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MINIMIZATION OF ENERGY LOSSES ON THE TRANSMISSION LINE

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The paper formulates and solves the problem of matching the receiver to the source in the way which minimizes the losses of active power on the transmission line at given active power supplied to the receiver. The existence and uniqueness of the solution have been shown and the convergence of iterative procedure has been proved. Thanks to functional analysis the problem has been solved for a large class of signals and linear systems. A simple example has been presented for sinusoidal signals in the system invariable in time.

1. Introduction

An important problem of the theory of any signal power is minimization of unfavorable effects of electrical energy transmission. It has been of a great interest for many years, but it still cannot be said that it has been solved satisfactorily. Previous works on it (Czarnecki, 1984; 1987; Kusters and Moore, 1980; Page, 1980) did not take into account the losses on the line of the transmission and they were limited to ideal sources. Their aim was to minimize apparent power of the source, and that was achieved by decomposition of the current into orthogonal components, from which only one ensured active power for the receiver. The current was determined by intuition, and it did not result from mathematical analysis of a given physical situation.

In subsequent works (Pasko, 1991; Siwczyński and Kłosiński, 1991; Walczak, 1991) the losses in the source due to adding inner impedance have been gradually considered. Variation methods applied there proved the uselessness of orthogonal current and reactive power notions connected with it. In these works unfavorable effects of energy transmission were determined in a certain square functional, which contained averaging current and its derivatives with weights (Walczak, 1991). Such a functional did not show properly the losses of active power on the transmission line, because it did not contain the receiver voltage and possessed undeterminate weights. There was no criterion of selecting these weights.

The present paper tries to determine the functional of the losses on the transmission line. The problem of the conditional minimization of the functional has been formulated and also possibility of the unique solution has been proved. Some iterative procedures of solving the minimization equations have been given and their convergence has been proved. The results obtained enable us to determine the signal

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of the receiver current called an *optimum current*, at which the losses in a two–port network modeling a transmission system are minimal. It is necessary to know this current in order to give the conditions of matching the electrical energy receivers, the source and the transmission line. A pair–optimum current and optimum voltage of the receiver–makes the selection of connected optimizing circuit possible.

A functional analysis applied here enables us to consider large classes of time signals and continuous or discrete frequency signals. The problem,which is formulated and solved here, may be applied not only for electroenergetic systems, but also for load optimization of high frequency systems in filters, wave–guides and parametric amplifiers.

2. Problem Formulation

A system under consideration consists of a signal source with voltage e_1 and with a positive definite linear operator of inner impedance Z_1, which transforms the signal space into itself. The transmission system is presented in the form of a two–port network described by a matrix of chain linear operators

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

The optimization problem consists in selecting the current so as to minimize total losses of active power inside the two–port network modeling the line at a given stream of active power P supplied to the receiver.

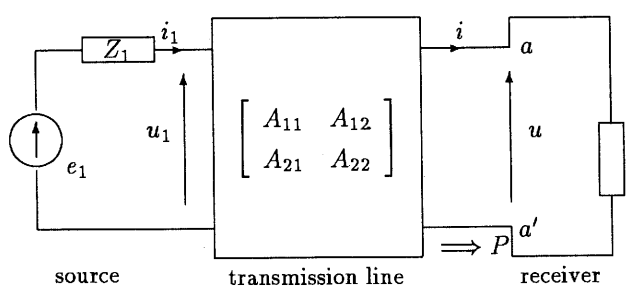

Fig. 1. Model of the source and the transmission line.

Active power losses inside the two–port network are defined by scalar products:

$$\Delta P = (u_1, i_1) - (u, i) \tag{1}$$

Taking into account that

$$\begin{aligned} u_1 &= A_{11}u + A_{12}i \\ i_1 &= A_{21}u + A_{22}i \end{aligned} \tag{2}$$

we obtain

$$\Delta P = (A_{11}u, A_{21}u) + (A_{12}i, A_{22}i) + (A_{11}u, A_{22}i) + (A_{21}u, A_{12}i) - (u, i) \tag{3}$$

After rearranging, formula (3) has the following form:

$$\Delta P = (A_{22}^* A_{12} i, i) + (A_{21}^* A_{11} u, u) + \left([A_{22}^* A_{11} + A_{12}^* A_{21} - \mathbb{1}]u, i\right) \tag{4}$$

where adjoint operators are marked with an asterisk and $\mathbb{1}$ is the identity operator.

