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STABILIZATION FOR NONLINEAR INTERCONNECTED SYSTEMS†

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The problem of decentralized stabilization for nonlinear interconnected systems coupled in the general form or, particularly, coupled in the lower triangular form are investigated. The Lyapunov–Like conditions of stabilization for these interconnected systems are established, and the decentralized feedback laws for stabilization are presented.

1. Introduction

In the 1970s the problem of stability of nonlinear large scale systems attracted much attention. A lot of research papers were presented and a great deal of criteria for the stability of nonlinear large scale systems were established. Among many significant achievements, the contribution of Michel and Miller (1977) is outstanding. For the nonlinear large scale systems in the general interconnected form, they established many criteria so that one may judge whether or not a nonlinear large scale system is stable from the properties of Lyapunov functions of its subsystems (Michel and Miller, 1977). For the nonlinear large scale systems in the lower triangularly interconnected form, they verified that stability of subsystems implies that large scale systems are stable if coupling functions satisfy certain requirements (Michel *et al.*, 1978). Their work laid a foundation for further investigation. As a matter of fact, the major results obtained in the 1970s were established on such a foundation.

However, the results of stabilization for nonlinear large scale systems obtained in the 1970s are not satisfying. It seems that the works carried out at that time were limited to the analysis of stability.

In the last decade, the research of stabilization for nonlinear systems was developed at speed. Many new stabilizing techniques have been presented, such as the method of center manifold (Aeyles, 1985; Pan *et al.*, 1990), factorization of nonlinear systems (Verma, 1988) and relaxed controls (see e.g. Artstein, 1983; Sontag, 1989a; 1989b; Tsinians, 1989). It seems that the technique of relaxed controls is more powerful than others, since it may stabilize some nonlinear systems which cannot be stabilized by smooth feedback (Sontag, 1989a). There is a number of contributed papers dealing with such a control technique. For instance, Artstein (1983) established a background of relaxed controls, Tsinians (1989) and Sontag (1989b) presented two types of relaxed feedback designs, a good survey was done by Sontag (1989a). These achievements make it possible to investigate the problem of decentralized stabilization of nonlinear large scale

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systems. In (Han *et al.*, 1992) the authors applied decentralized relaxed feedbacks to locally stabilize nonlinear large scale systems. This paper will apply such feedbacks to globally stabilize these systems. The outline of the designing scheme is that we design decentralized relaxed feedbacks such that every subsystem satisfies the conditions established in (Michel and Miller 1977, Michel *et al.*, 1978), then the theorems proposed in (Michel and Miller, 1977; Michel *et al.*, 1978) guarantee the stability of these nonlinear large scale systems.

This paper is organized as follows: Section 2 gives preliminaries, including notations and necessary lemmas. In Section 3, we deal with the problem of stabilization of nonlinear large scale systems in the general interconnected form. The stabilization for those with the lower triangular interconnection is treated in Section 4. Final comments are given in the last section.

2. Preliminaries

In this paper we consider nonlinear large scale systems of the form

$$\dot{x}_i = f_i(x_i,t) + \overline{f}_i(x,t) + g_i(x_i,t)u_i \qquad i \in \boldsymbol{\nu} \tag{1}$$

where $\boldsymbol{\nu} = \{1,2,...,\nu\}$ and $\boldsymbol{x}^T = [x_1^T,...,x_\nu^T]$. Equations (1) may be considered as a large scale system interconnected by ν subsystems:

$$\dot{x}_i = f_i(x_i,t) + g_i(x_i,t)u_i \qquad i \in \boldsymbol{\nu} \tag{2}$$

with the coupling functions $\overline{f}_i(x,t)$. In (1) and (2), $x \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are said to be the state and input vector of the i–th subsystem, respectively; $g_i(x_i,t) = [g_{i1}(x_i,t),...,g_{im_i}(x_i,t)]$ is an $n \times m_i$ matrix whose elements are assumed to be smooth functions for all $x_i \in \mathbb{R}^{n_i}$ and $t \in T$ where $T = [t_0,t_1]$ for some $t_1 > t_0 \geq 0$. If $u_i \equiv 0$ for all $i \in \boldsymbol{\nu}$, then the i–th subsystem is called a free subsystem, the word *free* means that this subsystem is not constrained. Let $n = \sum_{i=1}^{\gamma} n_i$. Suppose for all $i \in \boldsymbol{\nu}$ $f_i : \mathbb{R}^{n_i} \times T \to \mathbb{R}^{n_i}$ and $\overline{f}_i : \mathbb{R}^n \times T \to \mathbb{R}^{n_i}$ are smooth vector–value functions.

It is well–known that the interconnected system (1) corresponds to a directed graph (Michel *et al.*, 1978). If the directed graph is not strongly connected, then it can be decomposed into an interconnection of some strongly connected components with the interconnection having lower triangular form, a program has been presented in (Michel *et al.*, 1978) to fulfil such a decomposition. The interconnected systems in lower triangular form are described by

$$\dot{x}_i = f_i(x_i,t) + \sum_{j=1}^{i-1} \overline{f}_{ij}(x_j,t) + g_i(x_i,t)u_i \qquad i \in \boldsymbol{\nu} \tag{3}$$

where $\overline{f}_{ij} : \mathbb{R}^{n_j} \times T \to R^{n_i}$ are smooth vector–value functions for all $0 < j < i$ and for all $i \in \boldsymbol{\nu}$.

Assume that the origin of $\mathbb{R}^n$ is the unique equilibrium of both interconnected system (1) and system (3), i.e. for (1) we suppose $f_i(0,t) \equiv 0$ and $\overline{f}_i(0,t) \equiv 0$ for all $t \in T$, and for (3), $f_i(0,t) \equiv 0$ and $\overline{f}_{ij}(0,t) \equiv 0$ for all $t \in T$.

