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CONSTRAINED CONTROLLABILITY OF LINEAR RETARDED DYNAMICAL SYSTEMS

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The main purpose of the paper is to study the constrained controllability for linear, time–invariant retarded dynamical systems defined in infinite–dimensional Banach or Hilbert spaces. First the brief theory of such dynamical systems is recalled. Next, different definitions of controllability are presented and the relationships between them are explained. Using the general spectral theory of unbounded linear operators, many constrained controllability conditions are formulated and proved. Finally, a simple illustrative example is given. The results presented extend to the case of abstract retarded dynamical systems, constrained controllability conditions given in the literature for finite–dimensional case or systems without delays.

1. Introduction

Recent years have witnessed a good deal of research focussed on abstract control systems modelled by an ordinary differential equations in abstract spaces (see e.g. Carja, 1988; Fattorini, 1966; 1967; Klamka, 1991; 1992a; 1992b; Triggiani, 1975a; 1975b; 1976; 1978). Such abstract models are known to be a unified framework for studying a variety of different dynamical systems governed by integrodifferential or partial differntial equations both parabolic and hyperbolic. Regarding the controllability problems of such abstract dynamical systems, general results for different types of controllability can be found for example in the papers cited above and in (Farlov, 1986; Fabre, 1992; Fattorini, 1975; Fattorini and Russell,1971; Kabayashi, 1978; Krabs *et al.*, 1985; Lagnese, 1978; Lasiecka and Triggiani, 1983; Naito, 1987; Narukawa, 1982; 1984; Russell, 1973; Sakawa, 1974; Salamon, 1984; Schmidt, 1992; Weck, 1982; 1984; Yamamoto and Park, 1990; Zhou, 1983; 1984; Zuazua, 1990a; 1990b). So far most literature in this subject has been concerned, however, with unconstrained controllability. Only a few papers deal with the so called constrained controllability problems, i.e. with the case when the control is restricted to take the values in a preasigned set, (Korobov, 1979; Korobov *et al.*, 1975; Korobov and Rabah, 1979; Korobov and Son, 1980; Peichl and Schappacher, 1986; Seidman, 1979; 1987; Son, 1990; Szklar, 1985).

On the other hand, controllability theory for dynamical systems with delays has been developed in (Banks *et al.*, 1975; Chukwu, 1979; 1987; Colonius, 1984; Delfour and Mitter, 1972a; 1972b; Jacobs and Langenhop, 1976; Klamka, 1991; Manitius, 1980; 1981; 1982; Manitius and Triggiani, 1978a; 1978b; Naito and Park, 1989; Salamon, 1984; Szklar, 1985; Zmood, 1974), but only in (Nakagiri and Yamamoto, 1989) the abstract functional dynamical system defined in infinite–dimensional linear space is considered

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for the case of unconstrained controls. Therefore, it should be stressed that up to now, constrained controllability of abstract functional dynamical systems has not been considered in the literature. In order to fill this gap, the present paper studies the constrained controllability problem for abstract functional dynamical systems.

We shall formulate conditions for absolute and relatively exact and approximate controllability with constraints posed on the control for linear time–invariant dynamical systems with delays, defined in infinite–dimensional Banach or Hilbert spaces. The results presented extend controllability conditions given in the papers (Nakagiri and Yamamoto, 1989) and (Son, 1990) for the case of constrained controls and delayed systems, respectively.

2. Linear Retarded Systems in Banach Spaces

First we give the notation and terminology used throughout this paper. Let R be the set of real numbers and let R^+ be the set of non–negative numbers. Let X and U be real separable Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_U$, respectively and X is additionally reflexive. The adjoint spaces of X and U are denoted by X^* and U^*, respectively. The duality pairing between X and X is denoted by $\langle\cdot,\cdot\rangle$, i.e. $\langle x^*, x\rangle$ is the value of the functional $x^* \in X^*$ at the point $x \in X$. For the set $M \subset X$, we define the orthogonal complement by $M^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0, \text{ for all } x \in M\}$, and the polar cone by $M^0 = \{x^* \in X^* : \langle x^*, x\rangle \leq 0, \text{ for all } x \in M\}$. The convex hull, the interior, and the closure of $M \subset X$ are denoted respectively by $\mathrm{co}\{M\}$, $\mathrm{int}\{M\}$, and $\overline{M}$. The linear space spanned by M is denoted by $\mathrm{span}\{M\}$. For given Banach spaces U and $X, L(U, X)$ stands for the Banach space of linear bounded operators from U to X. When $U = X, L(X, X)$ is denoted shortly as $L(X)$ and the identity operator in $L(X)$ is denoted by I. For a densely defined closed linear operator A on X, its adjoint operator on X^* is denoted by A^*, and the null space, the range, and the spectrum of an operator A are denoted by $\mathrm{Ker}\, A$, $\mathrm{Im}\, A$, and $\sigma(A)$, respectively.

For a given interval $[a, b] \subset R$, we denote by $L_p([a, b], X)$ the usual Banach space of X–valued measurable functions which are p–Bochner integrable for $1 \leq p < \infty$, or essentially bounded for $p = \infty$ on $[a, b]$. Let $M_p([a, b], X)$ denote the product space $X \times L_p([a, b], X)$. Given an element $g \in M_p([a, b], X)$, then $g^0 \in X$ and $g^1(\cdot) \in L_p([a, b], X)$ will denote the two coordinates of $g = (g^0, g^1)$. The space $M_p([a, b], X)$ is the Banach space with the norm

$$\|g\|_{M_p([a,b],X)} = \begin{cases} \left(\|g^0\|_X^p + \|g^1\|^p_{L_p([a,b],X)}\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \|g^0\|_X + \|g^1\|_{L_\infty([a,b],X)} & \text{if } p = \infty \end{cases}$$

The spaces $L_p([a, b], X)$ and $M_p([a, b], X)$ are shortly denoted as L_p and M_p, respectively. Moreover, the symbol $\mathcal{X}_E$ means the characteristic function of the set E.

Throughout this paper, unless otherwise stated, the space X is always assumed to be infinite–dimensional.

Now, we shall review some basic results on linear retarded control systems in Banach spaces. Let us consider functional differential equation on a Banach space X: (Nakagiri, 1987; 1988; Nakagiri and Yamamoto, 1989; Webb, 1976):

$$\dot{x}(t) = A_0 x(t) + \int_{-h}^{0} d\eta(s) x(t+s) + B_0 u(t) \quad \text{a.e.} \quad t \geq 0 \tag{1}$$

$$x(0) = g^0 \in X, \quad x(s) = g^1(s) \qquad \text{a.e.} \quad s \in [-h, 0] \tag{2}$$

where $g = (g^0, g^1) \in M_p([a,b], X)$, $h > 0$ is a constant delay, $B_0 \in L(U, X)$, $A_0 : X \supset D(A_0) \to X$ is the infinitesimal generator of a strongly continuous semigroup of linear bounded operators $T(t), t \geq 0$ on the Banach space X. The Stieltjes measure η in (1) is given by

$$\eta(s) = -\sum_{i=1}^{i=m} \mathcal{X}_{(-\infty, -h_i]}(s) A_i - \int_s^0 A_I(t) dt, \qquad s \in [-h, 0] \tag{3}$$

where $0 < h_1 < \ldots < h_i < \ldots < h_m$, $A_i \in L(X)$, for $r = 1, 2, \ldots, m$ and $A_I \in L_1([-h, 0], L(X))$. Then, the delayed term in (1) is written by

$$\int_{-h}^{0} d\eta(s) x(t+s) = \sum_{i=1}^{i=m} A_i x(t - h_i) + \int_{-h}^{0} A_I(s) x(t+s) ds \tag{4}$$

If the condition $A_I(\cdot) \in L_q([-h, 0], L(X))$, $1/p + 1/q = 1$, is satisfied, then for each $t \in R^+$, $g = (g^0, g^1) \in M_p([-h, 0], X)$ and $u(\cdot) \in L_q^{loc}(R^+, U)$, there exists a unique solution $x(t; g, u) \in X$ of the autonomous linear equation (1) given in the integrated form by (Nakagiri, 1987; 1988; Nakagiri and Yamamoto, 1989):

$$x(t; g, u) = \begin{cases} T(t) g^0 + \int_0^t T(t-s) \Big(\int_{-h}^{0} d\eta(t) x(s+t) + B_0 u(s) \Big) ds, & t \geq 0 \\ g^1(t), & \text{a.e. } t \in [-h, 0] \end{cases} \tag{5}$$

In this sense, $x(t; g, u)$ is called the mild solution of problem (1) and (2). The detailed analysis of solution (5) is given in the papers (Nakagiri, 1987) and (Nakagiri, 1988).

