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ON TRACKING CONTROL OF RIGID AND FLEXIBLE JOINT ROBOTS

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The problem of (semi-global) trajectory tracking control of *rigid* and *flexible* joints robots using computed torque based controllers is considered. The paper contains two contributions; firstly, for rigid robots it is shown that velocity in the control algorithm can be replaced by the filtered position, thus obviating the explicit need for *state observers*. Secondly, it is shown that it is possible to achieve trajectory tracking of flexible joint robots *without calculating link jerk*. Removing the need for observers and the determination of link jerk reduces the controller complexity and enhances its robustness with respect to parameter uncertainty and unmodelled dynamics.

1. Introduction

Solutions to the tracking control problem of rigid robot manipulators have been known for many years now (see e.g. (Ortega and Spong, 1989) and (Wen and Bayard, 1988) and references therein). A drawback of these controllers is that they require the measurement of joint *velocities*, which is often contaminated with noise. Since, in practice, the values of the controller gains are limited by the noise level, the achievable performance is usually below par. In order to avoid the noise measurement problem, an *ad-hoc* solution is to numerically differentiate the position. Indeed, this solution is frequently employed in robotic applications today. However, besides the fact that there is no theoretical justification for such a solution, this reconstruction of velocity may be inadequate for low and high speeds (Belanger, 1992).

An alternative approach that has been considered in the literature is to design an observer that makes use of position information to reconstruct the velocity signal. Then, the controller is implemented replacing the velocity measurement by its estimate. It is interesting to note that even though it is well known that certainty equivalence does not apply to non-linear systems[1], the rationale behind this approach is precisely that the estimate will converge to the true signal and this will in turn entail stability of the closed loop. In (Nicosia and Tomei, 1990) a non-linear observer, that reproduces the whole robot dynamics, is used in a PD plus gravity

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[1] Specifically, an observer that asymptotically reconstructs the state of a non-linear system does not guarantee that a given stabilizing state feedback law will remain stable when using the estimated state instead of the true one.

compensation scheme. The authors prove that the equilibrium is *locally* asymptotically stable provided that the observer gain satisfies some lower bound determined by the robot parameters and the trajectories error norms (see also Canudas *et al.*, 1990). In (Berghuis and Nijmeijer, 1992) it is shown that adding a term proportional to the observation error to a PD plus gravity compensation controller allows us to use a *linear* observer still preserving the local asymptotic stability for sufficiently high gains. To the best of our knowledge, all existing solutions to the tracking control problem without velocity measurement require the use of observers and the injection of high gains to increase the basin of attraction.

Another factor that hampers the behaviour of robot controllers is the presence of *joint flexibilities* caused by harmonic drives, shaft win dup, bearing deformation, and compressibility of the fluid in hydraulic robots.

Several solutions to this problem have appeared in the litterature, for a good review of the state of the art see (Brogliato *et al.*, 1993). Among these, we can distinguish two different types of schemes which are the *computed torque* and the *sliding mode control* algotithm by Slotine and Li (1988); then, various variants stem from them. A family of controllers derived from the computed torque scheme is given in (Lanari and Wen, 1992; Lanari *et al.*, 1993). Nevertheless, any of these existing solutions, despite the theoretical interest they present, are quite complicated to be implemented.

Based on a controller first described in (Wen and Bayard, 1988) for the rigid case, we present in this paper two distinct results: firstly, we show that by simply adding a filter of the n-th order (n being the number of degrees of freedom), in the rigid case velocity measurements are no longer needed, thus obviating the necessity for observers. Secondly, we show that by applying a similar control law to flexible joints robots, the calculation of jerk can be removed. We prove in both cases that for any set of initial conditions it is always possible to find a controller such that the closed loop system tends exponentially to a unique equilibrium point.

For a good reference to this technique which is known as approximate differentiation (or "dirty" derivatives) see (Kelly *et al.*, 1994), where a global asymptotic stability has been proven for the regulation problem.

The organization of this paper is as follows. In the next section we formulate our problem, in Section 3 we give our main result, in Section 4 we develop the proof of our main result, in Section 5 we conclude with some important remarks.

Notation: $\|\cdot\|$ – Euclidean norm; $\mathcal{L}_2^n$ – space of n-dimensional square integrable functions, $\mathcal{L}_\infty^n$ – space of n-dimensional bounded functions. For further details and definitions see (Desoer and Vidyasagar, 1975). We will also define $\mathcal{D}^+ = \{M \in \mathbb{R}^{n \times n} | M_{i,j} = 0 \; \forall \, i \neq j, \;\; M_{i,j} > 0 \; \forall \, i = j\}$, $\underline{\lambda}$ denotes the smallest eigenvalue, whereas $\bar{\lambda}$ denotes the maximum one.