Let

$$u_i = \psi_i(x_i, t), \quad \text{and} \quad \psi_i(0,t) = 0 \quad \text{for all} \quad t \in T, \quad i \in \boldsymbol{\nu} \tag{4}$$

Equations (4) are decentralized feedbacks of (1) or (3). The feedbacks are said to be almost smooth if, for all $i \in \boldsymbol{\nu}$, ψ_i are continuous for all $(x_i, t) \in \mathbb{R}^{n_i} \times T$ and smooth for all $(x_i, t) \in (\mathbb{R}^{n_i} \setminus \{0\}) \times T$. Closing (1) or (3) by feedbacks (4) yields

$$\dot{x}_i = f_i(x_i, t) + \overline{f}_i(x, t) + g_i(x_i, t)\psi_i(x_i, t) \qquad i \in \boldsymbol{\nu} \tag{5}$$

or

$$\dot{x}_i = f_i(x_i, t) + \sum_{j=1}^{i-1} \overline{f}_{ij}(x_j, t) + g_j(x_i, t)\psi_i(x_i, t), \qquad i \in \boldsymbol{\nu} \tag{6}$$

Let $\mathbb{R}^+ = [0, \infty)$. A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called of class **K** if ϕ is continuous, strictly increasing and $\phi(0) = 0$. If in addition $\lim_{s\to\infty} \phi(s) = \infty$, then ϕ is called a function of class **KR**.

Let $\|\cdot\|$ denote Euclidean norm in $\mathbb{R}^n$, i.e. if $\boldsymbol{x}^T = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\|\boldsymbol{x}\| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$.

For the sake of convenience, we state some lemmas and their proofs are omitted and referred to the corresponding references.

Lemma 1. (Michel and Miller, 1977): *If for every $i \in \boldsymbol{\nu}$ there exists a continuous differential function $V_i : \mathbb{R}^{n_i} \times T \to \mathbb{R}^+$ and functions $\phi_{i1}, \phi_{i2} \in \mathbf{KR}$, $\phi_{i3} \in \mathbf{K}$, such that*

1. $\phi_{i1}(\|x_i\|) \le V_i(x_i, t) \le \phi_{i2}(\|x_i\|)$ *for all* $(x_i, t) \in \mathbb{R}^{n_i} \times T$;
2. $\nabla V_i(x_i, t) \cdot f_i(x_i, t) \le \sigma_i \phi_{i3}(\|x_i\|)$ *for some* $\sigma_i \in \mathbb{R}$ *and for all* $(x_i, t) \in \mathbb{R}^{n_i} \times T$;
3. $$\nabla V_i(x_i, t) \cdot \overline{f}_i(x, t) \le [\phi_{i3}(\|x_i\|)]^{\frac{1}{2}} \Big(\sum_{j=1}^{\nu} a_{ij} [\phi_{j3}(\|x_j\|)]^{\frac{1}{2}}\Big)$$
   for some $a_{ij} \in \mathbb{R}$, $j \in \boldsymbol{\nu}$, *and for all* $(x, t) \in \mathbb{R}^n \times T$;
4. *There exists a vector* $\boldsymbol{\alpha}^T = (\alpha_1, \ldots, \alpha_\nu)$ *with* $\alpha_i > 0$ *for all* $i \in \boldsymbol{\nu}$, *such that, the matrix* $S = (s_{ij})$ specified by
   $$s_{ij} = \begin{cases} \alpha_i(\sigma_i + a_{ii}) & \text{for } i = j \\ \frac{1}{2}(\alpha_i a_{ij} + \alpha_j a_{ji}) & \text{for } i \ne j \end{cases}$$
   is negative definite.

Then, when $u_i = 0$ for all $i \in \boldsymbol{\nu}$, the interconnected system (1) is globally asymptotically stable.

Remark 1. If conditions (3) and (4) are replaced by the following ones:

(3′) There exist almost smooth functions $a_{ij}(x, t)$ $i, j \in \boldsymbol{\nu}$, such that

$$\nabla V_i(x_i, t) \cdot \overline{f}_i(x, t) \le [\phi_{i3}(\|x_i\|)]^{\frac{1}{2}} \Big(\sum_{j=1}^{\nu} a_{ij}(x, t) [\phi_{j3}(\|x_j\|)]^{\frac{1}{2}}\Big)$$

(4′) There exists a vector $\boldsymbol{\alpha}^T = (\alpha_1, ..., \alpha_\nu)$ with $\alpha_i > 0$ $i \in \boldsymbol{\nu}$ and $\varepsilon > 0$, such that, the matrix $S(x,t) + \varepsilon I$ is negative definite where $S(x,t) = (s_{ij}(x,t))$ and

$$s_{ij} = \begin{cases} \alpha_i[\sigma_i + a_{ij}(x,t)] & i = j \\ \frac{1}{2}[\alpha_i a_{ij}(x,t) + \alpha_j a_{ji}(x,t)] & i \neq j \end{cases}$$

Then, the conclusion of Lemma 1 still holds.

Remark 2. In Lemma 1 and Remark 1, it is not required that the subsystems are stable, i.e. it is not assumed that $\sigma_i < 0$.

