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MONOMIAL SUBDIGRAPHS OF REACHABLE AND CONTROLLABLE POSITIVE DISCRETE–TIME SYSTEMS

RAFAEL BRU*, LOUIS CACCETTA**, VENTSI G. RUMCHEV **

* Departamento de Matemàtica Aplicada
Universidad Politècnica de València
Camí de Vera s/n, 46071 València, Spain
e-mail: rbru@mat.upv.es

** Department of Mathematics and Statistics
Curtin University of Technology
P.O. Box U 1987, Perth, WA 6845, Australia
e-mail: {caccetta, rumchevv}@maths.curtin.edu.au

A generic structure of reachable and controllable positive linear systems is given in terms of some characteristic components (monomial subdigraphs) of the digraph of a non-negative a pair. The properties of monomial subdigraphs are examined and used to derive reachability and controllability criteria in a digraph form for the general case when the system matrix A may contain zero columns. The graph-theoretic nature of these criteria makes them computationally more efficient than their known equivalents. The criteria identify not only the reachability and controllability properties of positive linear systems, but also their reachable and controllable parts (subsystems) when the system does not possess such properties.

Keywords: positive linear systems, reachability, controllability, system structure, monomial subdigraphs

1. Introduction

Positive discrete-time linear control systems are described by the equation

$$x(t+1) = Ax(t) + Bu(t), \quad t = 0, 1, 2, \ldots, \tag{1}$$

where $A = [a_{ij}] \in \mathbb{R}_+^{n\times n}$, $B = [b_{ij}] \in \mathbb{R}_+^{n\times m}$, $x \in \mathbb{R}_+^n$ is the state vector and $u \in \mathbb{R}_+^m$ is the control vector. The system (1) is denoted by the pair (A, B) and, when the system is positive, by $(A, B) \geq 0$.

A common property of positive systems is that their state evolution is always positive (or at least non-negative) whenever the initial state is positive (or at least non-negative). Note that A and B being non-negative matrices is a necessary and sufficient condition for a discrete-time linear system to have non-negative state evolution for any non-negative initial state, given that the controls are also non-negative.

The system (1) is said to be *reachable (controllable from the origin)* if, for any final state $x_f \geq 0$, there exist $k \in \mathbb{N}$ and a non-negative control sequence $u(t) \geq 0$, $t = 0, 1, 2, \ldots, k$, transferring the system from $x_0 = 0$ at $t = 0$ to x_f at $t = k$. The system (1) is called *null-controllable (controllable to the origin)* if, for any initial state $x_p \geq 0$, there exist $k \in \mathbb{N}$ and a non-negative control sequence $u(t) \geq 0$, $t = 0, 1, 2, \ldots, k-1$, transferring the system from $x_0 = x_p$ at $t = 0$ to $x_f = 0$ at $t = k$. The system (1) is *controllable* when it is reachable and null-controllable (Rumchev and James, 1989). Controllability is a fundamental property of the system that shows its ability to move in space. It has direct implications in many control problems such as optimal control, feedback stabilization, non-negative realizations and system minimality among others.

Characterizations of the reachability of the system (1) can be given in terms of the reachability matrix of the pair (A, B). The reachability matrix at time k is given by

$$\mathcal{R}_k(A, B) = [B|AB|A^2B|\ldots|A^{k-1}B]. \tag{2}$$

It is well known that the pair $(A, B) \geq 0$ is reachable if and only if the reachability matrix $\mathcal{R}_k(A, B)$ has a monomial submatrix of order n for some $k \leq n$. We recall that an n-dimensional vector is called *i-monomial* if it is a nonzero multiple of the i-th unit vector e_i of $\mathbb{R}^n$. A monomial matrix consists of n linearly independent monomial vectors. Throughtout this paper we consider non-negative vectors only.

