Appl. Math. and Comp. Sci., 1993, vol.3, No.4, 687–697

OPTIMAL CONTROL OF HYPERBOLIC SYSTEM WITH TIME LAGS†

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In this paper an optimal distributed control problem for a hyperbolic system in which time lags appear in the state equation and in the boundary condition simultaneously is considered. The necessary and sufficient conditions of optimality for the Neumann problem are derived making use of Lions' scheme.

1. Introduction

Various optimization problems associated with the optimal control of distributed – parameter systems with time lags appearing in the boundary conditions have been studied recently by Wang (1975), Knowles (1978), Wong (1987), Kowalewski (1987a; 1987b; 1988a; 1988b; 1988c; 1990a; 1990b; 1990c; 1990d; 1991), and Kowalewski and Duda (1992).

In this paper, we consider an optimal distributed control problem for a linear hyperbolic system in which time lags appear in the state equation and in the boundary condition simultaneously. Such an equation in a linear approximation constitutes a universal mathematical model for many processes in which transmission signals are sent at a certain distance with the electric, hydraulic and other long lines. In the processes mentioned above time delayed feedback signals are introduced at the boundary of a system's spatial domain. Then, the signal at the boundary of a system's spatial domain at any time depends on the signal which escaped earlier. This leads to the boundary conditions involving time lags.

The sufficient conditions for the existence of a unique solution of the hyperbolic lag equation with the Neumann boundary condition involving a constant time lag are proved. The performance functional has a quadratic form. The time horizon is fixed. Finally, we impose some constraints on the distributed control. The necessary and sufficient conditions of optimality with the quadratic performance functional and constrained control are derived for the Neumann problem. The flow chart of the algorithm which can be used in the numerical solution of certain optimization problems for distributed parameter systems with time lags is also presented.

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† This work was supported by the State Committee for Scientific Research under grant No. 3 02098 9101

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2. Preliminaries

Consider now the distributed–parameter system described by the following hyperbolic equation

$$\frac{\partial^2 y}{\partial t^2} + A(t)y + b(x,t)y(x,t-h) = u \qquad x \in \Omega, \;\; t \in (0,T) \tag{1}$$

$$y(x,t') = \Phi_0(x,t') \qquad x \in \Omega, \;\; t' \in [-h,0) \tag{2}$$

$$y(x,0) = y_0(x) \qquad x \in \Omega \tag{3}$$

$$y'(x,0) = y_I(x) \qquad x \in \Omega \tag{4}$$

$$\frac{\partial y}{\partial \eta_A} = c(x,t)y(x,t-h) + v \qquad x \in \Gamma, \;\; t \in (0,T) \tag{5}$$

$$y(x,t') = \Psi_0(x,t') \qquad x \in \Gamma, \;\; t' \in [-h,0) \tag{6}$$

where $\Omega \subset R^n$ is a bounded, open set with boundary Γ, which is a C^∞ – manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ.

$$y \equiv y(x,t;u), \;\; u \equiv u(x,t), \;\; v \equiv v(x,t), \;\; Q = \Omega \times (0,T)$$

$$\overline{Q} = \overline{\Omega} \times [0,T], \;\; Q_0 = \Omega \times [-h,0), \;\; \Sigma = \Gamma \times (0,T), \;\; \Sigma_0 = \Gamma \times [-h,0)$$

b is a given real C^∞ function defined on $\overline{Q}$; c – a given real C^∞ function defined on Σ; h – a specified positive number representing a time lag; Φ_0 – an initial function defined on Q_0; Ψ_0 – an initial function defined on Σ_0.
The operator $A(t)$ has the form

