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AN EQUIVALENT MATRIX PENCIL FOR BIVARIATE POLYNOMIAL MATRICES

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In this paper, we present a simple algorithm for the reduction of a given bivariate polynomial matrix to a pencil form which is encountered in Fornasini-Marchesini's type of singular systems. It is shown that the resulting matrix pencil is related to the original polynomial matrix by the transformation of zero coprime equivalence. The exact form of both the matrix pencil and the transformation connecting it to the original matrix are established.

Keywords: matrix pencils, 2-D singular systems, zero-coprime-equivalence, invariant polynomials, invariant zeros

1. Introduction

Polynomial systems theory for constant linear differential or difference systems is a well-established and efficient tool for the analysis and design of control systems (Blomberg and Ylinen, 1983; Rosenbrock, 1970; Wolovich, 1974). The approach utilizes algebraic properties of polynomial matrices with real or complex coefficients. The key for the success of this theory seems to be its computational nature, i.e., the ring $\mathbb{R}[s]$ of polynomials over a field $\mathbb{R}$ in an indeterminate s is a division algorithm which can be used to find common factors and to manipulate polynomial matrices into suitable canonical forms. Later on, polynomial systems theory was generalized to the so-called behavioral systems, where the variables are not *a-priori* divided into inputs and outputs (Polderman and Willems, 1998). The extension of polynomial systems theory to multidimensional (n-D) systems was proposed, e.g., in (Oberst, 1990). The resulting structure is a ring $\mathbb{R}[z_1, \dots, z_n]$ of polynomials over a field $\mathbb{R}$ in two or more indeterminates $z_1, \dots, z_n$ acting on a given signal space.

In 1-D systems theory, matrix pencils play an important role, see, for example, (Hayton *et al.*, 1990; Karampetakis *et al.*, 1995; Rosenbrock, 1970; Verghese, 1978). In the 2-D case, matrix pencils arise in the description of 2-D singular state space systems such as those studied by Kaczorek (1988).

One of the most basic procedures in systems theory is the transformation of a given system of differential or difference equations to a low order. In 2-D polynomial systems theory, this is related to the reduction of a bivariate polynomial matrix to a pencil form. The zeros of a polynomial matrix encapsulate the relevant properties of a system, such as controllability, observability and minimality. The reduction transformation must therefore preserve the zero structure of the original polynomial matrix.

The reduction of an arbitrary bivariate polynomial matrix to a pencil form was first studied by Pugh *et al.* (1998a). Their procedure consists in applying a two-stage algorithm which involves the removal of factors from certain matrices to ensure that the transformations linking the original matrix with the final matrix pencil are polynomial. The method gives *a priori* the form of neither the resulting pencil nor the transformation linking it to the original polynomial matrix. Pugh *et al.* (2005a) gave another two-step algorithm for the reduction of an arbitrary bivariate polynomial matrix to a pencil which is encountered in Roesser's type of singular 2-D systems.

In the present work, we propose a simple and direct procedure for the reduction of an arbitrary bivariate polynomial matrix to a pencil form which is encountered in Fornasini-Marchesini's type of 2-D singular systems. We will establish the exact nature of both the matrix pencil in terms of the coefficient matrices of a given matrix, and the transformation linking it to the original polynomial matrix. The paper further highlights the relevance of the transformation of zero coprime equivalence in n-D systems theory. This type of equivalence was studied by Levy (1981), Johnson (1993), and Pugh *et al.* (1998a). Pugh *et al.* (1998b) showed that it provides a basis for a 2-D generalization of Rosenbrock's least order characterization.

2. Preliminaries

Consider the following 2-D discrete system in a generalized state space form as given by Kaczorek (1988):

$$\left.\begin{aligned} Ex(i+1,j+1) &= A_1x(i+1,j) + A_2x(i,j+1) \\ &\quad +A_0x(i,j) + B_1u(i+1,j) \\ &\quad +B_2u(i,j+1) + B_0u(i,j), \\ y(i,j) &= Cx(i,j) + Du(i,j), \end{aligned}\right\} \tag{1}$$

where $x(i,j)$ is the state vector, $u(i,j)$ is the input vector, $y(i,j)$ is the output vector, E, A_0, A_1, A_2, B_0, B_1, B_2, C, D are constant real matrices of appropriate dimensions, and E may be singular. Then, taking the 2-D z-transform of (1) and assuming zero boundary conditions, we get

$$\begin{bmatrix} szE - sA_1 - zA_2 - A_0 & sB_0 + zB_1 + B_0 \\ -C & D \end{bmatrix} \begin{bmatrix} \overline{x}(s,z) \\ -\overline{u}(s,z) \end{bmatrix} = \begin{bmatrix} 0 \\ -\overline{y}(s,z) \end{bmatrix}. \tag{2}$$

The polynomial matrix

$$szE - sA_1 - zA_2 - A_0 \tag{3}$$

over $\mathbb{R}[s,z]$ in (2) is called a matrix pencil and can be regarded as an extension from 1-D of the matrix pencil $sE - A$.