Lemma 2. (Michel *et al.*, 1978) *If for every* $i \in \boldsymbol{\nu}$ *there exists a continuous differential function* $V_i : \mathbb{R}^{n_i} \times T \to \mathbb{R}^+$ *and functions* ϕ_{i1}, $\phi_{i2} \in \mathbf{KR}$, $\phi_{i3} \in \mathbf{K}$, *such that for all* $(x_i, t) \in \mathbb{R}^{n_i} \times T$, *the following requirements are satisfied simultaneously*

1. $\phi_{i1}(\|x_i\|) \leq V_i(x_i, t) \leq \phi_{i2}(\|x_i\|)$;
2. $\nabla V_i(x_i, t) \cdot f_i(x_i, t) \leq -\phi_{i3}(\|x_i\|)$;
3. $\lim_{\|x\| \to \infty} \|\nabla V_i(x_i, t)/\phi_{i3}(\|x_i\|)\| = 0$;
4. *For every* $0 < j < i$ *and for every* $i \in \boldsymbol{\nu}$, *there exists a function* $\varphi_{ij} \in \mathbf{K}$, *such that*

$$\|\overline{f}_{ij}(x_j, t)\| \leq \varphi_{ij}(\|x_j\|)$$

Then, when $u_i = 0$ *for all* $i \in \boldsymbol{\nu}$, *the interconnected system in the lower triangular form (3) is globally asymptotically stable.*

Lemma 3. (Sontag, 1989b) *Let* $a(x)$ *and* $b(x)$ *be almost smooth functions with the following properties:*

1. $b(0) = a(0) = 0$;
2. $a(x) \leq 0$ *for all* $x \neq 0$ *such that* $b(x) = 0$.

Then, the function $k(x)$ *is almost smooth where*

$$k(x) = \begin{cases} \dfrac{a(x) + \sqrt{a^2(x) + b^4(x)}}{b(x)} & \text{for} \quad b(x) \neq 0 \\ 0 & \text{for} \quad b(x) = 0 \end{cases}$$

3. Decentralized Stabilization for Interconnected Systems

In this section, the problem of stabilization of interconnected systems (1) by decentralized almost smooth feedbacks (4) is studied. To begin with we give the following definition.

Definition 1. If there exists an almost smooth function $V_i : \mathbb{R}^{n_i} \times T \to \mathbb{R}^+$ and almost smooth functions $\phi_{i1}, \phi_{i2} \in \mathbf{KR}$ and $\phi_{i3} \in \mathbf{K}$ such that

(1) $\phi_{i1}(\|x_i\|) \le V_i(x_i,t) \le \phi_{i2}(\|x_i\|)$ for all $(x_i,t) \in \mathbb{R}^{n_i} \times T$;

(2) For all $(x_i,t) \in \mathbb{R}^{n_i} \times T$ if $\nabla V_i(x_i,t) \cdot g(x_i,t) = 0$, then

$$\nabla V_i(x_i,t) \cdot f_i(x_i,t) \le \sigma_i \phi_{i3}(\|x_i\|)$$

for some $\sigma_i \in \mathbb{R}$.

Then, we say that the i–th subsystem has *a controllable Lyapunov function* $V_i(x_i,t)$.

The first condition of Definition 1 implies that $V_i(x_i,t) = 0$ iff $x_i = 0$. In Definition 1, we do not require that σ_i is a negative constant. If the i–th free subsystem is globally asymptotically stable, then it has a Lyapunov function $V_i(x_i,t)$ with $\sigma_i < 0$.

Theorem 1. *If the interconnected system (1) satisfies the conditions that*

(1) *Every subsystem has a controllable Lyapunov function* $V_i(x_i,t)$;

(2) *For every* $i \in \boldsymbol{\nu}$, *there exists* $a_{ij} \in \mathbb{R}$, *for all* $j \in \boldsymbol{\nu}$ *such that*

$$\nabla V_i(x_i,t) \cdot \overline{f}_i(x,t) \le [\phi_{i3}(\|x_i\|)]^{\frac{1}{2}} \Big(\sum_{j=1}^{\nu} a_{ij}[\phi_{j3}(\|x_j\|)]^{\frac{1}{2}}\Big)$$

(3) *There exists a vector* $\boldsymbol{\alpha}^T = (\alpha_1, \ldots, \alpha_\nu)$ *with* $\alpha_i > 0$ *for all* $i \in \boldsymbol{\nu}$ *such that the matrix* $S = (s_{ij})$ *specified by*

$$s_{ij} = \begin{cases} \alpha_i(\sigma_i + a_{ii}) & \text{for all } i = j \text{ and } i,j \in \boldsymbol{\nu} \\ \dfrac{1}{2}(\alpha_i a_{ij} + \alpha_j a_{ji}) & \text{for all } i \ne j \text{ and } i,j \in \boldsymbol{\nu} \end{cases}$$

is negative definite.

Then, there exist decentralized almost smooth feedbacks (4) such that the interconnected system (5) is globally asymptotically stable.

For the sake of simplicity of notations, Theorem 1 is only verified for the case of $\nu = 2$. Indeed, the general case can be proved in the same way.

Proof of Theorem 1 (for the case of $\nu = 2$):
Consider the following system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,t) + \overline{f}_1(x_1,x_2,t) + \sum_{j=1}^{m_1} g_{1j}(x_1,t)u_{1j} \\ \dot{x}_2 &= f_2(x_2,t) + \overline{f}_2(x_1,x_2,t) + \sum_{j=1}^{m_2} g_{2j}(x_2,t)u_{2j} \end{aligned} \tag{7}$$

Unlike (1), $g_i(x_i,t)u_i$ is now written in the form of $\sum_{j=1}^{\nu} g_{ij}(x_i,t)u_{ij}$. Let us introduce the following notations:

$$\begin{aligned} a_i(x_i,t) &= \nabla V_i(x_i,t) \cdot f_i(x_i,t) && \text{for all } i \in \boldsymbol{\nu} \\ b_{ij}(x_i,t) &= \nabla V_i(x_i,t) \cdot g_{ij}(x_i,t) && \text{for all } i,j \in \boldsymbol{\nu} \\ \beta_i(x_i,t) &= \sum_{j=1}^{\nu} b_{ij}^2(x_i,t) && \text{for all } i \in \boldsymbol{\nu} \end{aligned} \tag{8}$$

From the hypotheses about V_i, f_i and g_{ij}, it is easy to see that a_i, b_{ij} and β_i are almost smooth functions. By condition (2) of Definition 1, we have $a_i(x_i,t) \leq \sigma_i\phi_{i3}(\|x_i\|)$ for all $(x_i,t) \in \mathbb{R}^{n_i} \times T$ such that $\beta_i(x_i,t) = 0$.