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right) \tag{7}$$

and the functions $a_{ij}(x,t)$ satisfy the following conditions in $Q = \Omega \times (0,T)$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\varphi_i\varphi_j \geq \alpha \sum_{i=1}^{n} \varphi_i^2, \qquad \alpha > 0, \;\; \forall\,(x,t) \in \overline{Q}, \;\; \varphi_i \in R \tag{8}$$

$$a_{ij} = a_{ji} \quad \forall\, i,j \tag{9}$$

where $a_{ij}(x,t)$ are real C^∞ functions defined on $\overline{Q}$ (closure of Q).
It is easy to notice that equations (1)–(6) constitute the Neumann problem. The left–hand side of the Neumann boundary condition (5) is written in the following form

$$\frac{\partial y}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ij}(x,t)\cos(n,x_i)\frac{\partial y(x,t)}{\partial x_j} = q(x,t) \tag{10}$$

where $\partial y/\partial \eta_A$ is a normal deriviative at Γ, directed towards the exterior of Ω; $\cos(\boldsymbol{n}, x_i)$ – the i–th direction cosine of $\boldsymbol{n}$, $\boldsymbol{n}$–being the normal at Γ exterior to Ω;

$$q(x,t) = c(x,t)y(x,t-h) + v(x,t) \tag{11}$$

First we shall prove the existence of a unique solution of the mixed initial–boundary value problem (1)–(6). We shall consider the case where control u belongs to $H^{0,1}(Q)$. For simplicity, we shall introduce the following notations

$$T = K\,h \quad \text{where} \quad K - \text{ a positive integer}$$

$$I_j = \big((j-1)h, jh\big), \quad Q_j = \Omega \times I_j, \quad \Sigma_j = \Gamma \times I_j \quad \text{for } j = 0, 1, ..., K$$

The existence of a unique solution for the mixed initial–boundary value problem (1)–(6) on the cylinder Q can be proved using a constructive method, i.e. solving at first equations (1)–(6) on the subcylinder Q_1 and in turn on Q_2, etc. until the procedure covers the whole cylinder Q. In this way, the solution in the previous step determines the next one.

Using Theorem 3.1 of Lions and Magenes (1972, v.2, p.103), the following lemma may be proved.

Lemma 1. *Let*

$$u \in H^{0,1}(Q) \tag{12}$$

$$f_j \in H^{0,1}(Q_j) \tag{13}$$

where

$$f_j(x,t) = u(x,t) - b(x,t)y_{j-1}(x,t-h)$$

$$q_j \in H^{3/2,3/2}(\Sigma_j) \tag{14}$$

and

$$q_j(x,t) = c(x,t)y_{j-1}(x,t-h) + v(x,t)$$

$$y_{j-1}(\cdot,(j-1)h) \in H^2(\Omega) \tag{15}$$

$$y'_{j-1}(\cdot,(j-1)h) \in H^{3/2}(\Omega) \tag{16}$$

and the following compatibility relations be fulfilled

$$\frac{\partial y_{j-1}}{\partial \eta_A}(x,(j-1)h) = q_j(x,(j-1)h) \ \text{ on } \Gamma \tag{17}$$

$$\frac{\partial y'_{j-1}}{\partial \eta_A}(x,(j-1)h) + \left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \eta_A}\right)\right) y_{j-1}(x,(j-1)h) = \frac{\partial}{\partial t} q_j(x,(j-1)h) \ \text{ on } \Gamma \tag{18}$$

Then, there exists a unique solution $y_j \in H^{2,2}(Q_j)$ *for the mixed initial-boundary value problem (1), (5), (15), (16).*

Proof. For $j = 1$, $y_{j-1}\Big|_{Q_0}(x, t-h) = \phi_0(x, t-h)$ and $y_{j-1}\Big|_{\Sigma_0}(x, t-h) = \Psi_0(x, t-h)$ respectively. Then, assumptions (13),(14),(15) and (16) are fulfilled if we assume that $\phi_0 \in H^{2,2}(Q_0)$, $v \in H^{3/2,3/2}(\Sigma)$ and $\Psi_0 \in H^{3/2,3/2}(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in H^{2,2}(Q_1)$ if $y_0 \in H^2(\Omega)$, $y_1 \in H^{3/2}(\Omega)$ and the following compatibility conditions are satisfied

$$\frac{\partial y_0}{\partial \eta_A}(x, 0) = q_1(x, 0) \qquad \text{on } \Gamma \tag{19}$$

$$\frac{\partial y_I}{\partial \eta_A}(x, 0) + \left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \eta_A}\right)\right) y_0(x, 0) = \frac{\partial}{\partial t} q_1(x, 0) \qquad \text{on } \Gamma \tag{20}$$