Between the voltage signal u and the current signal i there is a relation

$$u = e - Zi \tag{5}$$

where e is a voltage signal on the open terminals $a - a'$, Z – an operator of impedance seen from the terminals $a - a'$ at the closed terminals source of the signal e_1. It can be shown that

$$e = (A_{11} + Z_1 A_{21})^{-1} e_1 \tag{6}$$

$$Z = (A_{11} + Z_1 A_{21})^{-1}(A_{12} + Z_1 A_{22}) \tag{7}$$

In formulae (6), (7) and later the symbol $(\cdot)^{-1}$ denotes the inverse operator. Moreover, multiplication should be done in sequence, because the operators may not commute in a general case (with respect to multiplication). In a particular case when Z_1 is a zero operator (the source on the left to the transmission line is ideal), we have

$$e = A_{11}^{-1} e_1 \tag{8}$$

$$Z = A_{11}^{-1} A_{12} \tag{9}$$

Formula (4) defines the functional $i \longrightarrow \Delta P(i)$, i.e. value of total losses inside the two-port network of the transmission system is assigned to each current signal i. The functional has the form:

$$\Delta P(i) = \frac{1}{2}(L_I i, i) + \frac{1}{2}(L_U u, u) + (L_{UI} u, i) \tag{10}$$

where L_I, L_U, L_{UI} are the operators described by the following formulae

$$L_I = A_{22}^* A_{12} + A_{12}^* A_{22} \tag{11}$$

$$L_U = A_{21}^* A_{11} + A_{11}^* A_{21} \tag{12}$$

$$L_{UI} = A_{22}^* A_{11} + A_{12}^* A_{21} - \mathbb{1} \tag{13}$$

Operators L_I, L_U are self–adjoint, i.e. $L_I = L_I^*$ and $L_U = L_U^*$. Minimization problem is as follows

$$\left[\frac{1}{2}(L_I i, i) + \frac{1}{2}(L_U u, u) + (L_{UI} u, i)\right] \longrightarrow \min \tag{14}$$

subject to

$$P - (u, i) \equiv P - (e - Zi, i) = 0 \tag{15}$$

This problem is solved by means of the Lagrange–multiplier method. The Lagrangian functional takes the form

$$\Phi(i,\lambda) = \frac{1}{2}(L_I i, i) + \frac{1}{2}(L_U u, u) + (L_{UI} u, i) + \lambda\big[P - (u,i)\big] \tag{16}$$

Eliminating the voltage u in formula (16) by using expression (5) we obtain

$$\Phi(i,\lambda) = \left(\left[\frac{1}{2}L_I + \frac{1}{2}Z^* L_U Z - L_{UI} Z + \lambda Z\right] i, i\right) - \left(\left[Z^* L_U - L_{UI} + \lambda \mathbb{1}\right] e, i\right) + \lambda P \tag{17}$$

The Frēchet differential of functional (17) has the form

$$\begin{aligned}\delta\Phi(i,\lambda) &= \Phi(i+\delta i,\lambda) - \Phi(i,\lambda) \\ &= \Big([K+K^*]i, \delta i\Big) - (g,\delta i) + \frac{1}{2}\Big([K+K^*]\delta i, \delta i\Big)\end{aligned} \tag{18}$$

where

$$K = \frac{1}{2}L_I + \frac{1}{2}Z^* L_U Z - L_{UI} Z + \lambda Z \tag{19}$$

$$g = (Z^* L_U - L_{UI} + \lambda \mathbb{1})e \tag{20}$$

From the condition of gradient vanishing and positive quadratic form in expression (18) the following necessary and sufficient conditions for a minimum result follow:

$$(A_1 + \lambda A)i = (A_2 + \lambda \mathbb{1})e \tag{21}$$

$$(A_1 + \lambda A) \geq 0 \tag{22}$$

where $A \geq 0$ denotes a positive–definite operator, i.e. $(Ax, x) > 0$ for any acceptable signal x, and

$$A = Z + Z^* \tag{23}$$

$$A_1 = L_I + Z^* L_U Z - L_{UI} Z - Z^* L_{UI}^* \tag{24}$$

$$A_2 = Z^* L_U - L_{UI} \tag{25}$$

From the condition of passivity of the operator Z_1 and the two–port network modeling the line it results that the operators A and A_1 are positive–definite: $A \geq 0$, $A_1 \geq 0$, and from the definition it results that they are self–adjoint. Thus, for $A \geq 0$ condition (22) is always fulfilled.

Equation (15) can be rearranged to the form:

$$(e,i) - \frac{1}{2}(Ai, i) = P \tag{26}$$

The process of calculating the Lagrange multiplier λ and optimum current is carried out in the following way: from operator equation (21) a family of signals $i(\lambda)$ is determined; putting it into equation (26) an algebraic nonlinear equation is obtained, from which a factor λ is determined. It should be checked whether condition (22) is fulfilled. It is illustrated in Figure 2.