Various authors have contributed to the characterization of positive reachability and controllability properties, including Coxson and Shapiro (1987), Coxson *et al.* (1987), Rumchev and James (1989), Murthy (1986), Muratori and Rinaldi (1991), Bru *et.al* (2000) and Caccetta and Rumchev (1998). At the same time, digraphs have been widely used in control theory. It is sufficient to mention only that the notion of the structural controllability of linear systems (Lin, 1974) and criteria to test this property have been formulated in terms of digraphs. However, algebraic methods have been used for the same problems, see, e.g., (Coxson and Shapiro, 1987; Muratori and Rinaldi, 1991; Rumchev and James, 1989). An overview of these results in both forms (algebraic and graph-theoretic) can be found in the very recent monograph by Kaczorek (2002). Moreover, original results on the reachability and controllability of continuous-time positive linear systems are also provided in that monograph.

In this paper, in order to increase the understanding of reachability and controllability properties of positive linear systems, the generic structure of reachable and controllable pairs $(A, B) \geq 0$, for the general case when A may contain zero columns is given in terms of the digraph of A. In this way, all possible structures (subdigraphs) of the digraph of A that can have a reachable or controllable pair (A, B) are detected and studied.

The paper is organized as follows: In Section 2 some basic combinatorial concepts are given as well as the reiteration of a basic but known lemma. The algebraic properties of all different monomial subdigraphs, which can be in the digraph of a reachable pair, are studied in Section 3. A characterization of reachability and controllability properties of the system (1) is obtained in Section 4. Finally, Section 5 contains conclusions.

2. Some Preliminaries

Let $A = [a_{ij}]$ be an $n \times n$ non-negative matrix. The *digraph* of A, denoted by $D(A)$, is defined as follows: The set of vertices of $D(A)$ is denoted by $N = \{1, 2, \dots, n\}$ and there is an arc in $D(A)$ from vertex i to vertex j if $a_{ji} > 0$. The set of all arcs is denoted by U. A *walk* in $D(A)$, from vertex i_1 to vertex i_k, is an alternating sequence of vertices and arcs, and we will denote it by $(i_1, \dots, i_k)$. A walk is called *closed* if the initial and final vertices coincide. The *length* of a walk is the number of arcs it contains. A walk is said to be a *path* if all its vertices are distinct, and a *cycle* if it is a closed path. The number of arcs directed away from a vertex i is called the *outdegree* of i and is presented as $\mathrm{od}(i)$. The number of ingoing arcs of a vertex i is called the *indegree* of i and is denoted by $\mathrm{id}(i)$. Note that the number of non-zero entries in the i-th column of A is $\mathrm{od}(i)$, while $\mathrm{id}(i)$ coincides with the number of non-zero entries of the i-th row.

The positive entries of the columns of the matrix B are associated with the corresponding vertices in $D(A)$. The vertices associated with the monomial columns of B are referred to as *origins*.

We have the following simple but basic result (Caccetta and Rumchev, 1998):

Lemma 1. *Let M be an $n \times n$ matrix whose j-th column is an i-monomial. Let b be an n-dimensional j-monomial vector. Then the product Mb is an i-monomial vector. In particular, if $M^s b$ is j-monomial, then $M^{s+1} b$ will be i-monomial.*

The above lemma tells us that if $\mathrm{od}(i) = 1$ and b is an i-monomial vector, then Mb is a monomial vector as well. However, if $\mathrm{od}(i) > 1$, then Mb is not monomial anymore. In fact, the number of non-zero entries of that product is exactly $\mathrm{od}(i)$.