In order to extend the result to Q_2 it is necessary to impose the compatibility relations

$$\frac{\partial y_1}{\partial \eta_A}(x, h) = q_2(x, h) \qquad \text{on } \Gamma \tag{21}$$

$$\frac{\partial y_1'}{\partial \eta_A}(x, h) + \left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \eta_A}\right)\right) y_0(x, 0) = \frac{\partial}{\partial t} q_2(x, h) \qquad \text{on } \Gamma \tag{22}$$

and it is sufficient to verify that

$$f_2 \in H^{0,1}(Q_2) \tag{23}$$

$$y_1(\cdot, h) \in H^2(\Omega) \tag{24}$$

$$y_1'(\cdot, h) \in H^{3/2}(\Omega) \tag{25}$$

$$q_2 \in H^{3/2,3/2}(\Sigma_2) \tag{26}$$

First, using the solution of the previous step and condition (12) we can prove immediately condition (23).

To verify (24) and (25) we use the fact (by Proposition 3.1 of (Lions and Magenes, 1972, v.2, p.100)) that the function w_1 has the following properties.

$$w_1 \in L^2(L_1; H^2(\Omega)), \quad w_1' \in L^2(L_1; H^2(\Omega)), \quad w_1''' \in L^2(L_1; H^0(\Omega))$$

Then, from Theorem 3.1 of Lions and Magenes (1972, v.1, p.19) it follows that the mappings $t \to w_1(\cdot, t)$ and $t \to w_1'(\cdot, t)$ are continuous from $[0, h] \to H^2(\Omega)$ and $[0, h] \to H^{3/2}(\Omega)$, respectively. Hence, $w_1(\cdot, h) \in H^2(\Omega)$ and $w_1'(\cdot, h) \in H^{3/2}(\Omega)$. But from Section 3 of Lions and Magenes (1972, v.2, p.99) it follows that $w_1(\cdot, h) = y_1(\cdot, h)$ and $w_1'(\cdot, h) = y_1'(\cdot, h)$. From the preceding results we can deduce that $y_1(\cdot, h) \in H^2(\Omega)$ and $y_1'(\cdot, h) \in H^{3/2}(\Omega)$. Again the trace theorem of Lions and Magenes (1972, v.2, p.9) $y_1 \in H^{2/2}(Q_1)$ implies that $y_1 \to y_1\Big|_{\Sigma_1}$ is a linear continuous mapping of $H^{2,2}(Q_1) \to H^{3/2,3/2}(\Sigma_1)$. Thus, $y_1\Big|_{\Sigma_1} \in H^{3/2,3/2}(\Sigma_1)$. Assuming that c is a C^∞ function and $v \in H^{3/2,3/2}(\Sigma)$, condition (26) is fulfilled. Then, there exists a unique solution $y_2 \in H^{2,2}(Q_2)$. We shall now extend our result to any Q_j, $2 < j \leq K$.

Theorem 1. *Let* $y_0, y_I, \phi_0, \Psi_0, v$ *and* u *be given with* $y_0 \in H^2(\Omega)$, $y_I \in H^{3/2}(\Omega)$, $\phi_0 \in H^{2,2}(Q_0)$, $\Psi_0 \in H^{3/2,3/2}(\Sigma_0)$, $v \in H^{3/2,3/2}(\Sigma)$, $u \in H^{0,1}(Q)$ *and the compatibility relations (19), (20) be fulfilled. Then, there exists a unique solution* $y \in H^{2,2}(Q)$ *for the mixed initial- boundary value problem (1)–(6) with* $y(\cdot, jh) \in H^2(\Omega)$ *and* $y'(\cdot, jh) \in H^{3/2}(\Omega)$ *for* $j = 1, ..., K$.