The following definitions are needed for the results of the paper:

Definition 1. Two polynomial matrices $P_1(s,z)$ and $S_1(s,z)$ of appropriate dimensions are said to be *zero left coprime* if the matrix

$$\begin{bmatrix} P_1(s,z) & S_1(s,z) \end{bmatrix} \tag{4}$$

has a full rank for all $(s,z) \in \mathbb{C}^2$. Similarly, $P_2(s,z)$ and $S_2(s,z)$ of appropriate dimensions are said to be *zero right coprime* if the matrix

$$\begin{bmatrix} P_2^T(s,z) & S_2^T(s,z) \end{bmatrix}^T \tag{5}$$

has a full rank for all $(s,z) \in \mathbb{C}^2$.

Following the results of (Youla and Gnavi, 1979), we obtain that the polynomial matrices $P_1(s,z)$ and $S_1(s,z)$ are zero left coprime if and only if there exist zero right coprime polynomial matrices $X(s,z)$ and $Y(s,z)$ of appropriate dimensions satisfying Bezout's relation

$$P_1(s,z)X(s,z) + S_1(s,z)Y(s,z) = I. \tag{6}$$

One immediate result given by Sontag (1980) is that a necessary and sufficient condition for the matrices $P_1(s,z)$ and $S_1(s,z)$ to be zero left coprime is that the matrix in (4) is unimodular equivalent to the matrix $\begin{bmatrix} I & 0 \end{bmatrix}$. Similar results can be stated for zero right coprime matrices.

Definition 2. Given a $p \times q$ polynomial matrix $P(s,z)$, the i-th order *invariant polynomial* $\Phi_i(s,z)$ of $P(s,z)$ is defined by

$$\Phi_i(s,z) = \begin{cases} \dfrac{D_i(s,z)}{D_{i-1}(s,z)} & \text{if } \ 1 \le i \le t, \\ 0 & \text{if } \ t \le i \le \min(p,q), \end{cases} \tag{7}$$

where t is the normal rank of $P(s,z)$, $d_0(s,z) = 1$ and $D_i(s,z)$ is the greatest common divisor of all the $i \times i$ minors of the given matrix $P(s,z)$.

As in the 1-D case, the zero structure of a bivariate polynomial matrix is a crucial indicator of system behavior. Zerz (1996) showed that the controllability and observability of a system in the behavioral setting is connected to the zero structure of the associated polynomial matrix. However, unlike in the 1-D case, the zero structure of a multivariate polynomial matrix is not completely captured by invariant polynomials. Therefore, the following concept of invariant zeros as given by Pugh *et al.* (2005b) is introduced.

Definition 3. Given a $p \times q$ polynomial matrix $P(s,z)$, the i-th order *invariant zeros* of $P(s,z)$ are the elements of the variety $\mathcal{V}_{\mathbb{R}}(\mathcal{I}_i^{[P]})$ defined by the ideal $\mathcal{I}_i^{[P]}$ generated by the $i \times i$ minors of $P(s,z)$.

An extension of Fuhrmann's strict system equivalence (Fuhrmann, 1977) from the 1-D to the 2-D setting is zero coprime equivalence and is defined by the following:

Definition 4. Let $\mathbb{P}(m,n)$ denote the class of $(r+m) \times (r+n)$ polynomial matrices where m, n are fixed positive integers and $r > -\min(m,n)$. Two polynomial system matrices $P_1(s,z)$ and $P_2(s,z)$ are said to be *zero coprime equivalent* if there exist polynomial matrices $S_1(s,z)$, $S_2(s,z)$ of appropriate dimensions such that

$$S_2(s,z)P_1(s,z) = P_2(s,z)S_1(s,z), \tag{8}$$

where $P_1(s,z), S_1(s,z)$ are zero left coprime and $P_2(s,z), S_2(s,z)$ are zero right coprime.