For each $\lambda \in C$ we define the densely defined closed linear operator $\Delta(\lambda) = \Delta(\lambda; A_0, \eta)$ by

$$\Delta(\lambda) = \lambda I - A_0 - \int_{-h}^{0} \exp(\lambda s) d\eta(s) \tag{6}$$

where I denotes the identity operator on X. The retarded resolvent set $\varphi(A_0, \eta)$ we understand as the set of all values $\lambda \in C$ for which the operator $\Delta(\lambda)$ has a bounded inverse with dense domain in X. In this case $\Delta(\lambda)^{-1}$ is denoted by $R(\lambda; A_0, \eta)$. The complement of $\varphi(A_0, \eta)$ in the complex plane is called the retarded spectrum and is denoted by $\sigma(A_0, \eta)$.

The mild solution $x(t; g, u)$ of the abstract retarded equation (1) given by formula (5) is expressed as a solution of the integral equation. In the applications it is more convenient to present this solution in a direct form, depending immediately on the given initial conditions (2) and control function $u(t)$, $t \geq 0$.

In order to do that, let us introduce operator $W(t) \in L(X)$, $t \geq 0$, which is the unique solution of the following integral equation (Nakagiri, 1987; 1988)

$$W(t) = \begin{cases} T(t) + \int_0^t T(t-s) \int_{-h}^0 \mathrm{d}\eta(l) W(l+s) \mathrm{d}s, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

The operator valued function $W(t)$ is strongly continuous on R^+ (Nakagiri, 1987; 1988).

Now, let us define the operator valued function $H_t(\cdot) \in L_q\big([-h,0], L(X)\big)$ given by the following formula

$$H_t(s) = \int_{-h}^s W(t-s+l) \mathrm{d}\eta(l), \qquad \text{a.e.} \quad s \in [-h,0]$$

Taking into account equality (3) we may express the function $H_t(\cdot)$ in a more concrete form, namely

$$H_t(s) = \sum_{r=1}^{r=m} W(t-s-h_r) A_r \mathcal{X}_{[-h_r,0]}(s) + \int_{-h}^s W(t-s+l) A_I(l) \mathrm{d}l, \quad \text{a.e.} \quad s \in [-h,0]$$

Therefore, using operator valued functions $W(t)$ and $H_t(s)$ we may express the solution $x(t;g,u) \in X$ as the following function (Nakagiri, 1987; 1988)

$$x(t;g,u) = \begin{cases} W(t)g^0 + \int_{-h}^0 H_t(s) g^1(s) \mathrm{d}s + \int_0^t W(t-s) u(s) \mathrm{d}s, & t \geq 0 \\ g^1(t), & \text{a.e. } t \in [-h,0] \end{cases}$$

The function $x(t;g,u)$ is well defined and is an element of the space $C(R^+,X) \cap L_p\big([-h,0],X\big)$, (Nakagiri, 1987; 1988).

More detailed analysis of the properties of the operator valued functions $W(t)$ and $H_t(s)$, including stability behavior, can be for example found in the publications (Nakagiri, 1981; 1987; 1988).

Finally, it should be stressed that operator valued functions $W(t)$ and $H_t(s)$ will play an important role in controllability considerations, which will be presented in the next sections of this paper.

Now, we shall recall basic properties of semigroup associated with the abstract equation (1), (for details see e.g. (Nakagiri, 1987; 1988; Nakagiri and Yamamoto, 1989)). In what follows we generally assume that $1 < p < \infty$.

Let $x(t;g,0)$ be the mild solution of equation (1) with $u = 0$ and $g = (g^0, g^1) \in M_p\big([-h,0],X\big)$, given by integral formula (5). With equation (1) we may associate the solution operator $S(t) : M_p \to M_p$, $t \geq 0$ defined by the following equality

$$S(t)g = \Big(x(t;g,0), x_t(\cdot;g,0)\Big) \in M_p, \qquad \text{for } g \in M_p \tag{7}$$

where $x_t(s;g,0) = x(t+s;g,0)$ for $s \in [-h,0]$.

The operator $S(t)$ is linear and bounded on M_p for $t \geq 0$, and has the following properties, listed in Lemma 1.

Lemma 1. (Nakagiri, 1987; 1988; Nakagiri and Yamamoto, 1989; Webb, 1976).

i) *The family of operators* $S(t) : t \geq 0$ *is* C_0*–semigroup on the Banach space* $M_p([-h,0],X)$.

ii) *If* $T(t)$ *is compact for all* $t > 0$, *then* $S(t)$ *is compact for* $t > h$.

iii) *The infinitesimal generator* A *of semigroup* $S(t)$, $t \geq 0$, *is given by*

$$D(A) = \left\{ g = (g^0, g^1) \in M_p([-h,0],X) : g^1 \in W_p^{(1)}([-h,0],X), g^1(0) = g^0 \in D(A_0) \right\} \tag{8}$$

$$Ag = \left(A_0 g^0 + \int_{-h}^{0} \mathrm{d}\eta(s) g^1(s), \frac{\mathrm{d}g^1}{\mathrm{d}s}(\cdot) \right), \quad for \quad g = (g^0, g^1) \in (A) \tag{9}$$

iv) $$\frac{\mathrm{d}S(t)g}{\mathrm{d}t} = AS(t)g = S(t)Ag, \quad for \quad g \in D(A) \ and \ t > 0$$

Since the space X is reflexive and $1 < p < \infty$, the adjoint space M_p^* of M_p is identified with the product space $X^* \times L_q([-h,0],X^*)$ via the following duality pairing

$$\langle g, f \rangle_{M_p} = \langle g^0, f^0 \rangle_X + \int_{-h}^{0} \langle g^1(s), f^1(s) \rangle_X \, \mathrm{d}s$$

for $g = (g^0, g^1) \in M_p, \qquad f = (f^0, f^1) \in M_p^*$

where $\langle \cdot, \cdot \rangle_X$ denotes the duality pairing between X and X^*, and $1/p + 1/q = 1$.

In control theory of retarded dynamical systems, it is desirable to consider so called retarded transposed dynamical systems in the space M_p^* (Delfour and Mitter, 1972a; Manitius, 1982; Nakagiri, 1981; 1988; Nakagiri and Yamamoto, 1989). This is strongly connected with the adjoint theory for retarded dynamical system (1), (Nakagiri, 1987; 1988; Nakagiri and Yamamoto, 1989).

The retarded transposed dynamical systems is defined by (Nakagiri, 1987; 1988)

$$\begin{array}{ll} \dot{x}(t) = A_0^*(t)x(t) + \int_{-h}^{0} \mathrm{d}\eta^*(s)x(t+s) & \text{a.e. } t \geq 0, \ x(t) \in X^* \\ x(0) = f^0, \quad x(s) = f^1(s) & \text{a.e. } s \in [-h,0], \ (f^0, f^1) \in M_p^* \end{array} \tag{10}$$

Since the space X is reflexive, the adjoint operator A_0^* generates a strongly continuous (of class C_0) semigroup $T^*(t)$ on X^* which is given by the adjoint of $T(t)$. Hence we can construct the operator valued function $W^*(t)$, which is strongly continuous on R^+ and $W^*(t)$ is the adjoint of $W(t)$ for $t \geq 0$, (Nakagiri, 1988). $W^*(t)$ is the unique solution of the following integral equation

$$W^*(t) = \begin{cases} T^*(t) + \int_0^t T^*(t-s) \int_{-h}^{0} \mathrm{d}\eta^*(l) W^*(l+s) \mathrm{d}s, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{11}$$

Using the solution of the retarded transposed dynamical system it is possble to construct, in a similar way as for system (1), the semigroup $S_T(t),\ t \geq 0$ of bounded linear operators on $M_p^*([-h,0],X)$ and its infinitesimal generator A_T, (Nakagiri, 1988). On the other hand, we may obtain semigroup $S^*(t),\ t \geq 0$, where $S^*(t)$ is the adjoint of $S(t)$ for $t \geq 0$, (Nakagiri, 1988) and its infinitesimal generator A^*. However, generally $S^*(t) \neq S_T(t)$ and $A^* \neq A_T$ (see e.g. (Nakagiri, 1987; 1988) and (Webb, 1976) for details). The connections between semigroups $S_T(t)$ and $S^*(t)$ can be easily explained by using structural operators F and $G(t)$, (Nakagiri, 1988).