3. Monomial Subdigraphs

In this section we construct special subdigraphs of the digraph $D(A)$ called monomial subdigraphs. The common property of monomial subdigraphs is the fact that from the column of B associated with the initial vertex of a monomial subdigraph one can obtain a maximal sequence of linearly independent monomial

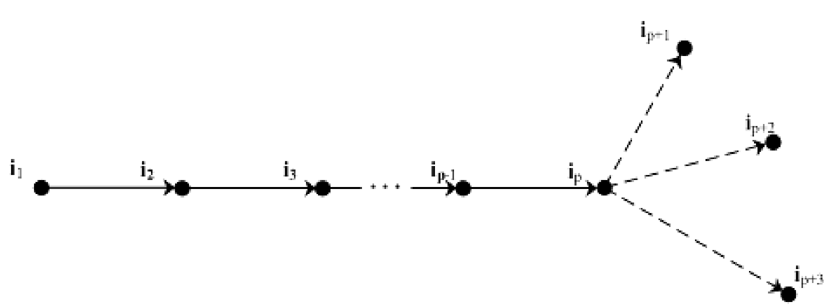

Fig. 1. Monomial path.

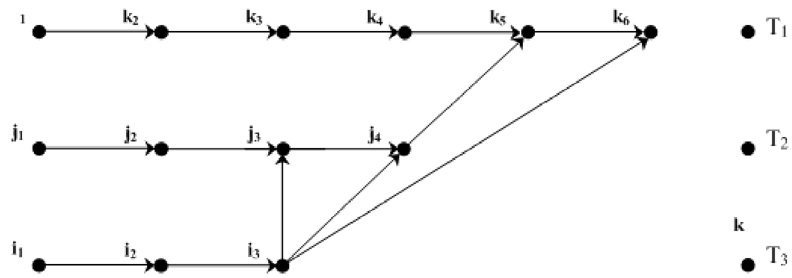

Fig. 2. Monomial tree.

vectors $b, Ab, A^2b, \ldots, A^{p-1}b$, where p is the number of the vertices of the subdigraph. Now, given a non-negative matrix A and a path $(i_1, i_2, \ldots, i_{p-1}, i_p)$ of length $p-1$ of a digraph $D(A)$, we will consider the following special paths:

Definition 1. (i) The above path is said to be an i_1-*monomial* path if its vertices have outdegree $\mathrm{od}(i_j) = 1$, for all $j = 1, 2, \ldots, p-1$, and $\mathrm{od}(i_p)$ is arbitrary, but i_p cannot be connected with any other vertex of the path.
(ii) When the last vertex of the monomial path has $\mathrm{od}(i_p) = 0$, then we have a *single* monomial path.
(iii) The path $(i_1, i_2, \ldots, i_{p-1}, i_p)$ of length $p-1$ with $\mathrm{od}(i_k) = 1$ for all $k = 1, 2, \ldots, p$ is called a *(monomial) cycle* if $i_1 = i_p$.

A monomial path $(i_1, \ldots, i_p)$ of lenth $p-1$ is presented in Fig. 1. When $D(A)$ consists of a monomial path or a (monomial) cycle, we have the following result:

Lemma 2. *Let $(A, B) \geq 0$ and let $D(A)$ be a (single) monomial path with vertices $(i_1, i_2, \ldots, i_p)$ of length $p-1$, where $p \leq n$, and let B have an i_1-monomial column b. Then*

(i) the p vectors

$$b, Ab, A^2b, \ldots, A^{p-1}b \tag{3}$$

are linearly independent monomial vectors, and

(ii) these p vectors constitute the maximal number of linearly independent monomial vectors generated by any column of B.

Proof. (i) Apply Lemma 1.
(ii) Since $\mathrm{od}(i_p) = 0$ (single monomial path) or $\mathrm{od}(i_p) > 1$ (monomial path), then the vector $A^p b$ is a zero vector or has, respectively, more than one positive entry, that is, it is not monomial anymore. By Lemma 4 and Remark 6 of (Caccetta and Rumchev, 1998), no non-monomial column b can generate in (3) as many monomial vectors as the i_1-monomial column. It is readily seen that the i_1-monomial column yields at least as many monomial columns as any other monomial column. ■

Remark 1. The results in Lemma 2 hold for monomial cycles. It is not difficult to see that monomial cycles raise a p-periodic sequence (3). That is, $A^{k+lp}b = A^k b$ (up to a scalar), $0 \leq k \leq p-1$ and $l = 0, 1, 2, \ldots$.