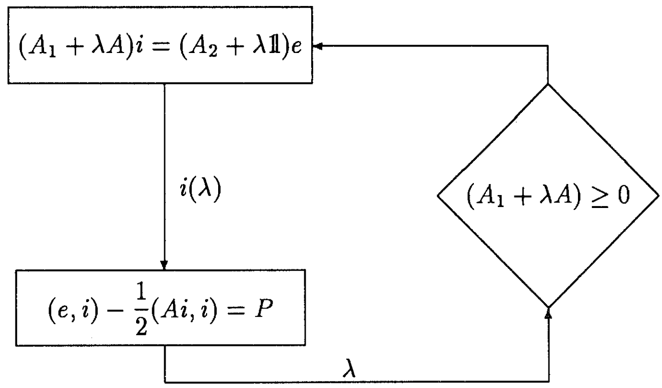

Fig. 2. Loop of the solution of minimization equations.

3. Existence and Uniqueness of Solutions

Solution of the optimization problem involves operator equation (21) and scalar equation (26). The latter can be rewritten as

$$F(\lambda) \equiv \big(e, i(\lambda)\big) - \frac{1}{2}\big(Ai(\lambda), i(\lambda)\big) = P \tag{27}$$

To prove the existence and uniqueness of the solution of the minimization problem the function F is differentiated with respect to λ

$$F'(\lambda) = (e, i^\lambda) - (Ai, i^\lambda) = (e - Ai, i^\lambda) \tag{28}$$

where i^λ is a signal of the so-called *functional derivative* of the current signal with respect to λ, i.e.

$$i^\lambda(t, \lambda) = \lim_{\delta\lambda \to 0} \frac{i(t, \lambda + \delta\lambda) - i(t, \lambda)}{\delta\lambda}$$

This derivative can be determined by differentiating operator equation (21) with respect to λ

$$(A_1 + \lambda A)i^\lambda = e - Ai \tag{29}$$

The operator equation obtained in this way enables us to determine the required signal of the derivative i^λ. It is worth noting that (29) and (21) have the same operator on their left-hand sides. From expressions (28) and (29) it follows that

$$F'(\lambda) = \big((A_1 + \lambda A)i^\lambda, i^\lambda\big) \tag{30}$$

Because the operators A_1, A are positive-definite, then the quadratic form (30) is also positive-definite for each $\lambda \geq 0$. Thus, the function $F(\lambda)$ is strictly increasing for $\lambda \geq 0$. For $\lambda \longrightarrow 0$ operator equation (21) takes the form $Ai = e$ so that $i = A^{-1}e$. Then

$$\big[F(\lambda)\big]_{\lambda \to \infty} = \frac{1}{2}(A^{-1}e, e) = P_{\max} \tag{31}$$

Formula (31) describes the maximum efficiency of the source active power.

On the basis of the considerations above the following theorem can be formulated.

Theorem 1. *For positive–defined operators A, A_1 there is always the same $P < P_{\max} = \frac{1}{2}(A^{-1}e, e)$ for which equation (26) has the only positive solution. Thus, condition (22) is fulfilled. So the minimization problem (14)–(15) has a unique solution.*

A practical solution of equation (26) requires an iterative process. Newton's process is assumed here as a model:

$$\lambda_{k+1} = \Gamma(\lambda_k) \qquad k = 0, 1, 2, \dots \tag{32}$$

where

$$\Gamma(\lambda) = \lambda + \frac{P - F(\lambda)}{F'(\lambda)} \tag{33}$$

Is this process convergent to λ_* such that $Ff(\lambda_*) - P = 0$? To check the procedure convergence it is necessary to determine the function derivative

$$\Gamma'(\lambda) = \left[F(\lambda) - P\right] \frac{F''(\lambda)}{\left[F'(\lambda)\right]^2} \tag{34}$$

By differentiating expression (28) again we obtain

$$F''(\lambda) = -(Ai^\lambda, i^\lambda) + (e - Ai, i^{\lambda\lambda}) = -(Ai^\lambda, i^\lambda) + (Bi^\lambda, i^{\lambda\lambda}) \tag{35}$$

where the operator $B = A_1 + \lambda A$ is self–adjoint. The second functional derivative $i^{\lambda\lambda}$ can be determined by another differentiation of operator equation (29). Then we obtain thé following operator equation:

$$(A_1 + \lambda A)i^{\lambda\lambda} = -2Ai^\lambda \tag{36}$$

Because for $\lambda > 0$ there exists an operator which solves equation (36), then expression (35) takes the form

$$F''(\lambda) = -(Ai^\lambda, i^\lambda) - 2(Bi^\lambda, B^{-1}Ai^\lambda) = -3(Ai^\lambda, i^\lambda) < 0 \tag{37}$$

for any $\lambda > 0$. From formula (34) it results that

$$\Gamma'(\lambda) > 0 \qquad \text{for} \quad \lambda < \lambda_*$$

$$\Gamma'(\lambda) > 0 \qquad \text{for} \quad \lambda > \lambda_*$$

Moreover, from formula (33) on the basis of the property of the functions $F(\lambda)$ and $F'(\lambda)$ it results that

$$\left[\Gamma(\lambda)\right]_{\lambda \downarrow 0} > 0$$

Thus, we obtain the following theorem:

Theorem 2. *There exists a* $\lambda_{**} > \lambda_*$ *such that for* $\lambda_0 \in [0, \lambda_{**}]$ *Newton's process is monotonically convergent to the solution of equation (26).*

4. Example

To avoid numerical methods the results have been illustrated by a very simple example of the time invariant system with sinusoidal forcing. The diagram of the system being considered is shown in Figure 3.

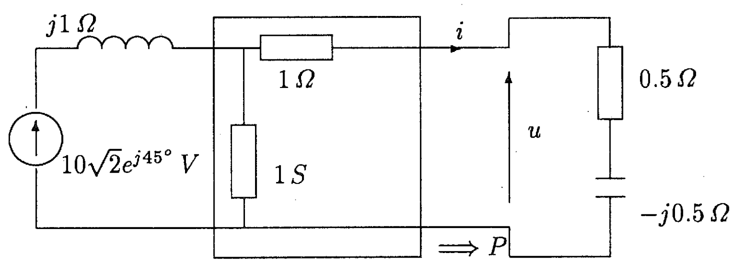

Fig. 3. Detailed diagram of the source and the transmission line.

In the case under consideration all the operators are represented by appropriate suitable complex numbers, while the signals by complex values. Then, equation (21) has a direct solution

$$I = \frac{A_2 + \lambda}{A_1 + \lambda A} E \tag{38}$$

while equation (26) takes the form

$$\mathrm{Re}\left[\frac{A_2^* + \lambda}{A_1 + \lambda A}|E|^2\right] - \frac{1}{2}A\frac{(A_2 + \lambda)(A_2^* + \lambda)}{(A_1 + \lambda A)^2}|E|^2 - P = 0$$

from which we obtain a quadratic equation for λ

$$\frac{1}{2}AD\lambda^2 + A_1 D\lambda + A_1 \mathrm{Re} A_2 - \frac{1}{2}A|A_2|^2 - \frac{P}{|E|^2}A_1^2 = 0 \tag{39}$$

where

$$D = 1 - 2A\frac{P}{|E|^2} \tag{40}$$

If $P < \frac{1}{2}A^{-1}|E|^2 = P_{\max}$, then $D > 0$. A discriminant of quadratic equation (39) is as follows

$$\Delta = D\left[(A_1 - A|A_2|)^2 + 4AA_1|A_2|\left(\sin\frac{1}{2} \sphericalangle A_2\right)^2\right]$$

It is positive, hence equation (39) has a real solution. The positive solution is of the form

$$\lambda_* = \frac{\sqrt{\Delta} - A_1 D}{AD} \tag{41}$$

An optimal current is described by the formula

$$I^{\mathrm{opt}} = \frac{A_2 + \lambda_*}{A_1 + \lambda_* A} E \tag{42}$$

and an optimum voltage by

$$U^{\mathrm{opt}} \doteq E - Z I^{\mathrm{opt}} = \frac{A_1 + \lambda_* Z^* - A_2 Z}{A + \lambda A} E \tag{43}$$

After applying numerical data of the system from Figure 3 we obtain

$$\begin{array}{lll} Z = 1.5 + j0.5\,\Omega & L_I = 4\,\Omega & A = 3\,\Omega \\ E = 10\,V & L_U = 2\,S & A_1 = 3\,\Omega \\ & L_{UI} = 2 & A_2 = 1 - j1 \end{array}$$

The current of unmatching receiver of the circuit shown in Figure 3 is $I = 5\,A$. The active power flux is $P = 12.5\,W$. The solution of equation (39) is $\lambda_* = 1$.
The optimum current and the optimum voltage are, respectively,

$$I^{\mathrm{opt}} = 3.3 - j1.7\,A, \qquad U^{\mathrm{opt}} = 4.2 + j0.8\,V$$

By applying formula (10) we can calculate that the losses on the transmission line before optimization are $87.5\,W$, and after optimization $70.84\,W$. Maximum reduction of losses is about 19%.

The optimum current can be achieved by means of a special matching circuit. It is another problem which is not considered in the present paper.

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