We write feedbacks (4) in the component form, i.e. $\psi_i(x_i,t) = [\psi_{i1}(x_i,t),...,\psi_{im_i}(x_i,t)]^T$. Define $u_{ij} = \psi_{ij}(x_i,t)$ as follows:

$$\psi_{ij}(x_i,t) = \begin{cases} -b_{ij}(x_i,t)\dfrac{c_i(x_i,t) + \sqrt{c_i^2(x_i,t) + \beta_i^4(x_i,t)}}{\beta_i(x_i,t)} & \text{if } \beta_i(x_i,t) \neq 0 \\ 0 & \text{if } \beta_i(x_i,t) = 0 \end{cases} \tag{9}$$

where $c_i(x_i,t) = a_i(x_i,t) - \sigma_i\phi_{i3}(\|x_i\|)$. As $a_i(x_i,t)$, $b_{ij}(x_i,t)$ and $\beta_i(x_i,t)$ are almost smooth, so is $\psi_{ij}(x_i,t)$. Moreover, $\psi_{ij}(0,t) = 0$ as $\beta_i(0,t) = 0$. Hence, $u_{ij} = \psi_{ij}(x_i,t)$ is an almost smooth feedback.

Now consider the closed loop system described as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,t) + \sum_{j=1}^{m_1} g_{1j}(x_1,t)\psi_{1j}(x_1,t) + \overline{f}_1(x_1,x_2,t) \\ \dot{x}_2 &= f_2(x_2,t) + \sum_{j=1}^{m_2} g_{2j}(x_2,t)\psi_{2j}(x_2,t) + \overline{f}_2(x_1,x_2,t) \end{aligned} \tag{10}$$

Equations (10) may be regarded as a system interconnected by the free subsystems

$$\dot{x}_i = f_i(x_i,t) + \sum_{j=1}^{m_i} g_{ij}(x_i,t)\psi_{ij}(x_i,t) \qquad \text{for} \quad i = 1,2$$

with the coupling functions $\overline{f}_i(x_1,x_2,t)$ for $i = 1,2$.

We now check that the conditions of Lemma 1 are satisfied for (10). By Definition 1, the first condition of Lemma 1 is obviously satisfied. For the second condition of Lemma 1, we first consider the case of $\beta_i(x_i,t) \neq 0$, it yields that

$$\begin{aligned} \nabla V_i(x_i,t) \cdot \Big[f_i(x_i,t) + \sum_{j=1}^{m_i} g_{ij}(x_i,t)\psi_{ij}(x_i,t)\Big] &= a_i(x_i,t) + \sum_{j=1}^{m_i} b_{ij}(x_i,t)\psi_{ij}(x_i,t) \\ \leq \sigma_i\phi_{i3}(\|x_i\|) - \sqrt{c_i^2(x_i,t) + \beta_i^4(x_i,t)} &\leq \sigma_i\phi_{i3}(\|x_i\|) \end{aligned} \tag{11a}$$

For the case of $\beta_i(x_i,t) = 0$, we have

$$\begin{aligned} \nabla V_i(x_i,t) \cdot \Big[f_i(x_i,t) \quad + \sum_{j=1}^{m_i} g_{ij}(x_i,t)\psi_{ij}(x_i,t)\Big] &= \\ \leq \nabla V_i(x_i,t) \cdot f_i(x_i,t) &\leq \sigma_i\phi_{i3}(\|x_i\|) \end{aligned} \tag{11b}$$

Inequalities (11a) and (11b) imply the second condition of Lemma 1. From the hypothesis of Theorem 1 the last two conditions of Lemma 1 are obviously satisfied. Hence, (10) is globally asymptotically stable. ■

If we substitute conditions (3′) and (4′) of Remark 1 for conditions (3) and (4) given in Lemma 1, then we can establish the following corollary.

Corollary 1. *If the interconnected system (1) satisfies the conditions that*

(1) *Every subsystem has a controllable Lyapunov function* $V_i(x_i, t)$;

(2) *For every* $i \in \boldsymbol{\nu}$, *there exist almost smooth functions* $a_{ij}(x,t)$, $j \in \boldsymbol{\nu}$, *such that for all* $(x,t) \in \mathbb{R}^n \times T$

$$\nabla V_i(x_i,t) \cdot \overline{f}_i(x,t) \le [\phi_{i3}(\|x_i\|)]^{\frac{1}{2}} \Big(\sum_{j=1}^{\nu} a_{ij}(x,t)[\phi_{j3}(\|x_j\|)]^{\frac{1}{2}}\Big)$$

(3) *There exists a vector* $\boldsymbol{\alpha}^T = (a_1, ..., \alpha_\nu)$ *with* $\alpha_i > 0$ *and a constant* $\varepsilon > 0$ *such that the matrix* $S(x,t) + \varepsilon I$ *is negative definite where* $S(x,t) = (s_{ij}(x,t))$ *and*

$$s_{ij} = \begin{cases} \alpha_i[\sigma_i + a_{ii}(x,t)] & \text{for all } i = j \text{ and } i,j \in \boldsymbol{\nu} \\ \frac{1}{2}[\alpha_i a_{ij}(x,t) + \alpha_j a_{ji}(x,t)] & \text{for all } i \ne j \text{ and } i,j \in \boldsymbol{\nu} \end{cases}$$

Then, there exist decentralized almost smooth feedbacks (4) such that the interconnected system (5) is globally asymptotically stable.