The transformation of zero coprime equivalence can be generated by the classical unimodular equivalence coupled with a trivial expansion or deflation of matrices. Pugh *et al.* (1996; 2005b) showed that zero coprime equivalence exhibits fundamental algebraic properties amongst its invariants.

Lemma 1. (Pugh *et al.*, 1996) *Suppose that two polynomial matrices $P(s,z)$ and $Q(s,z) \in \mathbb{P}(m,n)$ are related* *by zero coprime equivalence and let $\Phi_1^{[P]}, \Phi_2^{[P]}, \ldots, \Phi_h^{[P]}$, where $h = \min(r^{[P]} + m, r^{[P]} + n)$, denote the invariant polynomials of $P(s,z)$ and $\Phi_1^{[Q]}, \Phi_2^{[Q]}, \ldots, \Phi_k^{[Q]}$, where $k = \min(r^{[Q]} + m, r^{[Q]} + n)$, denote the invariant polynomials of $Q(s,z)$. Then*

$$\Phi_{h-i}^{[P]} = c_i \Phi_{k-i}^{[Q]} \quad \text{for} \quad i = 0, 1, \ldots, \max(k-1, h-1), \tag{9}$$

where

$$\Phi_j^{[P]} = 1, \Phi_j^{[Q]} = 1 \quad \text{for any} \;\; j < 1, c_i \in \mathbb{R}\backslash\{0\}.$$

Lemma 2. (Pugh *et al.*, 2005b) *Suppose that two polynomial matrices $P(s,z)$ and $Q(s,z) \in \mathbb{P}(m,n)$ are related by zero coprime equivalence and let $\mathcal{I}_j^{[P]}$ for $j = 1, \ldots, h = \min(r^{[P]} + m, r^{[P]} + n)$ denote the ideal generated by the $j \times j$ minors of $P(s,z)$ and $\mathcal{I}_i^{[Q]}$, for $i = 1, \ldots, k = \min(r^{[Q]} + m, r^{[Q]} + n)$, denote the ideal generated by the $i \times i$ minors of $Q(s,z)$. Then*

$$\mathcal{I}_{h-i}^{[P]} = \mathcal{I}_{k-i}^{[Q]}, \quad i = 0, \ldots, \bar{h}, \tag{10}$$

where

$$\bar{h} = \min(h-1, k-1)$$

and for any $i > h$,

$$\mathcal{I}_{h-i}^{[P]} = \langle 1 \rangle \;\; \text{or} \;\; \mathcal{I}_{k-i}^{[Q]} = \langle 1 \rangle \;\; \text{in case} \;\; i < h \;\; \text{or} \;\; i < k.$$

3. Bivariate Polynomial Matrix Reduction Procedure

A given $P(s,z) \in \mathbb{R}^{m \times n}[s,z]$ can be written as

$$\begin{aligned} P(s,z) &= \sum_{i=0}^{p} \sum_{j=0}^{q} P_{i,j} s^i z^j \\ &= P_{0,0} s^0 z^0 + P_{0,1} s^0 z^1 + P_{0,2} s^0 z^2 \\ &\quad + \cdots + P_{p,q} s^p z^q, \end{aligned} \tag{11}$$

where $P_{i,j},\ i = 0, 1, \ldots, p,\ j = 0, 1, \ldots, q$, are $m \times n$ real constant matrices: Now construct the following block real matrices,

$$E = \begin{bmatrix} & 0_{n(pq-1),npq} & & \\ E_q & E_{q-1} & \cdots & E_1 \end{bmatrix}, \tag{12}$$

where

$$E_j = \begin{bmatrix} P_{p,j} & P_{p-1,j} & \cdots & P_{1,j} \end{bmatrix}, \quad j = 1, 2, \ldots, q, \tag{13}$$

$$A_0 = \mathrm{Diag}(-I_{n(pq-1)}, P_{0,0}), \tag{14}$$

$$A_1 = \begin{bmatrix} & & 0_{np(q-1),npq} & & \\ & 0_{n(p-1),n(pq-p+1)} & I_{n(p-1)} & & \\ 0_{m,np(q-1)} & -P_{p,0} & -P_{p-1,0} & \cdots & -P_{1,0} \end{bmatrix}, \tag{15}$$

and

$$A_2 = \begin{bmatrix} 0_{np(q-1),np} & I_{np(q-1)} & & \\ & 0_{n(p-1),npq} & & \\ A_{2,q} & A_{2,q-1} & \cdots & A_{2,1} \end{bmatrix}, \tag{16}$$

where

$$A_{2,j} = \begin{bmatrix} 0_{m,n(p-1)} & -P_{0,j} \end{bmatrix}, \quad j = 1, 2, \ldots, q. \tag{17}$$

Then the $[n(pq-1) + m] \times npq$ polynomial matrix

$$Q(s,z) = szE - sA_1 - zA_2 - A_0 \tag{18}$$

in the form (3) is the matrix pencil associated with the polynomial matrix $P(s,z)$.