Definition 2. A subdigraph T of a digraph $D(A)$ is called a *monomial tree* if it is a union of different monomial paths, originating at different vertices and connected one to another from the last vertices only (in $D(A)$) *without forming cycles*.

Note that the existence of at least a single monomial path in a monomial tree results from the fact that there are no cycles. As cycles are not permitted in monomial trees, the monomial paths of any monomial tree can be grouped in levels as follows: At level T_1 we consider all single monomial paths of that monomial tree. Any monomial path connected from its last vertex only with that of a single monomial path will belong to level T_2. Any monomial path connected from its last vertex only with that of a monomial path from T_2, and possibly T_1, is in level T_3. By recursion, all levels in the monomial tree can be defined up to the last, which is denoted by T_{n-1}. The digraph of Fig. 2 is a monomial tree of three levels.

Let $\mathcal{T}$ be the index set of all initial vertices of all monomial paths of T. A similar result to Lemma 2 can be obtained for a monomial tree.

Lemma 3. *Let $(A, B) \geq 0$ and let $D(A)$ be a monomial tree T. Suppose that B has the i-monomial columns, for all $i \in \mathcal{T}$. Then*
(i) the vectors generated along each monomial path, as in (3), form a set of linearly independent monomial vectors; the union of all these sets is also linearly independent, and
(ii) this union is the maximal set of linearly independent monomial vectors generated by any column of B.

Definition 3. Let a digraph $D(A)$ contain at least one monomial path, one cycle and a tree T. A subdigraph

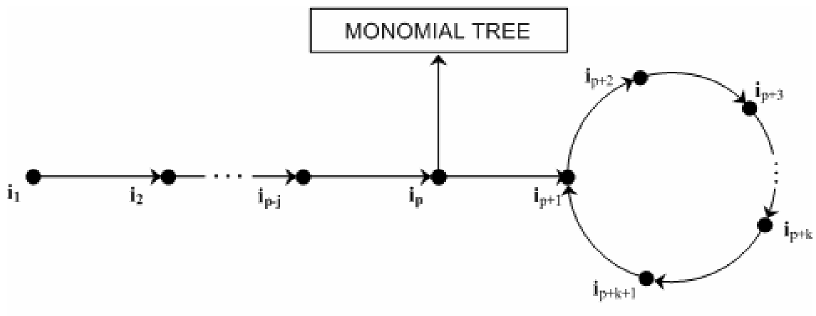

Fig. 3. Flower.

$F \subseteq D(A)$ is said to be a *flower* if it consists of a monomial path $(i_1, i_2, \ldots, i_p)$ of length $p-1$, linked to a (monomial) cycle $(i_{p+1}, i_{p+2}, \ldots, i_{p+k+1})$ with the arc (i_p, i_{p+1}) and, moreover, from the vertex i_p of the monomial path, there are arcs (i_p, t) for some $t \in T$.

Observe that there must be a tree in the digraph $D(A)$ for a flower to exist, but the flower itself contains only a monomial path and a connected (monomial) cycle. All vertices of a flower have $\mathrm{od}(i_s) = 1$, except for the vertex i_p, in which case $\mathrm{od}(i_p) \geq 2$. Figure 3 represents a flower.

Again, the following result is similar to Lemma 2.

Lemma 4. *Let $(A, B) \geq 0$ and let $D(A)$ be the digraph of A containing a flower F connected to a monomial tree T with q vertices. Assume that the flower has a monomial path $(i_1, i_2, \ldots, i_p)$, $p \leq n$ linked to the (monomial) cycle $(i_{p+1}, i_{p+2}, \ldots, i_{p+k+1})$, $k \leq n-p-q-1$. Suppose that B has an i_1-monomial column, namely, b. Then*

(i) the p vectors $\{b, Ab, A^2b, \ldots, A^{p-1}b\}$ are linearly independent and monomial. In addition, the $k+1$ vectors $\{A^{q+p}b, A^{q+p+1}b, \ldots, A^{q+p+(k+1)}b\}$ are linearly independent and monomial, and

(ii) the union of both sets gives the maximal number of linearly independent monomial vectors generated by any column of B along the flower F.