The proof of Corollary 1 is similar to that of Theorem 1 and omitted.

In the proof of the above theorems, the feedbacks are only used to improve the behavior of subsystems. Indeed, these feedbacks can be applied to improve the behavior of both subsystems and the interconnections. The further result is stated as Theorem 2. But before stating Theorem 2, we need a definition.

Definition 2. Given an almost smooth function $b(x,t)$, $a(x,t)$ is said to be *zero equivalent* to $b(x,t)$ if the limit $\lim_{x \to x_0} \frac{a(x,t)}{b(x,t)}$ exists for every x_0 such that $b(x_0,t) \equiv 0$ for all $t \in T$.

Theorem 2. *If the interconnected system (1) satisfies the conditions that*

(1) *Every subsystem has a controllable Lyapunov function* $V_i(x_i, t)$;

(2) *For every* $i \in \boldsymbol{\nu}$, *there exists* $a_{ij} \in \mathbb{R}$, $j \in \boldsymbol{\nu}$, *and an almost smooth function* $\gamma_i(x_i, t)$ *which is zero equivalent to* $\beta_i(x_i, t)$ *such that*

$$\nabla V_i(x_i,t) \cdot \overline{f}_i(x,t) \le [\phi_{i3}(\|x_i\|)]^{\frac{1}{2}} \Big(\sum_{j=1}^{\nu} a_{ij}[\phi_{j3}(\|x_j\|)]^{\frac{1}{2}}\Big) + \gamma_i(x_i,t)$$

(3) *There exists a vector* $\boldsymbol{\alpha}^T = (a_1, ..., \alpha_\nu)$ *with* $\alpha_i > 0$ *for all* $i \in \boldsymbol{\nu}$ *such that the matrix* $S = (s_{ij})$ *specified by*

$$s_{ij} = \begin{cases} \alpha_i(\sigma_i + a_{ii}) & \text{for all } i = j \text{ and } i,j \in \boldsymbol{\nu} \\ \frac{1}{2}(\alpha_i a_{ij} + \alpha_j a_{ji}) & \text{for all } i \ne j \text{ and } i,j \in \boldsymbol{\nu} \end{cases}$$

is negative definite.

Then, there exist decentralized almost smooth feedbacks (4) such that the interconnected system (5) is globally asymptotically stable.

Proof. For every $i \in \boldsymbol{\nu}$, we denote $\psi_i = [\psi_{i1}, ..., \psi_{im_i}]^T$. Let every component ψ_{ij} take the form of

$$u_{ij} = \psi_{ij}(x_i,t) = \psi_{ij}^{(1)}(x_i,t) + \psi_{ij}^{(2)}(x_i,t)$$

where $\psi_{ij}^{(1)}(x_i,t)$ is the same as (8), and

$$\psi_{ij}^{(2)}(x_i,t) = \begin{cases} -b_{ij}(x_i,t)\dfrac{\gamma_i(x_i,t)}{\beta_i(x_i,t)} & \text{if} \quad \beta_i(x_i,t) \neq 0 \\ 0 & \text{if} \quad \beta_i(x_i,t) = 0 \end{cases}$$

From the above equation, $\psi_{ij}^{(2)}(0,t) = 0$ for all $t \in T$ and from condition (2) of Theorem 2, $\psi_{ij}^{(2)}(x_i,t)$ is an almost smooth function, so is $\psi_{ij}(x_i,t)$.

For all $i \in \boldsymbol{\nu}$, we regard $\dot{x}_i = f_i(x_i,t) + \sum_{j=1}^{m_i} g_{ij}(x_i,t)\psi_{ij}^{(1)}(x_i,t)$ as the i–th subsystem and regard $\overline{f}_i(x_i,t) + \sum_{j=1}^{m_i} g_{ij}(x_i,t)\psi_{ij}^{(2)}(x_i,t)$ as the coupling functions. Repeating the same calculations as in (11), we have

$$\nabla V_i(x_i,t)\left[f_i(x_i,t) + \sum_{j=1}^{m_i} g_{ij}(x_i,t)\psi_{ij}^{(1)}(x_i,t)\right] \le \sigma_i\phi_{i3}(\|x_i\|) \tag{12}$$

We turn to consider the coupling functions. If $\beta_i(x_i,t) \neq 0$, then

$$\begin{aligned} &\nabla V_i(x_i,t)\left[\overline{f}_i(x,t) + \sum_{j=1}^{m_i} g_{ij}(x_i,t)\psi_{ij}^{(2)}(x_i,t)\right] \\ &= \nabla V_i(x_i,t)\overline{f}_i(x,t) - \sum_{j=1}^{m_i} b_{ij}^2(x_i,t)\frac{\gamma_i(x_i,t)}{\beta_i(x_i,t)} \\ &= \nabla V_i(x_i,t)\overline{f}_i(x,t) - \gamma_i(x_i,t) \le [\phi_{i3}(\|x_i\|)]^{\frac{1}{2}}\left(\sum_{j=1}^{\nu} a_{ij}[\phi_{j3}(\|x_j\|)]^{\frac{1}{2}}\right) \end{aligned} \tag{13a}$$