3. Problem Formulation. Optimization Theorem

Now we shall formulate the optimal control problem in the context of a case where $u \in H^{0,1}(Q)$. Let us denote by $U = H^{0,1}(Q)$ the space of controls. The time horizon T is fixed in our problem. The performance functional is given by

$$I(u) = \lambda_1 \int_Q \left| y(x,t;u) - z_d \right|^2 \mathrm{d}x\,\mathrm{d}t + \lambda_2 \left\| u \right\|^2_{H^{0,1}(Q)} \tag{27}$$

where $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 > 0$; z_d is a given element in $L^2(Q)$ and

$$\left\| u \right\|^2_{H^{0,1}(Q)} = \int_0^T \left\langle u, u \right\rangle_{L^2(\Omega)} \mathrm{d}t + \int_0^T \left\langle \frac{\partial u}{\partial t}, \frac{\partial u}{\partial t} \right\rangle_{L^2(\Omega)} \mathrm{d}t = \int_Q \left[\left(1 - \frac{\partial^2}{\partial t^2} \right) u \right] u \,\mathrm{d}x\,\mathrm{d}t$$

$$\forall\, u \in D\left(\frac{\partial^2}{\partial t^2} \right) \stackrel{\mathrm{df}}{=} \left\{ u : u \in H^2(0,T; L^2(\Omega)),\ u(x,0) = u'(x,T) = 0 \right\} \tag{28}$$

Finally, we assume the following constraint on controls

$$u \in U_{ad} \text{ is a closed, convex subset of } U \tag{29}$$

Let $y(x,t;u)$ denote the solution of (1)–(6) at (x,t) corresponding to a given control $u \in U_{ad}$. We note from Theorem 1 that for any $u \in U_{ad}$ performance functional (27) is well–defined since $y(u) \in H^{2,2}(Q) \subset L^2(Q)$. The solution of the stated optimal control problem is equivalent to the search for an $u_0 \in U_{ad}$ such that $I(u_0) \leq I(u)\ \forall\, u \in U_{ad}$.

From Lion's scheme (Theorem 1.3 of (Lions, 1971, p.10)) it follows that for $\lambda_2 > 0$ a unique optimal control u_0 exists; moreover u_0 is characterized by the following condition

$$I'(u_0) \cdot (u - u_0) \geq 0 \quad \forall\, u \in U_{ad} \tag{30}$$

Using the form of the cost function (27) we may express (30) in the following form

$$\lambda_1 \int_Q \big(y(u_0) - z_d \big) \big(y(u) - y(u_0) \big) \mathrm{d}x\,\mathrm{d}t + \lambda_2 \left\langle u_0, u - u_0 \right\rangle_{H^{0,1}(Q)} \geq 0 \quad \forall\, u \in U_{ad} \tag{31}$$

To simplify (31), we introduce the adjoint equation and for every $u \in U_{ad}$ we define the adjoint variable $p = p(u) = p(x,t;u)$ as the solution of the following equation

$$\frac{\partial^2 p(u)}{\partial t^2} + A(t)p(u) + b(x,t+h)p(x,t+h;u) = \lambda_1 (y(u) - z_d)$$

$$x \in \Omega,\ t \in (0, T-h) \tag{32}$$

$$\frac{\partial^2 p(u)}{\partial t^2} + A(t)p(u) = \lambda_1\big(y(u) - z_d\big) \qquad x \in \Omega,\ t \in (T-h,T) \tag{33}$$

$$p(x,T;u) = 0 \qquad x \in \Omega \tag{34}$$

$$p'(x,T;u) = 0 \qquad x \in \Omega \tag{35}$$

$$\frac{\partial p(u)}{\partial \eta_A}(x,t) = c(x,t+h)p(x,t+h;u) \qquad x \in \Gamma,\ t \in (0,T-h) \tag{36}$$

$$\frac{\partial p(u)}{\partial \eta_A}(x,t) = 0 \qquad x \in \Gamma,\ t \in (T-h,T) \tag{37}$$

where

$$\frac{\partial p(u)}{\partial \eta_A}(x,t) = \sum_{i,j=1}^{n} a_{ij}(x,t)\cos(\boldsymbol{n},x_i)\frac{\partial p(u)}{\partial x_j}(x,t) \tag{38}$$

It is easy to prove the following lemma.