Proof. (i) Since $\mathrm{od}(i_r) = 1$, $r = 1, 2, \ldots, p-1$, it is clear that the p vectors $\{b, Ab, A^2b, \ldots, A^{p-1}b\}$ are linearly independent monomial vectors, cf. Lemma 2.

The vector $A^p b$ will have at least two positive entries since $\mathrm{od}(i_p) \geq 2$. These positive entries correspond to the arcs from i_p to a vertex of T, and to a vertex of the cycle. The vectors $\{A^p b, A^{p+1}b, \ldots, A^{p+k}b\}$ will have at least a positive entry in addition to the i_s-th entry, $s = p+1, p+2, \ldots, p+k+1$, produced by the cycle. This additional positive entry, namely, the j-th one, is yielded by the link from i_p to the monomial tree. However, the j-th entry will eventually become zero, at least for the $(q+p)$-th power of A. This is because the entry will ultimately correspond to the final vertex of a single monomial path of T. So, the $k+1$ monomial vectors $\{A^{q+p}b, A^{q+p+1}b, \ldots, A^{q+p+(k+1)}b\}$ will be linearly independent. Clearly, the set $\{b, Ab, A^2b, \ldots, A^{p-1}b, A^{q+p}b, A^{q+p+1}b, \ldots, A^{q+p+(k+1)}b\}$ is formed by linearly independent monomial vectors, since the vertices in the flower are distinct.

(ii) Similar to the proof of Part (ii) of Lemma 2. ■

Cycles associated with monomial columns of B produce linearly independent monomial vectors, see Remark 1. In addition, (monomial) cycles may yield similar periodic sequences of linearly independent monomial vectors when they are associated with some special columns of B, called *proper*, as stated in Lemma 5, the proof being similar to that of Lemma 4.

Lemma 5. *Let $(A, B) \geq 0$ and let C be a (monomial) cycle with vertices $(i_1, i_2, \ldots, i_p = i_1)$, where $p < n$. Also, let T be a monomial tree with q vertices in $D(A)$. Suppose that B has a proper column, which can be written as $b = e_{i_k} + w$, where i_k is one of the indices of the cycle, e. g., $i_k = i_1$, and when $w_j > 0$, then j is a vertex of the monomial tree T. Then*

(i) the p vectors $\{A^q b, A^{q+1}b, \ldots, A^{q+p-1}b\}$ are linearly independent monomial vectors, and

(ii) these p vectors constitute the maximal number of linearly independent monomial vectors generated by any column of B associated with the cycle C.

We can weaken the definition of a monomial tree in order to obtain linearly independent monomial vectors.

Definition 4. A subdigraph P of a digraph $D(A)$ is called a *palm* if it is a path $(i_1, i_2, \ldots, i_p)$, such that $\mathrm{od}(i_k) = 1$, $k = 1, 2, \ldots, p-1$, and an arbitrary subset of arcs (i_p, i_k), $k = 1, 2, \ldots, p$.

From Definition 4 it follows that monomial paths can be considered as a special type of palm without

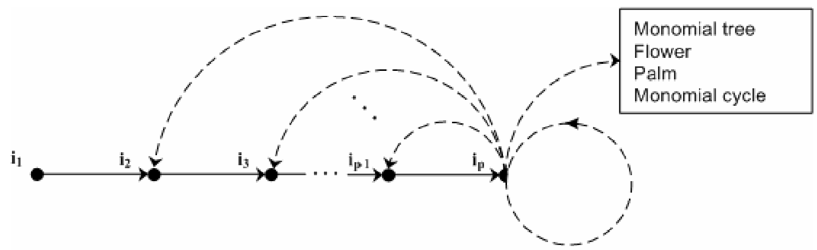

Fig. 4. Palm.