Theorem 1. *Let $P(s,z) \in \mathbb{R}^{m \times n}[s,z]$ be an arbitrary polynomial matrix as in (11), and let $Q(s,z) \in \mathbb{R}^{\bar{r} \times npq}[s,z]$ be the corresponding matrix pencil as in (18), where $\bar{r} = n(pq-1) + m$. Then $P(s,z)$ and $Q(s,z)$ are related by the following zero coprime equivalence transformation:*

$$M(s,z)P(s,z) = Q(s,z)N(s,z), \tag{19}$$

where

$$M(s,z) = \begin{bmatrix} 0_{n(pq-1),m} \\ I_m \end{bmatrix}, \quad N(s,z) = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_q \end{bmatrix} \otimes I_n, \tag{20}$$

the symbol '$\otimes$' denotes the Kronecker matrix product, and

$$N_j = \begin{bmatrix} s^{p-1} z^{q-j} & s^{p-2} z^{q-j} & \cdots & z^{q-j} \end{bmatrix}^T, \quad j = 1, 2, \ldots, q. \tag{21}$$

Proof. The matrix $Q(s,z)$ in (18) can be represented in the form

$$Q(s,z) = \begin{bmatrix} I_{np} & -zI_{np} & \cdots & 0 & 0 \\ 0 & I_{np} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{np} & -zI_{np} \\ Q_q & Q_{q-1} & \cdots & Q_2 & Q_1 \end{bmatrix}, \tag{22}$$

where the submatrices Q_j are given by (23) and (24), and

$$Q_1 = \begin{bmatrix} I_n & -sI_n & \cdots & 0 \\ 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -sI_n \\ szP_{p,1} + sP_{p,0} & szP_{p-1,1} + sP_{p-1,0} & \cdots & szP_{1,1} + sP_{1,0} + zP_{0,1} + P_{0,0} \end{bmatrix}, \tag{23}$$

$$Q_j = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ szP_{p,j} & szP_{p-1,j} & \cdots & szP_{2,j} & szP_{1,j} + zP_{0,j} \end{bmatrix}, \quad j = 2, 3, \dots, q. \tag{24}$$

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the matrix $N(s,z)$ is given by

$$N(s,z) = \begin{bmatrix} s^{p-1}z^{q-1}I_n \\ \vdots \\ z^{q-1}I_n \\ s^{p-1}z^{q-2}I_n \\ \vdots \\ z^{q-2}I_n \\ \vdots \\ \vdots \\ s^{p-1}I_n \\ \vdots \\ I_n \end{bmatrix}. \tag{25}$$

It follows that

$$M(s,z)P(s,z) = Q(s,z)N(s,z) \equiv \begin{bmatrix} 0_{n(pq-1),n} \\ P(s,z) \end{bmatrix}. \tag{26}$$

Now it remains to prove that the matrices $Q(s,z)$, $M(s,z)$ are zero left coprime and the matrices $P(s,z)$, $N(s,z)$ are zero right coprime.

The matrix $\begin{bmatrix} Q(s,z) & M(s,z) \end{bmatrix}$ is given by

$$\left[\begin{array}{ccccc|c} I_{np} & -zI_{np} & \cdots & 0 & 0 & 0 \\ 0 & I_{np} & \cdots & 0 & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{np} & -zI_{np} & 0 \\ Q_q & Q_{q-1} & \cdots & Q_2 & Q_1 & I_m \end{array}\right]. \tag{27}$$

It can be easily seen that the minor obtained by deleting the columns $n(pq-1)+1, \dots, npq$ from the matrix in (27) is equal to ± 1.

Similarly,

$$\begin{bmatrix} P(s,z) \\ N(s,z) \end{bmatrix} = \begin{bmatrix} P(s,z) \\ \hline s^{p-1}z^{q-1}I_n \\ \vdots \\ z^{q-1}I_n \\ s^{p-1}z^{q-2}I_n \\ \vdots \\ z^{q-2}I_n \\ \vdots \\ \vdots \\ s^{p-1}I_n \\ \vdots \\ I_n \end{bmatrix}, \tag{28}$$

and it is clear that the matrix in (28) contains a block identity matrix I_n and, therefore, it has an $n \times n$ highest order minor, which is equal to 1. ■

The zero coprime equivalence of $P(s,z)$ and $Q(s,z)$ implies that $P(s,z)$ and $Q(s,z)$ have the same invariant polynomials and invariant zeros in the sense described by Lemmas 1 and 2, respectively.