The retarded control system (1) can be transformed into an abstract differential dynamical system in the space $M_p([-h,0],X)$ (Nakagiri, 1987; 1988). In order to do that, let us define operator $B \in L(U, M_p)$ as follows (Manitius and Triggiani, 1978a):

$$Bu = (B_0 u, 0) \in M_p, \qquad \text{for} \quad u \in U \tag{12}$$

Hence the retarded dynamical system (1) may be expressed as an abstract differential dynamical system in a Banach space $M_p([-h,0],X)$

$$\dot{z}(t) = Az(t) + Bu(t), \qquad t \geq 0 \tag{13}$$

where $z(t) = (x(t;g,u),\ x_t(\cdot;g,u)) \in M_p([-h,0],X)$ and the operator A is defined by relations (8) and (9). This compact form of dynamical system is very useful in various research problems in modern control theory. Specially, it will be extensively used in controllability problems.

In recent years the state space theory for linear retarded functional differential equations has been developed in many publications (see e.g. the papers (Banks *et al.*, 1975; Delfour and Mitter, 1972a; 1980; Manitius, 1976; 1980; 1982; Nakagiri, 1981; 1987; 1988; Salamon, 198), and (Webb, 1976)). This theory is based on certain relations between semigroup $S(t),\ t \geq 0$ associated with equation (1) and the so called structural operators F and G. The structural operators have provided various new and efficient techniques for the study of control theory involving linear retarded functional equations (Delfour and Mitter, 1980; Delfour and Karrakchon, 1987; Manitius, 1976; 1982; Nakagiri, 1987; 1988).

Now, we introduce the concept of the structural operator $F : M_p \to M_p$, which is defined as follows:

$$F = \begin{bmatrix} I & 0 \\ 0 & F_1 \end{bmatrix}, \quad \text{i.e.} \quad [Fg]^0 = g^0, \ \ [Fg]^1 = F_1 g^1, \quad \text{for} \quad g = (g^0, g^1) \in M_p \tag{14}$$

where the operator $F_1 : L_p([-h,0],X) \to L_p([-h,0],X)$ is given by

$$[F_1 g^1](s) = \int_{-h}^{s} \mathrm{d}\eta(l) g^1(l-s) \qquad \text{a.e.} \quad s \in [-h,0]$$

It is easily verified (Nakagiri, 1987; 1988) that operator F_1 is linear, bounded and into the space $L_p([-h,0],X)$. Taking into account formula (4) we may express operator F_1 in a more convenient form given by

$$[F_1 g^1](s) = \sum_{i=1}^{i=m} A_r \mathcal{X}_{[-h_i,0]}(s) g^1(-h_i - s) + \int_{-h}^{s} A_I(l) g^1(l-s)\mathrm{d}l, \qquad \text{a.e.} \quad s \in [-h,0]$$

The adjoint operator $F^* : M_p^* \to M_p^*$ of F is given by Nakagiri (1988)

$$F^* = \begin{bmatrix} I & 0 \\ 0 & F_1^* \end{bmatrix} \tag{15}$$

where $F_1^* : L_q([-h,0],X^*) \to L_q([-h,0],X^*)$ denotes the adjoint of F_1 and is represented by the following formula

$$\begin{aligned}[F_1^* f](s) = \int_{-h}^{s} \mathrm{d}\eta^*(t) f^1(t-s) &= \sum_{i=1}^{i=m} A_r^* \mathcal{X}_{[-h_i,0]}(s) f^1(-h_i - s) \\ &+ \int_{-h}^{s} A_I^*(t) f^1(t-s)\mathrm{d}t \qquad \text{a.e.} \quad s \in [-h,0]\end{aligned} \tag{16}$$

For controllability investigations it is important to point the case, when the operator F_1 and hence of course F are onto the spaces $L_p([-h,0],X)$ and $M_p([-h,0],X)$, respectively. It is well known (Nakagiri, 1987; 1988) that if $0 \in \varphi(A_m)$, then $\operatorname{Im} F = M_p$, and moreover, $\operatorname{Im} F^* = M_p^*$. For the special case, when $A_I(s) = 0$, in the neighbourhood of $-h$, $\operatorname{Ker} F = \{0\}$ if and only if $\operatorname{Ker} A_m = \{0\}$.

In order to explain in detail the internal structure of dynamical system (1) it is necessary to introduce second structural operator $G(t) : M_p \to M_p$, $t \geq 0$, defined by:

$$[G(t)g]^1(s) = W(t+s)g^0 + \int_{-h}^{0} W(t+s+t)g^1(t)\mathrm{d}t, \qquad s \in [-h,0] \tag{17}$$

$$[G(t)g]^0 = [G(t)g]^1(0) \qquad g = (g^0, g^1) \in M_p \tag{18}$$

Especially we define structural operator $G : M_p \to M_p$ as

$$G = G(h) \tag{19}$$

It is easily verified that operators $G(t)$ are linear and bounded for $t \geq 0$, and hence of course G is linear and bounded.

In what follows, we shall give short comments on the spectral decomposition of dynamical system (1), (see e.g. (Nakagiri, 1987; 1988) for details). First of all let us observe that if the operator A_0 generates compact semigroup $T(t)$, for $t > 0$, then by (Nakagiri, 1987, Remarks 5.2) $\sigma(A) = \sigma_p(A)$, the point spectrum. Hence, our analysis will be simpler than in general case, (Nakagiri, 1987; 1988). Moreover, by Nakagiri (1987, Sec. 4) we have

$$\sigma(A) = \sigma_p(A) = \sigma_d(A) = \Big\{ \lambda \in C : \text{ is isolated eigenvalue of } A \text{ and } \dim M_\lambda = m_\lambda < \infty \Big\} \tag{20}$$

where $\sigma_d(A)$ is discrete spectrum and is countable, and the generalized eigenspace M_λ is given by

$$M_\lambda = \operatorname{Ker}(\lambda I - A)^{k_\lambda} \tag{21}$$

where k_λ is the index of the eigenvalue λ.

Definition 1. Dynamical system (1) is said to be spectrally complete if

$$\overline{\text{span}}\,\{M_\lambda : \lambda \in \sigma(A)\} = M_p \tag{22}$$

The completeness of dynamical system (1) plays an important role in the controllability analysis.

In a very similar way, we may define the so called F–completeness of dynamical system (1) (Manitius, 1982; Manitius and Triggiani, 1978a).

Definition 2. Dynamical system (1) is said to be spectrally F–complete if

$$\overline{\text{span}}\,\{FM_\lambda : \lambda \in \sigma(A)\} = \overline{\text{Im}}\,F \tag{23}$$

In other words, dynamical system (1) is spectrally complete if and only if the operator A is spectrally complete in the space M_p. Similarly, dynamical system (1) is spectrally F–complete if and only if the operator A is spectrally complete in the space $\text{Im}\,F \subset M_p$.

Under the assumption that the operator A_0 generates compact semigroup $T(t)$, for $t > 0$, then $\dim M_\lambda < \infty$ for all $\lambda \in \sigma(A)$ and M_λ is closed and invariant under $S(t)$ (Nakagiri, 1988, Sec. 7). Moreover, the restriction A_λ of A to M_λ and the restriction B_λ of B to M_λ are bounded operators and we have

$$\dot{x}_\lambda(t) = A_\lambda x_\lambda(t) + B_\lambda u(t), \qquad \text{for} \quad t \geq 0, \;\; x_\lambda \in R^{m_\lambda} \tag{24}$$

The detailed analysis for spectral decomposition of dynamical system (1) can be found in (Nakagiri, 1988, Sec. 7). However, it should be pointed out, that since the linear subspaces M_λ are finite–dimensional for all $\lambda \in \sigma(A)$, then the bounded linear operators A_λ and B_λ are in fact constant matrices of appropriate dimensions.

Taking into account the spectral theory of the operator A we may introduce the concept of so called spectral controllability of our dynamical system (1), which will be done in the next section.