$\{(i_p, i_k),\ k = 1, 2, \ldots, p\}$, but not every palm is a monomial path. If $\mathrm{od}(i_p) = 0$, the path is, indeed, a single monomial path in a monomial tree. Note that any connection from i_{p-1} to any monomial tree is excluded. It seems that a flower can be viewed as a particular type of palm, in which the last vertex i_p is connected with only one vertex $i_k, k = 2, \ldots, p$. For the existence of flowers, links $(i_{k-1}, t),\ t \in T$, must exist. But such links are not permitted in the palm. For this reason, we have considered the digraph flower independently. In addition, note that any monomial path or cycle that is not in a monomial tree, flower or (monomial) cycle is considered as a palm. An example of a palm is given in Fig. 4.

The following properties can be deduced in a similar way as in Lemma 2.

Lemma 6. *Let $(A, B) \geq 0$ and let $D(A)$ be a palm with vertices $i_1, i_2, \ldots, i_p$, where $p \leq n$. Suppose that B has a column b which is i_1-monomial. Then*

(i) the p vectors $\{b, Ab, A^2b, \ldots, A^{p-1}b\}$ are linearly independent monomial vectors, and

(ii) the set of those p vectors is the maximal set of linearly independent monomial vectors generated by any column of B.

Palms can be linked with one another by arcs (i_p, p) for some vertex p of another palm P, forming a family of palms, in which case *other cycles* can appear. Palms can also be linked to any other i_1-monomial subdigraph by arcs (i_p, t) for some vertex t in a monomial tree or flower or (monomial) cycle.

The above lemma is Theorem 3 of (Coxson *et al.*, 1987). In addition, for multi-input systems (A, B) a composition of palms is used in (Rumchev, 2000, Thm. 1) to study the case where A does not have any null columns.

4. Positive Reachability and Controllability

It is clear that when a pair (A, B) is such that $D(A)$ is one of the monomial subdigraphs introduced in the previous section and B contains all columns needed to generate the maximal number of linearly independent monomial vectors on $D(A)$, then the pair (A, B) is reachable. This is because one can obtain a monomial matrix of order n in the reachability matrix. With the above results, a characterization of reachable positive systems (A, B) is given in this section.

For this purpose, consider a non-negative pair (A, B) and the associated digraph $D(A)$. Recall that the positive entries of the monomial columns of B are identified with the corresponding vertices in $D(A)$ called *origins*. From these *origins*, construct the maximal monomial subdigraphs, without repeating vertices, in the following order: (i) all possible monomial trees; the initial vertices of all monomial paths of the monomial trees form the index set of *origins* $\mathcal{T}$; (ii) all possible flowers; the initial vertices of all monomial paths of the flowers form the index set of *origins* $\mathcal{F}$; (iii) all possible palms; the initial vertices of all paths of the palms form the index set of *origins* $\mathcal{P}$; (iv) all possible (monomial) cycles from the *proper columns* of B, $b_{l_r} = e_{l_r} + w$, where the indices of the positive components of vector w are vertices of a monomial tree; indices l_r form the set of *origins* $\mathcal{C}$.

Let

$$L = \Big\{(i_p, t), i_p \in F, t \in T \quad \text{and} \quad (i_p, t), i_p \in P, t \in T \text{ or } t \in F \text{ or } t \in P \text{ or } t \in C\Big\}$$

be the set of all arcs linking the formed monomial subdigraphs. Define $D'(A) = D(A) \setminus L = (N', U')$, where $N' = N$ and $U' = U \setminus L$. Thus, the monomial subdigraphs in $D'(A)$ are disjoint.

