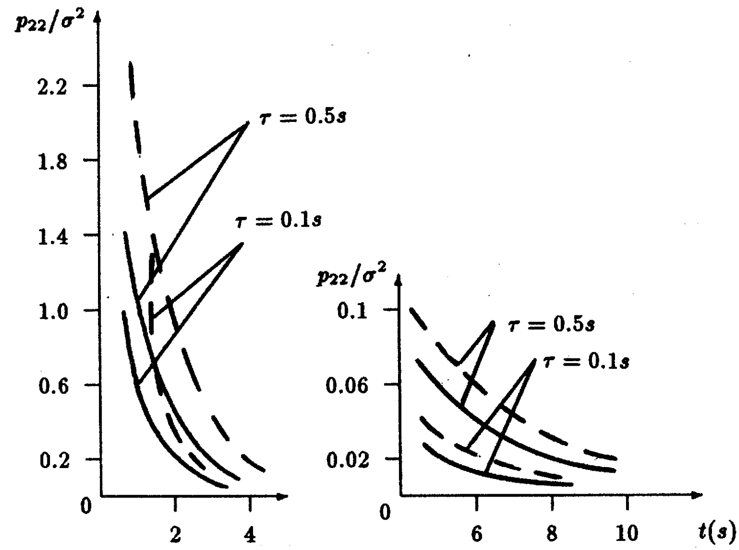

Fig. 2. Variance of filtering error for velocity estimation (ordinate axis scale was increased 10 times).

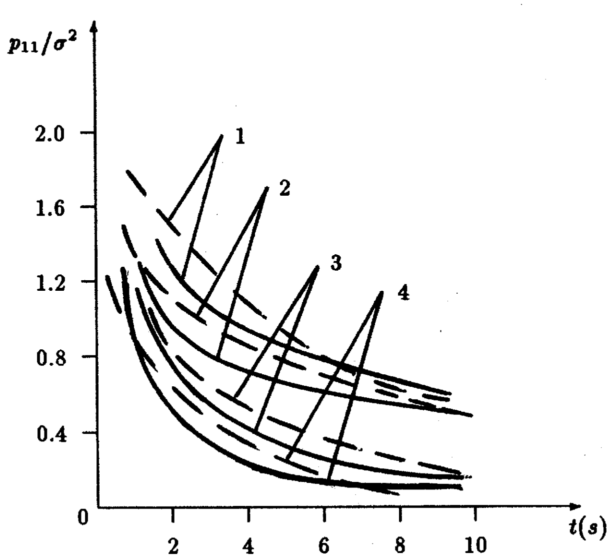

Fig. 3. Variance of extrapolation error for position versus observation time:
$1 - \Delta t = 0.5s,\ \tau = 0.5s$
$2 - \Delta t = 0.25s,\ \tau = 0.5s$
$3 - \Delta t = 0.5s,\ \tau = 0.1s$
$4 - \Delta t = 0.25s,\ \tau = 0.1s$

It can be seen from the figures that the larger is correlation interval τ, the slower is the convergence of variances of filtered position and velocity with respect to their steady–state values. Additional errors of filtering (in suboptimal case) are substantial at the beginning of the observation interval only $(t < 2s)$ for velocity estimates. But they are quickly decreasing to zero with extension of the observation interval.

Figure 3 illustrates normalized values of variances of estimate errors for extrapolation on an interval of time Δt of the moving object position for optimal and suboptimal cases. These variances were computed from expression

$$p_{11}^{e} = p_{11} + 2\Delta t p_{12} + (\Delta t)^2 p_{22},$$

with $\Delta t = 0.25s$, $0.5s$, and $\tau = 0.1s$, $0.5s$.

It can be seen from Figure 3 that with correlation interval $t = 0.5s$ the suboptimal filter requires $1 \div 0.3s$ more of observation time to reach the same estimation quality as optimal filter. If the correlation interval decreases to $\tau = 0.1s$, the maximum extra observation time for suboptimal filter does not exceed $0.3 \div 0.35s$ at the beginning of the observation period.

It can be concluded that processing of measurements can be performed without taking into account correlation of measurements if correlation interval $\tau \leq 0.1s$. If $\tau > 0.1s$, correlation of measurement errors should be taken into consideration to avoid a substantial decrease of filtering quality.

4. Conclusions

Analytical representations for the covariance matrices of estimation errors and Kalman filter transition matrix were derived in the paper. These expressions can be useful for the study of applied problems via analytical techniques, e.g. for determining observation time necessary for dynamic system state estimation.

Using analytical representation for the Kalman filter transition matrix, expressions for covariances of additional errors were derived, caused by *a priori* statistical ambiguities. Usefulness of analytical results was illustrated by an example of state estimation for the moving object. Thus, an approach was proposed for estimation of the Kalman filter characteristics by making use of analytical techniques. The approach allows us to calculate losses of quality of estimation due to application of suboptimal algorithms used in practice.

Using integral representation of the Kalman filter a comparative study of optimal and suboptimal filtering errors for the case of correlated measurement noise was performed. Expressions for actual errors of optimal and suboptimal filters were derived as explicit functions of correlation interval τ, and observation time t. The comparative study of filters showed that additional errors of estimation caused by correlation of measurement noise are increasing with the increase of correlation interval. It was shown that a less time consuming suboptimal filter can be used without a substantial loss of estimation quality if correlation interval t is less or equal $0.1s$.

References

Anderson B.D.O. and Moor J.B. (1979): *Optimal Filtering.* — New Jersey: Prentice Hall.

Andrejew Yu.N. (1976): *Control of Finite-Dimensional Linear Systems.* — Moscow: Physmathgiz, (in Russian).

Basin M.V. and Orlov Yu.V. (1992): *Guarantying estimation of dynamic linear system using discrete and continuous-time observations.* — *Automatics and Remote Control*, v.57, No.3, pp.36–45, (in Russian).

Bidyuk P.I. (1985): *Divergence analysis in the problems of stochastic processes.* — *Radio and Electronics*, (Kiev Polytechnic Institute), v.22, pp.12–17, (in Russian).

Brown R.G. (1983): *Introduction to Random Signals Analysis and Kalman Filtering.* — Now York: John Wiley & Sons.

Farber V.E. (1992): *Analysis of errors in digital filters with expanding finite memory.* — *Engineering Cybernetics*, v.30, No.1, pp.130–140, (in Russian).

Golubev G.A., Muravlev V.F. and Pisarev O.V. (1992): *Linear filtering of stationary random process in discrete time.* — *Engineering Cybernetics*, v.30, No.1, pp.141–147, (in Russian).

Korbicz J., Bidyuk P.I. and Podladchikov V.N. (1988): *Stability analysis of a discrete Kalman filter based on the use of the transition matrix.* — *Archiwum Automatyki i Telemechaniki*, v.33, No.3, pp.433–445, (in Polish).

Ljung L. and Kailath T. (1977): *Efficient change of initial conditions.* — *IEEE Trans. Automatic Control*, v.AC–22, No.3, pp.443–447.

Rivkin S.S., Ivanovsky R.I. and Kostrov A.B. (1976): *Statistical Optimization of Navigation Systems.* — Leningrad: Sudostroenie, (in Russian).

Roitenberg Yu.N. (1978): *Automatic Control.* — Moscow: Nauka, (in Russian).

Sage A.P. and Melsa J.L. (1971): *System Identification.* — New York: Academic Press.

Spencer R.V. (1981): *Sensitivities in parameter estimation.* — *IEEE Trans. Automatic Control*, v.AC–26, No.2.

Received October 27, 1992
Revised June 5, 1993