Now, let us concentrate on the investigation of the $\text{Im}\,S(t)$ for $t \geq 0$. It is an important feature from the controllability point of view. It follows from (24) , that (Son, 1990, Sec. 2)

$$S(t)x_\lambda(0) = \exp(A_\lambda t)x_\lambda(0), \qquad \text{for} \quad t \geq 0, \;\; x_\lambda \in R^{m_\lambda} \tag{25}$$

Hence, using spectral decomposition theory we see that

$$\overline{\text{span}}\,\{M_\lambda : \lambda \in \sigma(A)\} = \overline{\text{Im}}\,S(t) \qquad \text{for every} \quad t \geq 0 \tag{26}$$

On the other hand, from general semigroup theory it follows (Triggiani, 1976) that $\text{Im}\,S(t_1) \supset \text{Im}\,S(t_2)$, for $t_1 < t_2$. A more detailed analysis of the $\text{Im}\,S(t)$ is presented in (Nakagiri, 1987, Sec. 5), where some characterization of the $\text{Im}\,S(t)$ is given, using the adjoint theory. Finally, it should be mentioned, that in finite–dimensional case, i.e., when $X = R^n$ we have fine characterization of $\text{Im}\,S(t)$ for $t \geq nh$, namely (Nakagiri, 1987, p. 527)

$$\overline{\text{Im}}\,S(t) = \overline{\text{span}}\,\{M_\lambda : \lambda \in \sigma(A)\} \tag{27}$$

Hence, for finite–dimensional case with operator A with finite spectrum (Manitius and Triggiani, 1978a, Sec. 8) $\dim \operatorname{Im} S(t) < \infty$ for $t \geq nh$.

In the sequel, besides the general dynamical systems (1) we shall consider also some special cases, i.e. dynamical systems with lumped multiple delays described by the following retarded equation:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{i=m} A_i x(t - h_i) \tag{28}$$

where $A_i \in L(X)$ for $i = 1, 2, .., m$ and $0 < h_1 < h_2 < ... < h_i < ... < h_m$ are constant delays. The theory of such dynamical systems is studied in detail in the paper (Nakagiri, 1981). The special case $m = 1$ is of special interest.

The other possible special cases of dynamical system (1) are dynamical retarded systems with finite dimensional controls, i.e. the dynamical systems of the form (1) or (24) with $U = R^r$, $r < \infty$. In this case the linear bounded operator $B_0 : R^r \to X$ can be expressed as follows

$$B_0 u(t) = \sum_{j=1}^{j=r} b_{0j} u_j(t), \qquad \text{where} \quad b_{0j} \in X, \ \text{for} \ j = 1, 2, ..., r \tag{29}$$

and $u_j(t) \in R$, $t \geq 0$ are scalar controls.

The next possible special cases are dynamical systems of the form (1) or (24) with finite–dimensional controls and finite–dimensional space $X = R^n$, (Banks *et al.*, 1975; Chukwu, 1979; 1987; Colonius, 1984; Klamka, 1991; Manitius, 1980; 1981; 1982; Manitius and Triggiani, 1978a; 1978b; Szklar, 1985). In this case all the operators A_i, $i = 1, 2, ...m$ are of course bounded and given by the constant $n \times n$ dimensional matrices. Moreover, the operator B_0 is also $n \times r$ dimensional constant matrix.

We may also consider dynamical systems (1) or (24) in Hilbert space $M_2([-h, 0], X)$, where X is a Hilbert space of infinite dimension. Since in this case we may identify $X^* = X$, then we have also that $M_2^*([-h, 0], X) = X^* \times L_2([-h, 0], X^*) = X \times L_2([-h, 0], X) = M_2([-h, 0], X)$. For Hilbert space M_2 and specially X we can obtain more concrete and computable criteria for various kinds of controllability.

Finnaly, it should be pointed out that it is possible to consider more general retared systems than (1), i.e. retarded dynamical systems with linear unbounded operators A_i, $i = 1, 2, ..., m$. Theory of such dynamical systems can be found for example in the publication (Jeong, 1993; Nakagiri, 1981).

3. Basic Definitions

In this section we shall recall various definitions for controllability of dynamical system (1). As it was mentioned in the introduction, for retarded control systems in Banach spaces there exist many different notions of controllability. It follows from the fact that for dynamical systems with delays we may introduce different concepts of the state spaces, and on the other hand since Banach spaces are generally infinite–dimensional, then we must distinguish between exact and approximate controllability (Banks *et al.*, 1975;

Colonius, 1984; Delfour and Mitter, 1972a; 1972b; Fattorini, 1966; 1967; Klamka, 1991; Manitius, 1981; 1982; Manitius and Triggiani, 1978a; 1978b; Nakagiri and Yamamoto, 1989; Salamon, 1984; Triggiani, 1975; 1976; 1978; Yamamoto *et al.*, 1990; Zmood, 1974).

Let $V \subset U$ be a given set in the space U such that $0 \in V$. We define the set of admissible controls on $[0, t_1]$ by

$$V_{t_1} = \Big\{u(\cdot) \in L_p\big([0,t_1],U\big) : u(t) \in V, \qquad \text{a.e.} \quad t \in [0,t_1]\Big\} \tag{30}$$

We define the linear operators $P_t : L_p\big([0,t],U\big) \to X$ and $Q_t : M_p\big([-h,0],X\big) \to X$, respectively, by

$$P_t u = \int_0^t W(t-s)B_0 u(s)\mathrm{d}s, \qquad \text{for} \quad u \in L_p\big([0,t],U\big) \tag{31}$$

$$Q_t g = W(t)g^0 + \int_{-h}^0 H_t(s)g^1(s)\mathrm{d}s, \qquad \text{for} \quad g = (g^0, g^1) \in M_p \tag{32}$$

Similarly, we define linear operator $F_t : L_p\big([0,t],U\big) \to M_p$, by

$$F_t u = \int_0^t S(t-s)Bu(s)\mathrm{d}s, \qquad \text{for} \quad u \in L_p\big([0,t],U\big) \tag{33}$$

Linear operators P_t, Q_t, and F_t play an important role in the theory of attainable sets and in the controllability investigations.

First of all, let us define the so called attainable sets in the spaces X and M_p (Manitius and Triggiani, 1978a; Nakagiri and Yamamoto, 1989; Seidman, 1979; Triggiani, 1975a; 1975b; 1976; 1978).

$$C_t(V) = \Big\{x(t;0,u) \in X : u \in V_t\Big\}, \qquad t > 0 \tag{34}$$

$$C_\infty(V) = \bigcup_{t>o} C_t(V) \tag{35}$$

$$K_t(V) = \Big\{\big(x(t;0,u), x_t(\cdot;0,u)\big) \in M_p : u \in V_t\Big\}, \qquad t > 0 \tag{36}$$

$$K_\infty(V) = \bigcup_{t>0} K_t(V) \tag{37}$$

Hence, $C_t(V) = \operatorname{Im} P_t(V)$ and $K_t(V) = \operatorname{Im} F_t(V)$. Similarly, we define in the spaces X and M_p the sets reachable from nonzero initial conditions (2), (Chukwu, 1987; Colonius, 1984; Klamka, 1991; Manitius, 1980; Nakagiri, 1987; Zmood, 1974).

$$R_t = \Big\{x(t;g,0) \in X : g \in M_p\Big\}, \qquad t > 0 \tag{38}$$

$$R_\infty = \bigcup_{t>o} R_t \tag{39}$$

$$A_t = \left\{ \left(x(t;g,0), x_t(\cdot;g,0)\right) \in M_p : g \in M_p \right\}, \qquad t > 0 \tag{40}$$

$$A_\infty = \bigcup_{t>0} A_t \tag{41}$$

Therefore, $R_t = \operatorname{Im} Q_t$ and $A_t = \operatorname{Im} S(t)$, where $S(t)$ is defined by equality (7).

Now we are in the position to give formal definitions for various kinds of controllability for dynamical control system (1).