The following characterization follows from this construction:

Theorem 1. *Let $A \geq 0$ and let $D(A)$ be the associated digraph. Let $\mathcal{T}$, $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{C}$ be the index sets of the origins of the monomial subdigraphs, respectively, monomial trees T, flowers F, palms P, and (monomial) cycles C of $D(A)$ formed from the monomial and proper columns of B. Then the pair (A, B) is reachable if and only if*

$D'(A)$ is a union of these monomial subdigraphs, i.e.,

$$D'(A) = \bigcup_{t=1}^{c_t} T_t \bigcup_{f=1}^{c_f} F_f \bigcup_{p=1}^{c_p} P_p \bigcup_{c=1}^{c_p} C_c, \tag{4}$$

where c_t,c_f,c_k and c_p stand for the numbers of monomial trees, flowers, palms and (monomial) cycles, respectively.

Proof. Assume that all possible monomial subdigraphs are formed, without repeating vertices, in the order stated above, from the *origins* $\mathcal{T}$, $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{C}$ obtained from the monomial and proper ($b_{l_r} = e_{l_r} + w$) columns of B.

Suppose that $D'(A)$ is a union of monomial subdigraphs:

$$D'(A) = \bigcup_{t=1}^{c_t} T_t \bigcup_{f=1}^{c_f} F_f \bigcup_{p=1}^{c_p} P_p \bigcup_{c=1}^{c_p} C_c.$$

Since B has monomial columns corresponding to indices of $\mathcal{T}$, $\mathcal{F}$, and $\mathcal{P}$, and columns of type $b_{l_r} = e_{l_r} + w_{l_r}$ corresponding to $\mathcal{C}$, then by Lemmas 3–6 for each monomial subdigraph we obtain a maximal set of linearly independent monomial vectors. Since $D'(A)$ is a union of these monomial subdigraphs and it contains all vertices of $D(A)$, the union of all these vectors is a set of n linearly independent monomial vectors. This occurs because each vertex of $D'(A)$ is in one and only one monomial subdigraph, and so the pair (A, B) is reachable.

Conversely, assume now that (4) does not hold, i.e.,

$$\bigcup_{t=1}^{c_t} T_t \bigcup_{f=1}^{c_f} F_f \bigcup_{p=1}^{c_p} P_p \bigcup_{c=1}^{c_p} C_c \subset D'(A).$$

Then either $D'(A)$ has at least one vertex or an arc not included in the union. Denote by (N'', U'') the digraph formed by the union of those monomial subdigraphs, where N'' is the vertex set and U'' is the arc set.

Case 1. Suppose that $N'' \subset N'$. Hence the number of vertices in the union is smaller than the number n of vertices of $D'(A)$ and $D(A)$. Then the maximal number of linearly independent monomial vectors produced by the union of all monomial subdigraphs is smaller than n. Since the monomial subdigraphs give the maximal number of such vectors, according to Lemmas 3–6 the reachability matrix $\mathcal{R}_n$ will not contain a monomial submatrix of order n. The columns of B which are not monomial or proper are not used in the formation of the monomial subdigraphs of the union. They might produce with columns of A (corresponding to the vertices of certain subdigraphs, which are in $D'(A)$ and not in the union of monomial subdigraphs) some linearly independent monomial vectors in the sequence $b, Ab, A^2b, \dots, A^kb$. However, the number of such vectors is smaller than the number of linearly independent monomial vectors generated by the monomial column of B, corresponding to the *origins*, applied to the same subdigraph, cf. the proof of Lemma 2. Therefore $\mathcal{R}_n$ will not contain an $n \times n$ monomial submatrix and the pair (A, B) is not reachable.

Case 2. Now $U'' \subset U'$. Since all arcs connecting monomial subdigraphs are in L, the strict inclusion is due to the existence of an arc not included in L. Such an arc has a vertex in $D'(A)$, but not in the union, that is, $N'' \subset N'$, and then we can proceed as in Case 1. ■

The following examples illustrate the monomial subdigraphs of the digraph of a pair $(A, B) \geq 0$.