If $\beta_i(x_i,t) = 0$, then the zero equivalence of $\gamma_i(x_i,t)$ and $\beta_i(x_i,t)$ leads to

$$\begin{aligned} &\nabla V_i(x_i,t)\left[\overline{f}_i(x_i,t) + \sum_{j=1}^{m_i} g_{ij}(x_i,t)\psi_{ij}^{(2)}(x_i,t)\right] = \nabla V_i(x_i,t)\overline{f}_i(x_i,t) \\ &\le [\phi_{i3}(\|x_i\|)]^{\frac{1}{2}}\left(\sum_{j=1}^{\nu} a_{ij}[\phi_{j3}(\|x_j\|)]^{\frac{1}{2}}\right) \end{aligned} \tag{13b}$$

By the conditions of Theorem 2, along with (12) and (13), we get that the closed–loop system is globally asymptotically stable by using Lemma 1. ■

To close this section, we state a result corresponding to Corollary 1. Its proof is also omitted.

Corollary 2. *If the interconnected system (1) satisfies the conditions that*

(1) *Every subsystem has a controllable Lyapunov function* $V_i(x_i,t)$;

(2) *For every* $i \in \boldsymbol{\nu}$, *there exist almost smooth functions* $a_{ij}(x,t)$, $j \in \boldsymbol{\nu}$, *and* $\gamma_i(x_i,t)$ *which is zero equivalent to* $\beta_i(x_i,t)$ *such that for all* $(x,t) \in \mathbb{R}^n \times T$

$$\nabla V_i(x_i,t) \cdot \overline{f}_i(x,t) \le [\phi_{i3}(\|x_i\|)]^{\frac{1}{2}} \Big(\sum_{j=1}^{\nu} a_{ij}(x,t)[\phi_{j3}(\|x_j\|)]^{\frac{1}{2}}\Big) + \gamma_i(x_i,t)$$

(3) *There exists a vector* $\boldsymbol{\alpha}^T = (a_1,\dots,\alpha_\nu)$ *with* $\alpha_i > 0$ *and a constant* $\varepsilon > 0$ *such that the matrix* $S(x,t) + \varepsilon I$ *is negative definite where* $S(x,t) = (s_{ij}(x,t))$ *and*

$$s_{ij} = \begin{cases} \alpha_i[\sigma_i + a_{ii}(x,t)] & \text{for all } \ i = j \ \text{ and } \ i,j \in \boldsymbol{\nu} \\ \dfrac{1}{2}[\alpha_i a_{ij}(x,t) + \alpha_j a_{ji}(x,t)] & \text{for all } \ i \ne j \ \text{ and } \ i,j \in \boldsymbol{\nu} \end{cases}$$

Then, there exist decentralized almost smooth feedbacks (4) such that the interconnected system (5) is globally asymptotically stable.

4. Decentralized Stabilization for Lower Triangularly Interconnected Systems

This section turns to study the problem of decentralized stabilization for the lower triangularly interconnected systems (3). The designing process is quite similar to what we do in the last section, i.e. we design decentralized almost smooth feedback for every subsystem (2) such that the closed loop system (6) satisfies all of the conditions of Lemma 2. Then, its stability is implied.

Theorem 3. *If the interconnected system (3) satisfies the conditions that*

(1) *Every subsystem has a controllable Lyapunov function* $V_i(x_i,t)$ with $\sigma_i < 0$;

(2) $\lim_{\|x_i\| \to \infty} \|\nabla V_i(x,t)/\phi_{i3}(\|x_i\|)\| = 0$ for all $i \in \boldsymbol{\nu}$;

(3) *For every* $0 < j < i$ *and for every* $i \in \boldsymbol{\nu}$ *there exists a function* $\varphi_{ij} \in \mathbf{K}$ *such that* $\|\overline{f}_{ij}(x_j,t)\| \le \varphi_{ij}(\|x_j\|)$.

Then, there exist decentralized almost smooth feedbacks (4) such that the interconnected system (6) is globally asymptotically stable.

Proof. We rewrite (3) into the following form

$$\dot{x}_i = f_i(x_i,t) + \sum_{j=1}^{i-1} \overline{f}_{ij}(x_j,t) + \sum_{j=1}^{m_i} g_{ij}(x_i t)u_{ij} \tag{14}$$

Using the Lyapunov function $V_i(x_i,t)$, we construct feedbacks $u_{ij} = \psi_{ij}(x_i,t)$ as follows:

$$\psi_{ij}(x_i,t) = \begin{cases} -b_{ij}(x_i,t)\dfrac{c_i(x_i,t) + \sqrt{c_i^2(x_i,t) + \beta_i^4(x_i,t)}}{\beta_i(x_i,t)} & \text{if } \ \beta_i(x_i,t) \ne 0 \\ 0 & \text{if } \ \beta_i(x_i,t) = 0 \end{cases} \tag{15}$$

where $a_i(x_i,t)$, $\beta_i(x_i,t)$ and $c_i(x_i,t)$ are the same functions as those defined in thc proof of Theorem 1. Similarly, we can prove

$$\nabla V_i(x_i,t)\cdot\Big[f_i(x_i,t)+\sum_{j=1}^{m_i} g_{ij}(x_i,t)\psi_{ij}(x_i,t)\Big]\le \sigma_i\phi_{i3}(\|x_i\|) \tag{16}$$

Since $\sigma_i<0$, (16) together with conditions (2) and (3) of Theorem 3, the globally asymptotical stability is implied from Lemma 2. ■

Remark 3. The first condition of Theorem 3 is implies that every subsystem is asymptotically stable.