With a slight modification in the transformation matrix $M(s,z)$ and using the normalized form of the system matrix associated with $Q(s,z)$, the above procedure can be extended to polynomial system matrices, thereby obtaining a reduction of a given bivariate polynomial system matrix by zero coprime system equivalence to a system matrix associated with the Fornasini-Marchesini singular 2-D systems.

$$P = \begin{bmatrix} -2(z+1)s^2+(3z+2)s-z+2 & (z-4)s-z+4 & s^2-2 \\ 3s^2-1 & -zs-2z & (z+1)s^2-(z-3)s-3z+1 \\ (z-1)s^2-(z-2)s & (z+2)s^2-4z-1 & -2zs^2-(5z+2)s-2z+3 \end{bmatrix}. \quad (29)$$

4. Example

Consider the polynomial matrix $P(s,z) \in \mathbb{R}^{3\times 3}[s,z]$ given by (29). Here $m=n=3$, $p=2$ and $q=1$. Using a Maple procedure, the invariant polynomials of the matrix $P(s,z)$ are calculated as follows:

$$\begin{aligned} \Phi_1^{[P]} &= \Phi_2^{[P]} = 1, \\ \Phi_3^{[P]} &= 2z^3+8z^2+13z+10)s^6 \\ &\quad -(8z^3+12z^2+24z-12)s^5 \\ &\quad -(25z^3+25z^2+115z+55)s^4 \\ &\quad +(29z^3+14z^2+60z+16)s^3 \\ &\quad +(29z^3+61z^2+160z-3)s^2 \\ &\quad -(35z^3+z^2+31z+4)s \\ &\quad +8z^3-9z^2-30z+12. \end{aligned} \quad (30)$$

The ideals generated by the minors of the matrix $P(s,z)$ are given by

$$\begin{aligned} \mathcal{I}_1^{[P]} &= \mathcal{I}_2^{[P]} = \langle 1 \rangle, \\ \mathcal{I}_3^{[P]} &= \Big\langle -(2z^3-5z^2-12z-10)s^6 \\ &\quad -(8z^3+20z^2+25z-12)s^5 \\ &\quad -(12z^3-6z^2-101z+55)s^4 \\ &\quad +(21z^3-z^2+42z+16)s^3 \\ &\quad +(26z^3+63z^2+160z-3)s^2 \\ &\quad -(33z^3+8z^2+25z+4)s \\ &\quad +8z^3-9z^2-30z+12 \Big\rangle. \end{aligned} \quad (31)$$

Now,

$$\begin{aligned} P(s,z) &= \underbrace{\begin{bmatrix} 2 & 4 & -2 \\ -1 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix}}_{P_{0,0}} s^0 z^0 + \underbrace{\begin{bmatrix} -1 & -1 & 0 \\ 0 & -2 & -3 \\ 0 & -4 & -2 \end{bmatrix}}_{P_{0,1}} s^0 z^1 \\ &\quad + \underbrace{\begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 3 \\ 2 & 0 & -2 \end{bmatrix}}_{P_{1,0}} s^1 z^0 + \underbrace{\begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 3 \\ 2 & 0 & -2 \end{bmatrix}}_{P_{1,1}} s^1 z^1 \\ &\quad + \underbrace{\begin{bmatrix} -2 & 0 & 1 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}}_{P_{2,0}} s^2 z^0 + \underbrace{\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}}_{P_{2,1}} s^2 z^1 \end{aligned} \quad (32)$$

gives the matrix pencil $Q(s,z) = szE - sA_1 - zA_2 - A_0$, where the matrices E, A_0, A_1 and A_2 corresponding to (13)–(15) and (17) are given by

$$E \equiv \begin{bmatrix} 0_{3,6} \\ E_1 \end{bmatrix} \equiv \begin{bmatrix} 0_{3,3} & 0_{3,3} \\ P_{2,1} & P_{1,1} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -2 & 0 & 0 & 3 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & 1 & -2 & -1 & 0 & -5 \end{array}\right], \quad (33)$$

$$A_0 \equiv \begin{bmatrix} -I_3 & 0 \\ 0 & -P_{0,0} \end{bmatrix} = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 & -4 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right], \quad (34)$$