Lemma 2. *Let the hypothesis of Theorem 1 be satisfied. Then, for a given $z_d \in L^2(Q)$ and any $u \in H^{0,1}(Q)$ there exists a unique solution $p(u) \in H^{2,2}(Q)$ for (32)–(37).*

We simplify (31) using the adjoint equation (32)–(37), multiplying both sides of (32) and (33) by $\big(y(u) - y(u_0)\big)$, then integrating over $\Omega \times (0,T-h)$ and $\Omega \times (T-h,T)$, respectively, and then adding both sides of (32) and (33) we get

$$\begin{aligned}
&\lambda_1 \int_Q \big(y(u_0) - z_d\big)\big(y(u) - y(u_0)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \Big(\frac{\partial^2 p(u)}{\partial t^2} + A(t)p(u_0)\Big)\Big(y(u) - y(u_0)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^{T-h}\int_\Omega b(x,t+h)p(x,t+h;u_0)\big(y(x,t;u) - y(x,t;u_0)\big)\,\mathrm{d}x\,\mathrm{d}t \qquad (39)\\
&= \int_Q p(u)\frac{\partial^2}{\partial t^2}\big(y(u) - y(u_0)\big)\,\mathrm{d}x\,\mathrm{d}t + \int_Q A(t)p(u_0)\big(y(u) - y(u_0)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^{T-h}\int_\Omega b(x,t+h)p(x,t+h;u_0)\big(y(x,t;u) - y(x,t;u_0)\big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

Using equation (1), the first integral on the right–hand side of (39) can be rewritten as

$$\begin{aligned}
&\int_Q p(u_0)\frac{\partial^2}{\partial t^2}\big(y(u) - y(u_0)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q p(u_0)(u - u_0)\,\mathrm{d}x\,\mathrm{d}t - \int_Q p(u_0)A(t)\big(y(u) - y(u_0)\big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$-\int_0^T \int_\Omega p(x,t;u_0)b(x,t)\big(y(x,t-h;u)-y(x,t-h;u_0)\big)\mathrm{d}x\,\mathrm{d}t \tag{40}$$

$$= \int_Q p(u_0)(u-u_0)\mathrm{d}x\,\mathrm{d}t - \int_Q p(u_0)A(t)\big(y(u)-y(u_0)\big)\mathrm{d}x\,\mathrm{d}t$$

$$-\int_{-h}^{T-h} \int_Q p(x,t'+h;u_0)b(x,t'+h)\big(y(x,t';u)-y(x,t';u_0)\big)\mathrm{d}x\,\mathrm{d}t'$$

Using the Green's formula, the second component in (39) can be written as

$$\int_Q A(t)p(u_0)\big(y(u)-y(u_0)\big)\mathrm{d}x\,\mathrm{d}t = \int_Q p(u)A(t)\big(y(u)-y(u_0)\big)\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^T \int_\Gamma p(u_0)\Big[\frac{\partial y(u)}{\partial \eta_A} - \frac{\partial y(u_0)}{\partial \eta_A}\Big]\mathrm{d}\Gamma\,\mathrm{d}t - \int_0^T \int_\Gamma \frac{\partial p(u_0)}{\partial \eta_A}\Big[y(u)-y(u_0)\Big]\mathrm{d}\Gamma\,\mathrm{d}t \tag{41}$$

Using boundary condition (5), the second integral in the right–hand side of (41) can be expressed as

$$\int_0^T \int_\Gamma p(u_0)\Big[\frac{\partial y(u)}{\partial \eta_A} - \frac{\partial y(u_0)}{\partial \eta_A}\Big]\mathrm{d}\Gamma\,\mathrm{d}t$$

$$= \int_0^T \int_\Gamma p(x,t;u_0)c(x,t)\Big[y(x,t-h;u)-y(x,t-h;u_0)\Big]\mathrm{d}\Gamma\,\mathrm{d}t \tag{42}$$

$$= \int_{-h}^{T-h} \int_\Gamma p(x,t'+h;u_0)c(x,t'+h)\Big[y(x,t';u)-y(x,t';u_0)\Big]\mathrm{d}\Gamma\,\mathrm{d}t'$$