2. Problem Formulation

Throughout this paper we will consider the *simplified model* of an n-link robot with flexible joints proposed in (Spong, 1987), which assumes that the angular part of the kinetic energy of each rotor is due only to its own rotation[2], and is given by

$$\begin{aligned} D_l(q_1)\ddot{q}_1 + C(q_1,\dot{q}_1)\dot{q}_1 + g(q_1) &= K(q_2 - q_1) \\ J\ddot{q}_2 + K(q_2 - q_1) &= u \end{aligned} \tag{1}$$

where $q_1 \in \mathbb{R}^n$ and $q_2 \in \mathbb{R}^n$ represent the link angles and motor angles, respectively, $D(q_1) > 0$ is the $n \times n$ inertia matrix for the rigid links, $J \in \mathcal{D}^+$ is the matrix of actuator inertias reflected to the link side of the gears, $C(q_1,\dot{q}_1)\dot{q}_1$ represents the Coriolis and centrifugal forces, the vector $g(q_1)$ contains the gravitational terms, and $K \in \mathcal{D}^+$ contains the joint stiffness coefficients. As suggested in (Spong and Vidyasagar, 1989) $C(q_1,\dot{q}_1)$ is defined via the Christoffel symbols of the first kind so model (1) has the following properties:

P1. The matrix $D(q_1)$ is positive definite and the matrix $N = \dot{D}(q_1) - 2C(q_1,\dot{q}_1)$ is skew symmetric.

P2. Since the matrix $C(x,y)$ is bounded in x and linear in y, for a vector $z \in \mathbb{R}^n$ we are able to write:

$$C(x,y)z = C(x,z)y$$

$$C(x,y) \le k_c\|y\|, \qquad k_c > 0$$

In the case of negligible flexibility $(K \to \infty)$ it is shown in (Spong, 1987) that model (1) reduces to

$$D(q_1)\ddot{q}_1 + C(q_1,\dot{q}_1)\dot{q}_1 + g(q_1) = u \tag{2}$$

where $D(q_1) = D_l(q_1) + J$.

Once we have defined the rigid and flexible joint robot model we will use throughout this paper, let us define the problems we deal with in the following sections.

Output Feedback Tracking Control of Rigid Joint Robots (OF/RR)
For system (2) assume that *only the link position* is available for measurement. Under these conditions, define an internally stable (smooth) control law (whose gains may depend on the system initial conditions) that insures

$$\lim_{t\to\infty} \tilde{q}_1(t) = \lim_{t\to\infty} (q_1(t) - q_{1d}(t)) = 0 \tag{3}$$

for all $q_{1d} \in \mathcal{C}^4$, $\|q_{1d}(t)\|$, $\|\dot{q}_{1d}(t)\|$, $\|\ddot{q}_{1d}(t)\| < B_d$

State Feedback Tracking Control of Flexible Joints Robots
(SF/FR) For system (1) assume that the full state is available for measurement. Then, find an internally stable control law (whose gains may depend on the system's initial conditions) that, without the requirement of calculating $q_1^{(3)}$, insures (3) for all $q_{1d} \in \mathcal{C}^4$, $\|q_{1d}(t)\|$, $\|\dot{q}_{1d}(t)\|$, $\|\ddot{q}_{1d}(t)\| < B_d$

[2] See e.g. (Tomei, 1991b) for a model that relaxes this assumption.

Remark 1. In modern terminology control laws which satisfy the conditions of the above definition are referred to as *semi-globally stable.*

3. Main Result

Considering the model described in the previous section and properties P1 and P2 let us enunciate the following proposition

Proposition 1. *Consider model (2) in closed loop with the control law*

$$u = D(q_1)\ddot{q}_{1d} + C(q_1, \dot{q}_{1d})\dot{q}_{1d} + g(q_1) - K_P\tilde{q}_1 - K_D\tilde{\vartheta} \tag{4}$$

$$\tilde{\vartheta} = \operatorname{diag}\left\{\frac{b_i p}{p + a_i}\right\}\tilde{q}_1 \tag{5}$$

where

$$A = \operatorname{diag}\{a_i\}, \qquad K_P,\ K_D \in \mathcal{D}^+ \tag{6}$$

and[3]

$$B = \operatorname{diag}\{b_i\}, \qquad b_i > \frac{\bar{\lambda}(D)}{\beta_2 \underline{\lambda}(D)} \tag{7}$$

and β_2 a constant such that $0 < \beta_2 < 1$. Then, for any (bounded) initial condition $x_0 = [\tilde{q}_1(0)^T, \dot{\tilde{q}}_1(0)^T, \tilde{\vartheta}(0)^T]^T$ some sufficiently large (bounded) gains of controller (4), (5) always exist such that (3) holds with a domain of attraction including

$$\{x \in \mathbb{R}^{3n} : \|x\| < c_1\} \tag{8}$$

where $\lim_{\underline{\lambda}(B)\to\infty} c_1 = \infty$.