Definition 3. Dynamical system (1) is said to be absolutely exactly (approximately) V–controllable if

$$K_\infty(V) = M_p, \qquad (\overline{K_\infty(V)} = M_p) \tag{42}$$

Definition 4. Dynamical system (1) is said to be relatively exactly (approximately) V–controllable if

$$C_\infty(V) = X, \qquad (\overline{C_\infty(V)} = X) \tag{43}$$

In the literature absolute controllability is named also as function space or M_p–controllability (Banks *et al.*, 1975; Manitius, 1981; Manitius and Triggiani, 1978a; 1978b). Similarly, the terms X–controllability are frequently used instead of relative controllability (Chukwu, 1979; Klamka, 1991; Manitius and Triggiani, 1978a; 1978b; Zmood, 1974). For a finite–dimensional case, when $X = R^n$, exact and approximate relative V–controllability are equivalent. For the case, when the final time $t = t_1$ is given *a priori* we may formulate the definitions of absolute and relative controllability in time interval $[0, t_1]$. However, in the sequel we do not consider such kinds of controllability.

In many practical situations, it is important to steer our dynamical system to zero final state. This is strongly connected with the concepts of absolute and relative null controllability (Chukwu, 1979; 1987; Colonius, 1984).

Definition 5. Dynamical system (1) is said to be absolutely exactly (approximately) null V–controllable if

$$A_\infty \subset K_\infty(V), \qquad (A_\infty \subset \overline{K_\infty(V)}) \tag{44}$$

Definition 6. Dynamical system (1) is said to be relatively exactly (approximately) null V–controllable if

$$R_\infty \subset C_\infty(V), \qquad (R_\infty \subset \overline{C_\infty(V)}) \tag{45}$$

All the remarks and comments following definitions of absolute and relative controllability are also valid for relative and absolute null controllability. Moreover, it should be stressed that the concepts of absolute and absolute null controllability are essentially stronger than the concepts of relative and relative null controllability, respectively (Klamka, 1991; Manitius and Triggiani, 1978a; 1978b; Nakagiri, and Yamamoto, 1989). Finally, it should be pointed out that generally each notion of controllability is essentially stronger than the corresponding notion of null controllability. The equivalence relations between these two concepts will be considered in the next sections.

In retarded functional differential control systems the concept of controllability is strongly related to the choice of a state space. In many cases controllability in the full state space is a very restrictive assumption, too restrictive from the practical point of view (Manitius, 1982). This suggests to define the weaker concept of controllability, i.e. controllability in certain nontrivial linear subspaces of the state space. This observation leads directly to the notion of the so called F–controllability (Manitius, 1982), i.e. controllability in a linear subspace $\operatorname{Im} F$.

Definition 7. Dynamical system (1) is said to be $F-V$–controllable if

$$\overline{F K_\infty(V)} = \overline{\operatorname{Im} F} \tag{46}$$

The notion of F–controllability is generally weaker than the notion of approximate absolute controllability (Manitius, 1982). The equivalence of these two concepts depends on the internal structure of dynamical system (1) represented by the linear operator F.

The link between absolute approximate controllability and F–controllability is provided by the operator F which contains the information about the structure of the hereditary part of the dynamical system. The "true" state of the dynamical system is Fz, $z \in M_p$ rather than z itself. This is especially important when the structure of the dynamical systems is such that there are nonzero z for which $Fz = 0$, i.e., $\operatorname{Ker} F \neq \{0\}$. Then the introduction of the operator F shows explicitely that some nontrivial reduction of the state space is possible. For finite–dimensional case, when $X = R^n$, it means that there are state variables which are not delayed (Manitius, 1982).

Spectral decomposition of the state space M_p presented in Section 2 gives us the opportunity to define the next kind of controllability, namely, spectral controllability (Manitius and Triggiani, 1978a; 1978b).

Definition 8. Dynamical system (1) is said to be spectrally V–controllable if for each $\lambda \in \sigma(A)$ dynamical systems (24) are V–controllable.

Spectral V–controllability is in general an essentially weaker concept than absolute approximate V–controllability, i.e. absolute approximate V–controllability always implies spectral V–controllability. The converse statement holds if and only if operator A is spectrally complete in the space M_p.

Similar considerations are valid for F–controllability and spectral controllability. More precisely, $F-V$–controllability always implies spectral controllability. The converse implication holds if and only if dynamical system (1) is spectrally F–complete.

Finally, let us return to the concepts of absolute exact V–controllability and relative exact V–controllability. Since by assumption the operator A_0 has compact resolvent, then by the results of (Nakagiri, 1987; Nakagiri and Yamamoto,1989; Triggiani, 1975a; 1975b; 1976; 1978) dynamical system (1) is never absolutely exactly V–controllable and relatively exactly V–controllable in the case when X is infinite–dimensional.

For the finite–dimensional case, when $X = B^n$ absolute and relative exact controllability are possible (see e.g. (Banks *et al.*, 1975; Colonius, 1984; Jacobs and Langenhop, 1976; Manitius and Triggiani, 1978a; 1978b) and (Zmood, 1974)). For example in the paper (Banks *et al.*, 1975) rather a very restrictive condition rank $B_0 = n$

for absolute exact R^r–controllability has been formulated and proved. The conditions for relative exact controllability are less restrictive, because for $X = R^n$ exact and approximate relative controllability are the same concept. It follows from the fact that in finite–dimensional linear spaces all linear subspaces are closed.

Taking into account previous considerations, in the next sections of this paper we shall concentrate on approximate absolute and relative controllability.

4. Constrained Approximate Controllability

In this section we shall solve the problem to find characterization of various kinds of approximate V–controllability, expressed in terms of the operators connected with the retarded abstract equation (1) and the restraint set V. The results extend to infinite–dimensional case the approximate controllability criteria given in the papers (Son, 1990; Szklar, 1985) for retarded dynamical systems with $X = R^n$. The results of the paper (Son, 1990) are based on certain general theorem (Son, 1990, Thm. 3.3). For the sake of convenience, we formulate here the main theorem of that work. However, in order to do that let us formulate the so called spectral decomposition property for the linear operator A (Son, 1990).

Assumption H1: (spectral decomposition property). For every $\alpha \in R$ the spectral set $\sigma_\alpha = \sigma(A) \cap \{z \in C : \mathrm{Re}\, z > \alpha\}$ consists of a finite number of eigenvalues of the linear operator A with finite multiplicities.

Proposition 1. (Son, 1990, Thm. 3.3) *Let Z and U be separable Banach spaces and Z reflexive. Let A be a generator of C_0–semigroup of linear bounded operators $S(t)$, $t \geq 0$, on Z, and let additionally operator A satisfy assumption H1. Let V be a cone with vertex at the origin in U such that intco $V \neq \emptyset$. If the operator A is spectrally complete then the dynamical system*

$$\dot{z}(t) = Az(t) + Bu(t)$$

is approximately V–controllable if and only if it is approximately U–controllable (i.e. without any constraints) and moreover

$$\mathrm{Ker}\,(\lambda I - A^*) \cap (BV)^0 = \{0\} \qquad \text{for every} \quad \lambda \in R \tag{47}$$

From the proof of Proposition 1 which is presented in the paper (Son, 1990) it immediately follows that without assumption about spectrall completeness of the operator A, that proposition is the necessary and sufficient condition for spectral V–controllability. Moreover, from the spectral decomposition property (assumption H1) it follows that linear subspaces corresponding to each spectral set in the decomposition procedure are finite dimensional (Son, 1990).

Now, we are in the position to formulate main theorem concerning absolute approximate V–controllability of dynamical system (1).

Theorem 1. *Suppose that:*

i) *Operator A_0 generates compact semigroup $T(t)$, for $t > 0$.*

ii) *Operator A is spectrally complete.*

iii) *The set V is a cone with vertex at the origin in U and such that* $\mathrm{int}\,(\mathrm{co}\,\{V\}) \neq \emptyset$.