The last component in (41) can be rewritten as

$$\int_0^T \int_\Gamma \frac{\partial p(u_0)}{\partial \eta_A}\Big[y(u)-y(u_0)\Big]\mathrm{d}\Gamma\,\mathrm{d}t = \int_0^{T-h} \int_\Gamma \frac{\partial p(u_0)}{\partial \eta_A}\Big[y(u)-y(u_0)\Big]\mathrm{d}\Gamma\,\mathrm{d}t$$

$$+\int_{T-h}^T \int_\Gamma \frac{\partial p(u_0)}{\partial \eta_A}\Big[y(u)-y(u_0)\Big]\mathrm{d}\Gamma\,\mathrm{d}t \tag{43}$$

Substituting (42), (43) into (41) and then (40), (41) into (39) we get

$$\lambda_1 \int_Q \big(y(u_0)-z_d\big)\big(y(u)-y(u_0)\big)\mathrm{d}x\,\mathrm{d}t$$

$$= \int_Q p(u_0)(u-u_0)\mathrm{d}x\,\mathrm{d}t - \int_Q p(u_0)A(t)\big(y(u)-y(u_0)\big)\mathrm{d}x\,\mathrm{d}t$$

$$-\int_{-h}^0 \int_\Omega b(x,t+h)p(x,t+h;u_0)\Big[y(x,t;u)-y(x,t;u_0)\Big]\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&-\int_0^{T-h}\int_\Omega b(x,t+h)p(x,t+h;u_0)\Big[y(x,t;u)-y(x,t;u_0)\Big]\mathrm{d}x\,\mathrm{d}t \\
&+\int_Q p(u_0)A(t)\big(y(u)-y(u_0)\big)\mathrm{d}x\,\mathrm{d}t \\
&+\int_{-h}^{0}\int_\Gamma c(x,t+h)p(x,t+h;u_0)\big(y(x,t;u)-y(x,t;u_0)\big)\mathrm{d}\Gamma\,\mathrm{d}t \\
&+\int_0^{T-h}\int_\Gamma c(x,t+h)p(x,t+h;u_0)\big(y(x,t;u)-y(x,t;u_0)\big)\mathrm{d}\Gamma\,\mathrm{d}t \\
&-\int_0^{T-h}\int_\Gamma \frac{\partial p(u_0)}{\partial\eta_A}\Big[y(x,t;u)-y(x,t;u_0)\Big]\mathrm{d}\Gamma\,\mathrm{d}t \\
&-\int_{T-h}^{T}\int_\Gamma \frac{\partial p(u_0)}{\partial\eta_A}\Big[y(x,t;u)-y(x,t;u_0)\Big]\mathrm{d}\Gamma\,\mathrm{d}t \\
&+\int_0^{T-h}\int_\Gamma b(x,t+h)p(x,t+h;u_0)\big(y(x,t;u)-y(x,t;u_0)\big)\mathrm{d}x\,\mathrm{d}t \\
&=\int_\Omega p(x,t;u_0)(u-u_0)\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{44}
$$

Using formula (28) and substituting (44) into (31) gives

$$
\int_Q\Big[p(u_0)+\lambda_2\Big(1-\frac{\partial^2}{\partial t^2}\Big)u_0\Big](u-u_0)\mathrm{d}x\,\mathrm{d}t\ge 0 \tag{45}
$$

$$\forall\, u\in U_{ad}$$

$$\forall\, u\in D\Big(\frac{\partial^2}{\partial t^2}\Big)\overset{\mathrm{df}}{=}\Big\{u: u\in H^2(0,T;L^2(\Omega)),u(x,0)=u'(x,T)=0\Big\}$$