$$A_1 \equiv \begin{bmatrix} 0_{3,3} & I_3 \\ -P_{2,0} & -P_{1,0} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 2 & 0 & -1 & -2 & 4 & 0 \\ -3 & 0 & -1 & 0 & 0 & -3 \\ 1 & -2 & 0 & -2 & 0 & 2 \end{array}\right], \quad (35)$$

$$A_2 \equiv \begin{bmatrix} 0_{3,6} \\ A_{2,1} \end{bmatrix} \equiv \begin{bmatrix} 0_{3,3} & 0_{3,3} \\ 0_{3,3} & -P_{0,1} \end{bmatrix}$$

$$= \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 & 2 \end{array}\right] \tag{36}$$

By virtue of Theorem 1, the polynomial matrix $P(s,z)$ in (29) and the associated matrix pencil $Q(s,z)$ are related to the zero coprime equivalence transformation

$$M(s,z)P(s,z) = Q(s,z)N(s,z), \tag{37}$$

where

$$M(s,z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N(s,z) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

In fact, using polynomial matrix multiplication in Maple, it can be easily verified that

$$M(s,z)P(s,z) = Q(s,z)N(s,z). \tag{39}$$

The matrices $Q(s,z)$ and $M(s,z)$ are zero left coprime and the matrices $P(s,z)$ and $N(s,z)$ are zero right coprime since the matrices

$$\begin{bmatrix} Q(s,z) & M(s,z) \end{bmatrix}, \quad \begin{bmatrix} P(s,z) \\ N(s,z) \end{bmatrix} \tag{40}$$

have respectively a 6×6 and a 2×2 minor equal to 1.

The invariant polynomials of the matrix $Q(s,z)$ are given by

$$\begin{aligned} \Phi_1^{[Q]} &= \Phi_2^{[Q]} = \Phi_3^{[Q]} = 1, \\ \Phi_4^{[Q]} &= 1 = \Phi_1^{[P]}, \\ \Phi_5^{[Q]} &= 1 = \Phi_2^{[P]}, \\ \Phi_6^{[Q]} &= 2z^3 + 8z^2 + 13z + 10)s^6 \\ &\quad - (8z^3 + 12z^2 + 24z - 12)s^5 \\ &\quad - (25z^3 + 25z^2 + 115z + 55)s^4 \\ &\quad + (29z^3 + 14z^2 + 60z + 16)s^3 \\ &\quad + (29z^3 + 61z^2 + 160z - 3)s^2 \\ &\quad - (35z^3 + z^2 + 31z + 4)s \\ &\quad + 8z^3 - 9z^2 - 30z + 12 \\ &= \Phi_3^{[P]}, \end{aligned} \tag{41}$$

which agrees with Lemma 1.

The ideals generated by the minors of the matrix $Q(s,z)$ are given by

$$\begin{aligned} \mathcal{I}_1^{[Q]} &= \mathcal{I}_2^{[Q]} = \mathcal{I}_3^{[Q]} = \langle 1 \rangle, \\ \mathcal{I}_4^{[Q]} &= \langle 1 \rangle = \mathcal{I}_1^{[P]}, \\ \mathcal{I}_5^{[Q]} &= \langle 1 \rangle = \mathcal{I}_2^{[P]}, \\ \mathcal{I}_6^{[Q]} &= \big\langle -(2z^3 - 5z^2 - 12z - 10)s^6 \\ &\quad - (8z^3 + 20z^2 + 25z - 12)s^5 \\ &\quad - (12z^3 - 6z^2 - 101z + 55)s^4 \\ &\quad + (21z^3 - z^2 + 42z + 16)s^3 \\ &\quad + (26z^3 + 63z^2 + 160z - 3)s^2 \\ &\quad - (33z^3 + 8z^2 + 25z + 4)s \\ &\quad + 8z^3 - 9z^2 - 30z + 12 \big\rangle \\ &= \mathcal{I}_3^{[P]}, \end{aligned}$$

which is also in accordance with Lemma 2.

5. Conclusions

In this paper, a simple algorithm is presented for the computation of a matrix pencil which is equivalent to a given bivariate polynomial matrix. The resulting matrix pencil arises in the context of the theory of singular 2-D linear systems. The type and exact form of the equivalence linking the original matrix with its associated pencil were set out and shown to be of zero coprime equivalence. This transformation preserves the zero structure of the original polynomial matrix, making it possible to analyze the polynomial matrix in terms of its associated pencil form.

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