Proof. See Section 4.1.1.

Remark 2. Let us notice that (4) is controller (4.2) proposed in (Wen and Bayard, 1988) where we have simply replaced the measurement of velocity in the last term by its approximate derivative (5). Furthermore, in the case of an invariant reference $\dot{q}_{1d}(t) = 0$, control law (4) reduces to (2) proposed in (Kelly *et al.*, 1994) which has been proven to be *globally asymptotically stable.*

Proposition 2. *Consider model (1) in a closed loop with the control law*

$$u = J\ddot{q}_{2d} + K(q_{2d} - q_{1d}) - K_{P2}\tilde{q}_2 - K_{D2}\tilde{\vartheta}_2 \tag{9}$$

$$\tilde{\vartheta}_j = \operatorname{diag}\left\{\frac{b_{ij} p}{p + a_{ij}}\right\}\tilde{q}_j, \qquad j = 1,2 \tag{10}$$

[3] There is an additional technical requirement that will always be satisfied in practice, namely that $|\bar{\lambda}(\cdot) - \underline{\lambda}(\cdot)|$ be uniformly bounded for A, B, K_P, K_D.

where we define

$$q_{2d} = K^{-1}\Big[D(q_1)\ddot{q}_{1d} + C(q_1, \dot{q}_{1d})\dot{q}_{1d} + g(q_1) - K_{P1}\tilde{q}_1 - K_{D1}\tilde{\vartheta}_1\Big] + q_{1d}$$

and

$$A_j = \operatorname{diag}\{a_{ij}\}, \qquad K_{Pj},\ K_{Dj} \in \mathcal{D}^+ \tag{11}$$

$$B_j = \operatorname{diag}\{b_i\}, \qquad b_i > \frac{\bar{\lambda}(\mathcal{D})}{\beta_2 \underline{\lambda}(\mathcal{D})}, \qquad \mathcal{D} := \operatorname{blockdiag}[D_l \ \ J] \tag{12}$$

Then, for any (bounded) initial conditions $x_0 = [\tilde{q}(0)^T, \dot{\tilde{q}}(0)^T, \tilde{\vartheta}(0)^T]^T$ *where* $q = [q_1^T, q_2^T]^T$ *and* $\tilde{\vartheta} = [\tilde{\vartheta}_1^T, \tilde{\vartheta}_2^T]^T$ *some sufficiently large (bounded) gains for controller (9) always exist such that (3) holds. Furthermore, a domain of atraction is defined by*

$$\left\{x \in \mathbb{R}^{6n} : \|x\| < c_2\right\} \tag{13}$$

where $\lim_{\underline{\lambda}(B)\to\infty} c_2 = \infty$.

Proof. See Section 4.1.5.

Remark 3. Let us notice that the calculation of u in (9) requires $\ddot{q}_{2d}$, however, in contrast to all existing solutions to this problem, our controller does not require the calculation of $q_1^{(3)}$. This stems from the use of $\dot{q}_{1d}$ instead of $\dot{q}_1$ in the second right-hand term of q_{2d}, and the use of the filter. The second derivative of q_{2d} still needs link acceleration and velocity. Yet, only link velocity is considered to be available for measurement and acceleration can be computed using the first equation of (1).

4. Proof of the Main Result

4.1. Proof of Proposition 1

The proof relies on the *classical Lyapunov theory* and is divided into four parts. First, we define a suitable error equation for the closed loop system, whose (unique) equilibrium is at the desired value. Then, we propose a Lyapunov function candidate. Thereafter, we prove that under the conditions of the theorem the proposed function is a Lyapunov function, and establish the exponential stability of the equilibrium invoking Lyapunov's second method. Finally, we define the domain of attraction.

4.1.1. Error Equation

Using property P2 from Section 2 we write the error equation of (2), (4) and (5) as

$$\begin{aligned} D\ddot{\tilde{q}}_1 + (C + C_d)\dot{\tilde{q}}_1 + K_P\tilde{q}_1 + K_D\tilde{\vartheta} &= 0 \\ \dot{\tilde{\vartheta}} &= -A\tilde{\vartheta} + B\dot{\tilde{q}}_1 \end{aligned} \tag{14}$$

where for simplicity we have omitted the arguments and $C_d = C(q_1, \dot{q}_{1d})$. Let us notice that a solution of (14) is $x = [\tilde{q}_1^T,\ \dot{\tilde{q}}_1^T,\ \tilde{\vartheta}^T]^T = [0\ 0\ 0]^T$. We will now construct a Lyapunov function for (14) which has an absolute minimum at the origin and whose time derivative is negative definite in the whole domain.