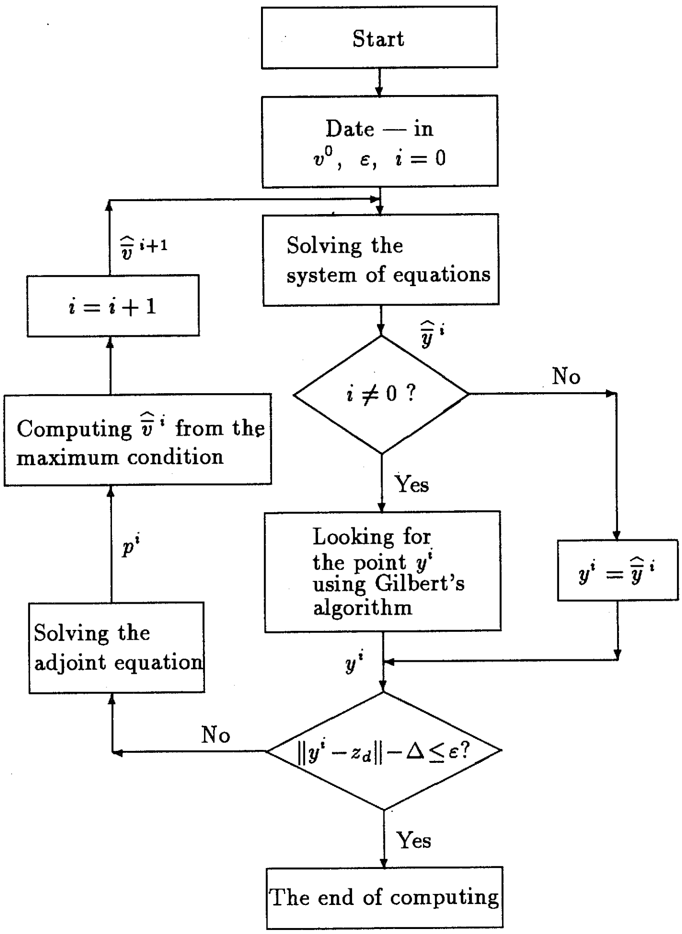
Theorem 2. *For problem (1)–(6) with cost function (27) with* $z_d\in L^2(\Omega)$ *and* $\lambda_2>0$ *and with constraints on controls (29), there exists a unique optimal control* u_0 *which satisfies condition (45).*

We must notice that the conditions of optimality derived above (Theorem 2) allow us to obtain an analytical formula for the optimal control in particular cases only (e.g. there are no constraints on controls). This results from the following: determination of the function $p(x,t)$ in the maximum condition is possible from the adjoint equation if and only if we know y_0 which corresponds to the control u_0.

These mutual connections make practical use of the derived optimization formulae difficult. Thus, we resign from the exact determination of the optimal control and we use approximation methods.

In the case of performance functional (27) with $\lambda_1>0$ and $\lambda_2=0$, the optimal control problem is reduced to minimizing the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to quadratic programming which can be solved by the use of the well–known algorithms, e.g. Gilbert's (1966). In this case we can show the calculation procedure on the flow chart (see Fig. 1).

The practical application of Gilbert's algorithm to an optimal control problem for a parabolic system with boundary condition involving a time lag is presented in Kowalewski and Duda (1992). Using Gilbert's algorithm, a one-dimensional numerical example of the plasma control process is solved.



v^0 – an arbitrary initial control ($v^0 \in U_{ad}$)

$\|y^i - z_d\| - \Delta$ – an error in the i–th iteration

Fig. 1. The flow chart of the algorithm which can be used in the numerical solving of optimization problems for distributed systems.

4. Conclusions

The results presented in the paper can be treated as an extension of the results obtained by Kowalewski (1988b) onto the case of additional constant time lags appearing in the state equation.

In this paper we have considered the optimal hyperbolic system where time lags appear both in the state equation and in the Neumann boundary condition. We can also derive conditions of optimality for a more complex case of such a hyperbolic system with the Dirichlet boundary condition.

We can also obtain estimates and sufficient conditions for the boundedness of solutions for such a hyperbolic system with specified forms of feedback control.

Finally, we can consider a more complex case of optimal boundary control for a hyperbolic system in which constant time lags appear in the state equation and in the boundary condition simultaneously.

The ideas mentioned above will be developed in forthcoming papers.

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Received July 16, 1993
Revised December 31, 1993