It is obvious that the conditions of Theorem 3 are much weaker than those of Theorem 1 and Theorem 2. If $\nabla V_i(x_i,t)\cdot g(x_i,t)\neq 0$ for all $(x_i,t)\in \mathbb{R}^{n_i}\times T$ and for all $i\in\boldsymbol{\nu}$, then we can further weaken the stabilizing conditions.

Theorem 4. *If every subsystem of the interconnected system (3) has a continuous differential function* $V_i(x_i,t):\mathbb{R}^{n_i}\times T\to\mathbb{R}^+$ *and functions* $\phi_{i1},\ \phi_{i2}\in\mathbf{KR}$ *such that*

(1) $\phi_{i1}(\|x_i\|)\le V_i(x_i,t)\le\phi_{i2}(\|x_i\|)$ *for all* $(x_i,t)\in\mathbb{R}^{n_i}\times T$;

(2) $\|\nabla V_i(x_i,t)\cdot g_i(x_i,t)\|\neq 0$ *for all* $(x_i,t)\in\mathbb{R}^{n_i}\times T\setminus\{0\}$;

(3) *For every* $j\in\underline{i-1}$ *and* $i\in\boldsymbol{\nu}$ *there exists a function* $\varphi_{ij}\in\mathbf{K}$ *such that* $\|\bar{f}_{ij}(x_i,t)\|\le\varphi_{ij}(\|x_i\|)$ *for all* $(x_i,t)\in\mathbb{R}^{n_i}\times T$.

Then, there exist decentralized almost smooth feedbacks (4) such that the interconnected system (6) is globally asymptotically stable.

Proof. Let $\phi_{i3}(r)=\max\limits_{\|x_i\|\le r}\|\nabla V_i(x_i,t)\|\cdot e^r$. Then $\phi_{i3}\in\mathbf{K}$ and $\|\nabla V_i(x_i,t)\|\le\phi_{i3}(\|x_i\|)$. Moreover,

$$\lim_{\|x_i\|\to\infty}\|\nabla V_i(x_i,t)\|/\phi_{i3}(\|x_i\|)\le\lim_{\|x_i\|\to\infty}e^{-\|x_i\|}=0 \tag{17}$$

Let us take the decentralized feedbacks as follows:

$$\psi_{ij}(x_i,t)=-b_{ij}(x_i,t)\frac{a_i(x_i,t)+\phi_{i3}(\|x_i\|)}{\beta_i(x_i,t)}$$

where $a_i(x_i,t)$, $b_{ij}(x_i,t)$ and $\beta_i(x_i,t)$ are the same as those defined in the proof of Theorem 1. Thus,

$$\nabla V_i(x_i,t)\Big[f_i(x_i,t)+\sum_{j=1}^{m_i} g_{ij}(x_i,t)\psi_{ij}(x_i,t)]=-\phi_{i3}(\|x_i\|) \tag{18}$$

From condition (1) together with inequalities (17) and (18), the validness of Theorem 4 is verified. ■

5. An Example

This section will give an example to illustrate the application of Theorem 1. Although only an example whose calculation is quite tedious is presented, the readers can construct the examples for other Theorems by following this way.

Consider the decentralized system described as

$$\dot{x}_1 = \begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} -x_{11}^3 - 5x_{12}^3 \\ x_{11}x_{12}^2 + x_{12}^3 \end{bmatrix} + \begin{bmatrix} x_{11}x_{22} \\ x_{12}x_{21} \end{bmatrix} + \begin{bmatrix} x_{11}^2 - x_{12} \\ x_{11} - x_{12}^2 \end{bmatrix} u_1 \tag{14a}$$

$$\dot{x}_2 = \begin{bmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} x_{21}^2 \\ -x_{22} \end{bmatrix} + \begin{bmatrix} \frac{1}{4}x_{11}^2 \\ \frac{1}{2}x_{12}^2 \end{bmatrix} + \begin{bmatrix} 2x_{21} + x_{22} \\ x_{22} \end{bmatrix} u_2 \tag{14b}$$

Using the notations defined in Equation (1), we denote

$$f_1(x_1) = \begin{bmatrix} -x_{11}^3 - 5x_{12}^3 \\ x_{11}x_{12}^2 + x_{12}^3 \end{bmatrix}, \quad \overline{f}_1(x_1, x_2) = \begin{bmatrix} x_{11}x_{22} \\ x_{12}x_{21} \end{bmatrix}, \quad g_1(x_1) = \begin{bmatrix} x_{11}^2 - x_{12} \\ x_{11} - x_{12}^2 \end{bmatrix}$$

$$f_2(x_2) = \begin{bmatrix} x_{21}^2 \\ -x_{22} \end{bmatrix}, \quad \overline{f}_2(x_1, x_2) = \begin{bmatrix} \frac{1}{4}x_{11}^2 \\ \frac{1}{2}x_{12}^2 \end{bmatrix}, \quad g_2(x_2) = \begin{bmatrix} 2x_{21} + x_{22} \\ x_{22} \end{bmatrix}$$

Let $V_1(x_1) = \frac{1}{2}(x_{11}^2 + x_{12}^2)$ and $V_2(x_2) = x_{21}^2 + (x_{21} - x_{22})^2$. We now use the notations established in the proof of Theorem 1. The functions $b_1(x_1)$ and $b_2(x_2)$ are calculated as follows:

$$b_1(x_1) = \nabla V_1(x_1) g_1(x_1) = x_{11}^3 - x_{12}^3$$

$$b_2(x_2) = \nabla V_2(x_2) g_2(x_2) = 4x_{21}^2$$

Obviously, $b_1(x_1) = 0$ iff $x_{11} = x_{12}$ (note: x_1 is a real vector) and $b_2(x_2) = 0$ iff $x_{21} = 0$. The functions $a_1(x_1)$ and $a_2(x_2)$ are

$$a_1(x_1) = \nabla V_1(x_1) f_1(x_1) = -x_{11}^4 - 4x_{11}x_{12}^3 + x_{22}^4$$

$$a_2(x_2) = \nabla V_2(x_2) f_2(x_2) = 2x_{21}^3 - x_{21}^2 x_{22} - x_{22}^2$$

When $b_1(x_1) = 0$, we have

$$a_1(x_1) = -4x_{11}^4 = -2(x_{11}^4 + x_{12}^4) \leq -(x_{11}^2 + x_{12}^2)^2 = -\|x_1\|^2$$