Then, dynamical system (1) is absolutely approximately V-controllable if and only if it is absolutely approximately U-controllable and

$$\operatorname{Ker} \Delta_T(\lambda) \cap (B_0 V)^0 = \{0\} \qquad \textit{for every} \quad \lambda \in R \tag{48}$$

Proof. Proof of Theorem 1 is based on Proposition 1. First of all, let us verify the assumptions of Proposition 1. It is evident that all the assumptions, besides assumption H1 are satisfied. Hence, this verification is reduced to showing that assumption H1 holds. In order to do that, let us observe, that since by assumption i) operator A_0 generates a compact semigroup $T(t)$ for $t > 0$, then A_0 has compact resolvent (Triggiani, 1976), and moreover the retarded resolvent $R(\lambda; A_0, \eta)$ is also compact for all $\lambda \in \varphi(A_0.\eta)$ (Nakagiri, 1988, Remark 7.3). Furthermore, $\sigma(A)$ is a countable set consisting entirely of eigenvalues of the operator A (Nakagiri and Yamamoto,1989, Sec. 4; Nakagiri, 1987, Rem. 5.2) and for each $\lambda \in \sigma(A)$, the corresponding generalized eigenspace M_λ is finite–dimensional (Nakagiri and Yamamoto, 1989, Sec. 4). Hence, we have proved, that operator A satisfies assumptions of (Webb, 1976, Lemma 3.4 and Remark 3.6), which states that spectral sets σ_α consist of finite number of eigenvalues for each $\alpha \in R$. Therefore, operator A satisfies spectral decomposition property given in assumption H1.

Now, let us concentrate on the equivalencee between conditions (47) and (48). First of all, let us observe, that for each eigenvalue $\lambda \in C$, the corresponding eigenspaces of A_T and A^* are related through the so called structural operator G by the following formula (Nakagiri, 1988, Thm. 8.2)

$$\operatorname{Ker} (\lambda I - A_T)^i = G^* \operatorname{Ker} (\lambda I - A^*)^i, \qquad \text{for} \quad i = 1, 2, \ldots \tag{49}$$

Moreover, by (Nakagiri, 1988, Prop. 4.5) $\operatorname{Ker} G^* = \{0\}$ and $\overline{\operatorname{Im} G^*} = M_p^*$. Hence, repeating the proof of (Son, 1990, Thm. 4.3), we conclude, that conditions (47) and (48) are equivalent. Thus, the proof of our Theorem 1 is complete.

Corollary 1. *Suppose that:*

i) *Operator A_0 generates compact semigroup $T(t)$ for $t > 0$.*

ii) $\operatorname{Ker} A_m = \{0\}$.

iii) *The set V is a cone with vertex at the origin in U and such that* $\operatorname{int}\big(\operatorname{co}\{V\}\big) \neq \emptyset$.

Then, dynamical system (28) is absolutely approximately V-controllable if and only if it is absolutely approximately U-controllable and condition (48) holds.

Proof. Since $\operatorname{Ker} A_m = \{0\}$, then the operator A associated with the dynamical system (28) is spectrally complete (see (Nakagiri, 1987) or (Nakagiri, 1988)). Hence, all the assumptions of Theorem 1 are satisfied for dynamical system (28). Thus, our Corollary 1 follows.

For the finite–dimensional case, i.e. $X = R^n$, from Theorem 1 and Corollary 1 we obtain immediately the well–known results (Son, 1990; Szklar, 1985).

Corollary 2. *Suppose that:*

i) $X = R^n$.

ii) *Operator A is spectrally complete.*

iii) *The set* V *satisfies assumption iii) of Theorem 1.*

iv) $$\operatorname{rank}[\Delta(\lambda)|B_0] = n, \qquad for\ every\ \ \lambda \in C \tag{50}$$

Then, dynamical system (1) is absolutely approximately V*-controllable if and only if condition (48) holds.*

Proof. Since $X = R^n$, then operator A_0 generates compact semigroup $T(t)$ for $t > 0$ (Nakagiri, 1987; 1988). Moreover, condition (50) is equivalent to absolute approximate U-controllability of dynamical system (1) (Manitius, 1981; Manitius and Triggiani, 1978a; 1978b). Hence, all the assumptions of Theorem 1 are satisfied and then our Corollary immediately follows.

Corollary 3. *Suppose that:*

i) $X = R^n$.

ii) $\operatorname{rank} A_m = n$.

iii) *The set* V *satisfies the assumption iii) of Theorem 1..*

iv) $$\operatorname{rank}[\lambda I - A_0 - \sum_{i=1}^{i=m} \exp(-\lambda h_i) A_i | B_0] = n, \qquad for\ every\ \ \lambda \in C \tag{51}$$

Then, dynamical system (28) is absolutely approximately V*-controllable if and only if condition (48) holds.*

Proof. First of all, let us observe, that condition (51) is the necessary and sufficient condition for absolute approximate U-controllability of dynamical system (28) for the case $X = R^n$, (Manitius, 1981; Manitius and Triggiani, 1978a; 1978b). On the other hand, assumption ii) is equivalent to spectral completeness of the operator A (Manitius, 1980; 1982). Hence, all assumptions of Theorem 1 are satisfied and then our Corollary follows.

Finally, let us consider the problem of $F-V$-controllability of dynamical system (1) (Definition 8), which is strongly connected with spectral F-completeness of our dynamical system (Definition 2). We assume that the set V satisfies the assumption iii) in Theorem 1.

Corollary 4. *If dynamical system (1) is spectrally* F*-complete, then absolute approximate* V*-controllability is equivalent to* $F - V$*-controllability.*

Proof. Using the same arguments as in (Manitius, 1982, Proposition 2) it is easy to show that under the assumption of spectral F-completeness dynamical system (1) is absolute approximate U-controllable if and only if it is $F - U$-controllable. Hence, repeating the proof of Theorem 1, but in the space $\overline{\operatorname{Im} F}$ we conclude the equivalence stated in our Corollary.

Many results concerning F-controllability and relationships between F-controllability and other types of controllability can be found in the paper (Manitius, 1982), but only for the finite dimensional case, i.e. $X = R^n$.

The connections between F-controllability and other kinds of controllability depend mainly on the concept of F-completeness of dynamical system (1) or (28), (Manitius, 1980). Moreover, it should be stressed that conditions for F-completenes of dynamical system (1) or (28) for infinite–dimensional case are rather complicated and difficult in practice to verify (Nakagiri, 1988).

5. Constrained Relative Controllability

In this section we shall study various kinds of constrained relative controllability for general dynamical system (1) and its special cases considered in section 2.

Generally, relative controllability of dynamical system (1) is strongly related to the investigation of the properties of the linear operator $P_t : L_p([0,t],U) \to X$, $t > 0$, defined by (31), and the attainable sets $C_t(V)$, $t > 0$ and $C_\infty(V)$, given by (34) and (35), respectively.

First of all, let us observe, that unfortunately, in contrast to the situation for absolute controllability, the linear operator $W(t)$, $t \geq 0$, in (31) does not possess semigroup property (Nakagiri, 1988, Sec. 2 and 4).

On the other hand, as has been stated in section 3, absolute exact (approximate) V–controllability implies always relative exact (approximate) V–controllability.

Firstly, in this section we concentrate on negative results for relative exact U–controllability for infinite–dimensional space X. Such a fact for mild solutions in infinite–dimensional control systems without delays has been proved in (Triggiani, 1975a; 1975b; 1976), where some types of compactness of operators are assumed. A similar conclusion has been stated in the paper (Nakagiri, 1988, Sec. 3) for retarded dynamical systems. Now, we recall without proof the main result on the lack of relative exact U–controllability.

Proposition 2. (Triggiani, 1975a; 1975b; 1977) *Let X be infinite–dimensional. If $T(t)$ is semigroup of compact operators for all $t > 0$, then dynamical system (1) is never relatively exactly U– controllable.*

Hence, if the assumptions of Proposition 2 are satisfied, then dynamical system (1) is never relatively exactly U–controllable for any set $V \subset U$. Therefore, since we assumed that operator A_0 generates compact semigroup $T(t)$, $t > 0$, in the sequel we shall concentrate only on the study of relative approximate V–controllability of dynamical system (1) and its special cases given in the section 2.

In order to do that, let us recall some well–known results (Nakagiri, 1987, Sec. 5) concerning the pointwise completeness of dynamical system (1). Combining the results stated in (Nakagiri, 1987, Corollary 5.1, Remark 5.2) we obtain the following special version of (Nakagiri, 1987, Corollary 5.1).

Proposition 3. *Suppose that the operator A_0 generates compact semigroup $T(t)$, for $t > 0$. If there exists a set $\Lambda \subset \sigma(A)$ such that*

$$\overline{\text{span}}\left\{\pi_0 M_\lambda : \lambda \in \Lambda\right\} = X \tag{52}$$

where $\pi_0 : M_p \to X$ is the projection operator then dynamical system (1) is approximately pointwise complete in each time $t > 0$.