Example 1. Let

$$A = \begin{bmatrix} 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0&1\\ 0&1&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0 \end{bmatrix},$$

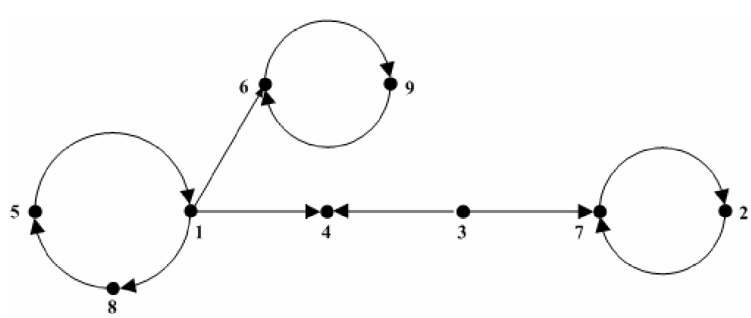

Fig. 5. Diagraph D(A).

and

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The digraph of A is given in Fig. 5.

First note that the matrix B_1 has unit vectors e_4, e_8 and e_3. It is clear that starting from:

- vector e_4, a single monomial path (4), of length zero, is found and it will be in the monomial tree T;
- vector e_3, a flower, F, is found and it is formed by the monomial path (3), of length zero, and the (monomial) cycle $(7, 2)$; note that vertex 3 is connected with this cycle and there is an arc from vertex 3 to T, the arc $(3, 4)$;
- vector e_8, the monomial path $(8, 5, 1)$, of length two, is obtained and it will be in the set of palms P; it cannot be in T because this would produce a cycle in T;
- vector $b = e_6 + w$, where the positive components of w are just the 4-th component and this index is a vertex of T; therefore, from that vector, one can consider the cycle $(6, 9)$, which is a (monomial) cycle C.

It is clear that $D'(A) = T \bigcup P \bigcup F \bigcup C$ and thus the pair (A, B_1) is reachable. The arcs of $D(A)$ not included in $D'(A)$ are $L = \{(1, 4), (1, 6), (3, 4)\}$. It is worth noticing that the same decomposition can be obtained if the matrix B_1 has the monomial column e_6 instead of the column b_4.

Example 2. Let A be the matrix of Example 1 and let

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, the monomial tree T, the flower F and the palm P previously described are obtained starting from the vertices which correspond to the first three columns of B_2. However, from the fourth column $b = e_6 + e_8 + e_1$ one cannot obtain monomial vectors because vectors $A^k b$, $k = 0, 1, 2 \ldots$, have more than one positive component. The cycle $(6, 9)$ cannot be obtained and $T \bigcup P \bigcup F \subset D'(A)$. Hence the pair (A, B_2) is not reachable.

Following the approach proposed in this paper we can identify reachable parts (monomial trees, flowers, palms, cycles) of the system matrix A even when the pair (A, B) is not reachable. The positive system can be made reachable by applying suitable controls (see Examples 1 and 2).

As is well known, reachability from zero plus nilpotence is equivalent to controllability (Coxson and Shapiro, 1987; Rumchev and James, 1989)). It is thereby sufficient to eliminate all possible monomial subdigraphs of $D(A)$ with cycles for obtaining the controllability property. In fact, we can establish the following result:

Theorem 2. *Let $(A, B) \geq 0$ and let $D(A)$ be the associated digraph of A. Then the pair (A, B) is controllable if, and only if, $D(A) = \bigcup_{t=1}^{c_T} T_t$ and B contains all monomial columns corresponding to the origins of the monomial trees.*

5. Conclusion

In this paper, reachability and controllability properties of discrete-time positive linear systems, in the more general case when the system matrix contains zero columns, are established in terms of the digraph of the pair (A, B). Monomial subgraphs of reachable and controllable non-negative pairs (A, B) are identified and their properties studied. Criteria in a digraph form recognising the reachability and controllability properties of such pairs are obtained in the paper. These criteria give a better understanding of the structure of reachable and controllable discrete-time positive linear systems than the corresponding criteria in an algebraic form. The results obtained in this paper can be used to develop computationally efficient combinatorial algorithms for revealing fundalmental properties of discrete-time positive linear systems such as reachability and controllability.

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