Similarily, provided that $b_2(x_2) = 0$,

$$a_2(x_2) = -x_{22}^2 = -(x_{21}^2 + x_{22}^2) = -\|x_2\|^2$$

The above discussion shows that $V_1(x_1)$ and $V_2(x_2)$ are exactly the controllable Lyapunov functions with $\sigma_1 = \sigma_2 = -1$, $\varphi_{13}(\|x_1\|) = \|x_1\|^4$ and $\varphi_{23}(\|x_2\|) = \|x_2\|^2$.

The interconnected functions are now estimated by using Cauchy–Schwarz inequality.

$$\begin{aligned} \nabla V_1(x_1)\overline{f}_1(x_1, x_2) &= x_{11}^2 x_{22} + x_{12}^2 x_{21} \\ &\leq \sqrt{x_{11}^4 + x_{12}^4} \cdot \sqrt{x_{21}^2 + x_{22}^2} \leq (x_{11}^2 + x_{12}^2) \cdot \sqrt{x_{21}^2 + x_{22}^2} \\ &= \varphi_{13}^{\frac{1}{2}}(\|x_1\|) \cdot \varphi_{23}^{\frac{1}{2}}(\|x_2\|) \end{aligned} \tag{15a}$$

and

$$\begin{aligned} \nabla V_2(x_2)\overline{f}_2(x_1,x_2) &= \tfrac{1}{2}x_{11}^2x_{21} - \tfrac{1}{4}x_{11}^2x_{22} + \tfrac{1}{2}x_{12}^2x_{22} \\ &\le \sqrt{\tfrac{1}{4}x_{11}^4 + \tfrac{1}{16}x_{11}^4 + \tfrac{1}{4}x_{12}^4} \cdot \sqrt{x_{21}^2 + x_{22}^2 + x_{22}^2} \\ &\le \tfrac{\sqrt{10}}{4}\sqrt{x_{11}^4 + x_{12}^4} \cdot \sqrt{x_{21}^2 + x_{22}^2} \le \tfrac{\sqrt{10}}{4}(x_{11}^2 + x_{12}^2) \cdot \sqrt{x_{21}^2 + x_{22}^2} \\ &= \tfrac{\sqrt{10}}{4}\varphi_{13}^{\frac{1}{2}}(\|x_1\|) \cdot \varphi_{23}^{\frac{1}{2}}(\|x_2\|) \end{aligned} \tag{15b}$$

From (15a) and (15b), we obtain that $a_{11} = a_{22} = 0$, $a_{12} = 1$ and $a_{21} = \frac{\sqrt{10}}{4}$. Taking $\alpha = (1,1)^T$, the discriminating matrix S is

$$S = \begin{bmatrix} -1 & \frac{1}{2}\left(1 + \frac{\sqrt{10}}{4}\right) \\ \frac{1}{2}\left(1 + \frac{\sqrt{10}}{4}\right) & -1 \end{bmatrix}$$

It is easily verified that matrix S is negative finite, hence by Lemma 1, system (14) with the following feedbacks

$$u_1(x_1) = \begin{cases} \frac{2x_{11}^2x_{12}^2 - 4x_{11}x_{12}^3 + 2x_{12}^4 + \sqrt{(2x_{11}^2x_{12}^2 - 4x_{11}x_{12}^3 + 2x_{12}^4)^2 + (x_{11}^3 - x_{12}^3)^4}}{x_{11}^3 - x_{12}^3} & x_{11} \neq x_{12} \\ 0 & x_{11} = x_{12} \end{cases}$$

and

$$u_2(x_2) = \begin{cases} \frac{2x_{21}^3 - x_{21}^2x_{22} + x_{21}^2 + \sqrt{(2x_{21}^3 - x_{21}^2x_{22} + 2x_{21}^2)^2 + (4x_{21}^2)^4}}{4x_{21}^2} & x_{21} \neq 0 \\ 0 & x_{21} = 0 \end{cases}$$

is asymptotically stable. ■

Remark. If we add $\begin{bmatrix} x_{11} - x_{12} \\ x_{12}^2 - x_{11}^2 \end{bmatrix}$ to the right side of Equation (14a) and/or add $\begin{bmatrix} x_{21}^4 \\ x_{21}^2 - x_{22} \end{bmatrix}$ to the right side of Equation (14b), then this system may be stabilized by using decentralized feedback from Theorem 2.

6. Conclusions

In the paper, the problem of decentralized stabilization by using almost smooth feedbacks for two classes of interconnected systems has been studied. The Lyapunov–like conditions for such stabilization were obtained and the decentralized feedbacks were given using these Lyapunov functions. Although only the globally asymptotical stability is considered and only four theorems are verified, the authors are sure that there is no difficulty to extend the designing technique developed in the paper to the decentralized stabilization for other stable requirements such as Lyapunov stability, bounded stability, exponent stability and so on. With the use of such a technique, the most results established for the analysis of stability of interconnected systems may also be applied to design decentralized feedback laws.

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