It should be stressed that condition (52) is satisfied if the generalized eigenfunctions associated with dynamical system (1) form a complete set in the space M_p (Nakagiri, 1987, Corollary 5.1).

Theorem 2. *Suppose that:*

i) *Operator* A_0 *generates compact semigroup* $T(t)$, *for* $t > 0$.

ii) *Assumption (52) holds.*

iii) *Assumption iii) of Theorem 1 is satisfied.*

Then, dynamical system (1) is relatively aproximately V*-controllable if it is spectrally* U*-controllable for every* $\lambda \in \Lambda$, *and*

$$\operatorname{Ker} \Delta_T(\lambda) \cap (B_0 V)^0 = \{0\} \quad \text{for every} \quad \lambda \in \Lambda \cap R \tag{53}$$

Proof. From assumptions i) and ii) and Proposition 3 it immediately follows that dynamical system (1) is approximately pointwise complete in each time $t > 0$. Approximate pointwise completeness for the set $\Lambda \subset \sigma(A)$, and spectral U–controllability for every $\lambda \in \Lambda$, imply relative approximate U–controllability of dynamical system (1). On the other hand, from the proof of Theorem 3.3 in (Son, 1990) it follows that under assumption iii) condition (53) is in fact the sufficient condition for spectral V–controllability for $\lambda \in \Lambda$, and hence taking into account the approximate pointwise completeness, it is also the sufficient condition for relative approximate V–controllability of dynamical system (1). Hence, our Theorem 2 follows.

Now, let us consider the problem of relative approximate null V–controllability of dynamical system (1). Definition 4 implies that if dynamical system (1) is relatively approximately V–controllable, then we can steer the response of our system arbitrarily close to any given point in the space X. Hence, of course to point zero. Therefore, taking into account Definition 6 we see that generally relative approximate V–controllability always implies relative approximate null V–controllability. However, the converse statement is not always true, even for finite–dimensional case $X = R^n$ (Klamka, 1991, Chapter 4).

Another important point is that relative approximate null V–controllability is always implied by absolute approximate null V–controllability. However, the converse statement is not always true. There are dynamical retarded systems which are relatively approximately null V–controllable, but not absolutely approximately null V–controllable (Klamka, 1991, Chapter 4).

The relationships between relative approximate V–controllability and relative approximate null V–controllability depend strongly on the concept of degeneration of retarded dynamical system (1) and the reachable set R_∞ given by (38). Relative approximate null V–controllability means, that $\overline{C_\infty(V)} \supset R_\infty$. Therefore, it is obvious that preceeding inclusion may hold even $\overline{C_\infty(V)} \neq X$, i.e., even dynamical system (1) is not relatively approximately V–controllable. Hence, we have the following simple Corollary.

Corollary 5. *If dynamical system (1) is approximately, pointwise complete, then relative approximate* V*–controllability is equivalent to relative approximate null* V*–controllability.*

Proof. If dynamical system (1) is approximately pointwise complete, then $\overline{R_\infty} = X$. Therefore, dynamical system (1) is relatively approximately null V–controllable if

$\overline{C_\infty(V)} = \overline{R_\infty} = X$, i.e. if it is relatively approximately V–controllable. The converse implication is evident. Hence, our Corollary 5 follows.

Taking into account Proposition 3 we can formulate the next Corollary on the equivalence between relative approximate V–controllability and relative approximate null V–controllability.

Corollary 6. *Suppose that A_0 generates compact semigroup $T(t)$, for $t > 0$ and condition (52) holds. Then, relative approximate V–controllability is equivalent to relative approximate null V– controllability.*

Proof. Since by Proposition 3 condition (52) implies approximate pointwise completeness of dynamical system (1), then by Corollary 5 we have the desired equivalence between relative approximate V–controllability and relative approximate null V–controllability. Hence, our Corollary 6 follows.

Remark 1. Condition (52) stated in Proposition 3 is essentially stronger than the approximate pointwise completeness of dynamical system (1), (see e.g. (Nakagiri, 1987, Sec. 4 and 5) for detailed study of the approximate pointwise completeness).

Remark 2. Theorem 2 is only sufficient but not the necessary condition for relative approximate V–controllability. This is a consequence of Proposition 3 and the fact that Theorem 2 is formulated and proved in terms of spectral properties of dynamical system (1). Moreover, we used in fact spectral properties of the operator A in the space $M_p([-h,0],X)$, and semigroup $S(t)$, $t > 0$ in the space $M_p([-h,0],X)$ and we did not use directly the spectral properties of the operators $W(t)$, $t > 0$, which do not form semigroup (Nakagiri, 1988, Sec. 3). Therefore, we cannot derive explicit necessary and sufficient conditions for relative approximate V—controllability of dynamical system (1) similar to those given in Theorem 1 for absolute approximate V–controllability.

Remark 3. Similarly as in the finite–dimensional case (Manitius and Triggiani, 1978a; 1978b) approximate V–controllability of dynamical system without delays (i.e. $\eta(s) = 0$, for $s \in [-h,0]$) implies relative approximate V–controllability of dynamical system (1).

Finally, it should be mentioned that many algebraic conditions for relative controllability for finite–dimensional case (i.e. $X = R^n$), can be found in (Klamka, 1991, Chapter 4; Manitius and Triggiani, 1978a, Sec. 6) and (Zmood, 1974), and for relative approximate controllability in (Nakagiri and Yamamoto, 1989) for infinite–dimensional case.

In what follows, we shall concentrate on the special case of dynamical system (1). First, we shall list the desired assumptions.

A1) X is separable Hilbert space and the state space for dynamical system (1) is a Hilbert space $M_2([-h,0],X)$.

A2) U is finite–dimensional space, i.e. $U = R^r$.

A3) The operator A_0 is selfadjoint and generates compact semigroup $T(t)$, for $t > 0$.

In view of the assumptions on X and A_0 there exists a set of eigenvalues s_i, $i = 1,2,...$, which are distinct from each other and real and of multiplicity d_i, $i = 1,2,...$, respectively. Moreover, the corresponding set of eigenvectors x_{ij}, $i = 1,2,...$, $j = 1,2,...d_i$, $x_{ij} \in X$ is a complete orthonormal system in the space X.

Under the above assumptions it is possible to formulate the suffcient condition for relative approximate V–controllability.

Theorem 3. *Suppose that the assumptions A1, A2, A3 and assertion iii) from Theorem 1 are satisfied. Then, dynamical system (1) is relatively approximately V-controllable if*

$$B_{0i} \operatorname{co} V = R^{d_i}, \qquad \text{for all} \quad i = 1, 2, \dots \tag{54}$$

where B_{0i}, $i = 1, 2, \dots$ *are* $d_i \times r$ *dimensional constant matrices given by*

$$B_{0i} = \begin{bmatrix} \langle b_{01}, x_{i1} \rangle, & \langle b_{02}, x_{i1} \rangle, & \cdots, & \langle b_{0r}, x_{i1} \rangle \\ \langle b_{01}, x_{i2} \rangle, & \langle b_{02}, x_{i2} \rangle, & \cdots, & \langle b_{0r}, x_{i2} \rangle \\ \dots, & \dots, & \dots, & \dots \\ \langle b_{01}, x_{id_i} \rangle, & \langle b_{02}, x_{id_i} \rangle & \cdots, & \langle b_{0r}, x_{id_i} \rangle, \end{bmatrix} \tag{55}$$

Proof. If the assumptions A1, A2, A3, and assertion iii) from Theorem 1 are satisfied, then condition (53) is the necessary and sufficient condition for approximate V–controllability of the dynamical system without delays ($\eta(s) = 0$, for $s \in [-h, 0]$)) (Son, 1990, Thm. 4.1). Therefore, since approximate V–controllability implies relative approximate V–controllability of dynamical system (1) our Theorem 3 follows.

Remark 4. Theorem 3 implies that the number of controls required for relative approximate V–controllability ($V \neq R^r = U$) is at least that of the highest multiplicity of the eigenvalues plus one (Son, 1990).

Theorem 3 is only sufficient but not the necessary condition for relative approximate V–controllability of dynamical system (1). However, under some additional assumptions on the Stieltjes measure $\eta(s)$, $s \in [-h, 0]$, it is possible to formulate the necessary and sufficient condition for relative approximate V–controllability.

Corollary 7. *Let the assumptions of Theorem 3 be satisfied. Suppose that the measure η is scalar–valued, that is $A_i = a_i I$, $a_i \in R$, for $i = 1, 2, ..., m$ and $A_I(s) = a_I(s) I$, where $a_I(s) \in L_2([-h, 0], X)$.*

Then, condition (54) is the necessary and sufficient condition for relative approximate V-controllability of dynamical system (1).

Proof. Under the assumptions stated in Corollary 7 dynamical system (1) is relatively approximately U–controllable if and only if (Nakagiri and Yamamoto, 1989, Thm. 4.2)

$$\operatorname{rank} B_{0i} = d_i, \qquad \text{for all} \quad i = 1, 2, \dots \tag{56}$$

On the other hand, condition (55) is the necessary and sufficient condition for approximate controllability of dynamical system without delays ($\eta(s) = 0$, $s \in [-h, 0]$), (Triggiani, 1976, Proposition 3.1). Moreover, condition (54) is the necessary and sufficient condition for approximate V–controllability of dynamical system without delays (Son, 1990, Thm. 4.1). Therefore, combining the above assertions, we deduce that (54) is the necessary and sufficient condition for relative approximate V–controllability of dynamical system (1).

Remark 5. In the proof of the above theorem it is essential that the retarded resolvent $R(\lambda; A_0, \eta)$ is not so different from the resolvent of the operator A_0, $R(\lambda; A_0)$ (Nakagiri and Yamamoto, 1989, Remark 3). Moreover, it should be stressed that in applications, many dynamical systems satisfy the assumptions listed in Corollary 7. This is especially true for distributed parameter systems defined in bounded region.

Remark 6. In the proof of Corollary 7 we can use also the results on approximate controllability for dynamical system without delays given in (Fattorini, 1966, Proposition 2.3).

6. Example

We illustrate the theory we have developed on a simple example of linear partial differential equation of parabolic type with Dirichlet type boundary conditions and with lumped constant single delay.

Let us consider the following equation:

$$\frac{\partial}{\partial t}w(t,y) = \frac{\partial^2}{\partial y^2}w(t,y) + aw(t-h,y) + b_1(y)u_1(t) + b_2(y)u_2(t) \tag{57}$$

defined for $y \in [0,1]$, $t \in [0,\infty)$ and satisfying the homogeneous boundary conditions

$$w(t,0) = w(t,1) = 0, \qquad \text{for} \quad t \in [0,\infty) \tag{58}$$

Initial conditions for equation (57) are as follows:

$$\begin{aligned} &w(0,y) = w^0(y) \in L_2([0,1],R) = X \\ &w(t,y) = w_0(y)(t), \qquad \text{for} \quad t \in [-h,0] \quad \text{and} \quad y \in [0,1] \end{aligned} \tag{59}$$

where $w_0(y)(t) \in L_2\Big([-h,0],\, L_2([0,1],R)\Big)$.

Moreover, it is assumed that $a \in R$ is a constant coefficient, $h > 0$, $b_1 \in L_2([0,1],R)$, $b_2 \in L_2([0,1],R)$ and the controls are nonnegative, i.e. $u_1(t) \geq 0$, $u_2(t) \geq 0$ for $t \in [0,\infty)$.

For dynamical system with delay given by equation (57) the instantaneous state at time t is a function $w(t,y) \in L_2([0,1],R)$, and the complete state at time t is a pair of functions i.e. $z(t) = \{w(t,y),\, w_t(y)\} \in M_2([-h,0],X) = X \times L_2([-h,0],X)$. Moreover, in our example the cone $V \subset U = R^2$ is of the following form:

$$V = \{u \in R^2 : u_1 \geq 0, u_2 \geq 0\} \tag{60}$$

In the abstract setting, equation (57) can be described by the ordinary differential equation with delay

$$\dot{x}(t) = A_0x(t) + ax(t-h) + b_1u_1(t) + b_2u_2(t), \quad t \geq 0 \tag{61}$$

where A_0x is the Laplacian, i.e.

$$A_0x = A_0w(y) = \frac{\partial^2}{\partial y^2}w(y), \qquad \text{for} \quad x \in D(A_0) \subset X = L_2([0,1],R) \tag{62}$$

$$D(A_0) = \{x \in X : A_0x \in X, x(0) = x(1) = 0\} \tag{63}$$

Then, it is well known (see e.g. Fattorini and Russell (1971), Klamka (1991, Chapter 3), Sakawa (1974) or Son (1990)) that the operator A_0 is self adjoint and generates compact semigroup $T(t)$, for $t > 0$. Moreover, the eigenvalues are $s_i = -i^2\pi^2$, $i = 1, 2, ..$ with multiplicity $d_i = 1$, $i = 1, 2, \ldots$. The corresponding eigenfunctions $x_i(y) = \sqrt{2}\sin(i\pi y)$, $i = 1, 2, \ldots$ $y \in [0, 1]$ form an orthonormal complete set in the Hilbert space X.

Hence, taking into account equality (55) we have

$$B_{0i} = \Big[\langle b_{01}, x_i\rangle, \langle b_{02}, x_i\rangle\Big] = \Big[\int_0^1 b_1(y)\sin(i\pi y)\mathrm{d}y, \int_0^1 b_2(y)\sin(i\pi y)\mathrm{d}y\Big] \quad \text{for } i = 1, 2, \ldots \tag{64}$$

Now, we shall verify the relative V–controllability of dynamical system (57) with boundary conditions (58).

First of all, let us observe that the linear operator A_0 generates compact semigroup $T(t)$, for $t > 0$. Hence, since the space X is infinite–dimensional, then by Proposition 2 dynamical system (57) is never relatively exactly U–controllable and in consequence it is never relatively exactly V–controllable.

In what follows we shall concentrate on relative approximate V–controllability of dynamical system (57). In order to do that it is necessary to verify condition (54) in Theorem 3. Taking into account the form of the vectors B_{0i}, $i = 1, 2, \ldots$ given by equality (64) we immediately conclude that condition (54) is satisfied if and only if

$$\Big(\int_0^1 b_1(y)\sin(i\pi y)\mathrm{d}y, \Big)\cdot\Big(\int_0^1 b_2(y)\sin(i\pi y)\mathrm{d}y\Big) < 0 \qquad \text{for } i = 1, 2, \ldots \tag{65}$$

Therefore, by Corollary 7 dynamical system (57) is relatively approximately V–controllable if and only if inequalities (65) hold.

Finally, it should be pointed out that for the case $U = V = R^2$ i.e. for the case of unconstrained controls, dynamical system (57) is relatively approximately U–controllable if and only if (Klamka, 1991; Triggiani, 1976)

$$\langle b_{01}, x_i\rangle^2 + \langle b_{02}, x_i\rangle^2 \neq 0 \qquad \text{for } \quad i = 1, 2, \ldots \tag{66}$$

7. Conclusions

In the paper constrained controllability problem for linear abstract retarded dynamical systems has been investigated. The retarded dynamical systems with distributed and lumps delays in the state variables and defined in infinite–dimensional Banach or Hilbert spaces have been considered using the general methods of linear operators and semigroup theory. The structural properties of the solution both in the space X and in the state space M_p have been listed using the concepts and notations taken directly from the functional analysis.

Next, several definitions of different kinds of controllability, as absolute (relative) exact V–controllability or absolute (relative) approximate V–controllability have been introduced. The relationships between these different types of controllability have been explained using the notions of completeness and structural operators.

In section 4 necessary and sufficient conditions for absotute approximate V–controllability have been formulated and proved. These conditions are generalizations to infinite–dimensional case and constrained controls, the results given in (Son, 1990; Szklar, 1985) and (Nakagiri and Yamamoto, 1989), respectively. As a very special case, constrained controllability conditions for finite–dimensional case have been recalled.

Section 5 contains conditions for constrained relative approximate controllability for various types of retarded abstract dynamical systems. Constrained relative approximate controllability conditions for retarded systems with distributed delays in the state variables and with lumped delays have been formulated and proved using some previous results taken from the literature.

The last section contains simple illustrative example of dynamical system described by partial differential equation of parabolic type and with Dirichlet type boundary conditions and lumped constant delay, which is constrained relatively approximately controllable.

The results presented in the paper can be extended to cover the case of linear abstract retarded dynamical systems but with time–depended coefficients.

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