4.1.2. Lyapunov Function Candidate

Consider the function

$$V(\tilde{x}(t),t) = \frac{1}{2}\dot{\tilde{q}}_1^T D\dot{\tilde{q}}_1 + \frac{1}{2}\tilde{q}_1^T K_P \tilde{q}_1 + \frac{1}{2}\tilde{\vartheta}^T K_D B^{-1}\tilde{\vartheta} + \epsilon\tilde{q}_1^T D\dot{\tilde{q}}_1 - \epsilon\tilde{\vartheta}^T D\dot{\tilde{q}}_1 \quad (15)$$

We will now give sufficient conditions in order to guarantee *positive definiteness* of V. To make the proof easy, we will partition V as $V = W_1 + W_2$ where

$$W_1 = \frac{1}{4}\dot{\tilde{q}}_1^T D\dot{\tilde{q}}_1 + \frac{1}{4}\tilde{q}_1^T K_P \tilde{q}_1 + \frac{1}{4}\tilde{\vartheta}^T K_D B^{-1}\tilde{\vartheta} + \epsilon\tilde{q}_1^T D\dot{\tilde{q}}_1 - \epsilon\tilde{\vartheta}^T D\dot{\tilde{q}}_1 \quad (16)$$

$$W_2 = \frac{1}{4}\dot{\tilde{q}}_1^T D\dot{\tilde{q}}_1 + \frac{1}{4}\tilde{q}_1^T K_P \tilde{q}_1 + \frac{1}{4}\tilde{\vartheta}^T K_D B^{-1}\tilde{\vartheta} \quad (17)$$

Let us notice that eqn. (16) can be rewritten in the matrix form as

$$\begin{aligned} W_1 &= \frac{1}{4}\begin{bmatrix}\tilde{q}_1\\ \dot{\tilde{q}}_1\end{bmatrix}^T \underbrace{\begin{bmatrix}K_P & 2\epsilon D\\ 2\epsilon D & \frac{1}{2}D\end{bmatrix}}_{P_1}\begin{bmatrix}\tilde{q}_1\\ \dot{\tilde{q}}_1\end{bmatrix} \\ &+\frac{1}{4}\begin{bmatrix}\dot{\tilde{q}}_1\\ \tilde{\vartheta}\end{bmatrix}^T \underbrace{\begin{bmatrix}\frac{1}{2}D & -2\epsilon D\\ -2\epsilon D & K_D B^{-1}\end{bmatrix}}_{P_2}\begin{bmatrix}\dot{\tilde{q}}_1\\ \tilde{\vartheta}\end{bmatrix}\end{aligned} \quad (18)$$

From the definition of K_P, P_1 is positive definite if

$$\frac{1}{2}\sqrt{\frac{\underline{\lambda}(K_P)}{2\overline{\lambda}(D)}} > \epsilon \quad (19)$$

Similarly, from the definitions of K_D and B, P_2 is positive definite if

$$\frac{1}{2}\sqrt{\frac{\underline{\lambda}(K_D B^{-1})}{2\overline{\lambda}(D)}} > \epsilon \quad (20)$$

while W_2 is trivially positive definite.

Furthermore, by virtue of (19) and (20) and the nature of D we can prove that W_1 is strictly convex, hence radially unbounded by simply looking at the Hessian matrix which satisfies

$$\frac{\partial^2 W_1}{\partial x^2} = \begin{bmatrix}K_P & 2\epsilon D & 0\\ 2\epsilon D & D & -2\epsilon D\\ 0 & -2\epsilon D & K_D B^{-1}\end{bmatrix} \geq \eta I_{3n}, \qquad \forall x \in \mathbb{R}^{3n}$$

where $\eta > 0$. In a similar way, from (6), and (7) W_2 is trivially strictly convex.

4.1.3. Lyapunov Function Derivative and Exponential Stability

In this subsection we show that the time derivative of (15) along the trajectories of (14) is *locally* negative definite in the whole state $x = [q^T,\ \dot{q}^T,\ \tilde{\vartheta}^T]^T$ so it is exponentially stable. To this end, we first use properties **P1** and **P2** of Section 2 to write

$$
\begin{aligned}
\dot{V} \leq & -\tilde{\vartheta}^T K_D B^{-1} A \tilde{\vartheta} + \tilde{q}_1 C_d \dot{\tilde{q}}_1 + \epsilon \dot{\tilde{q}}_1^T D \dot{\tilde{q}}_1 + \epsilon \tilde{q}_1^T C(q_1, \dot{\tilde{q}}_1)\dot{\tilde{q}}_1 - \epsilon \tilde{q}_1^T K_P \tilde{q}_1 \\
& -\epsilon \tilde{q}_1^T K_D \tilde{\vartheta} + +\epsilon \tilde{\vartheta}^T A^T D \dot{\tilde{q}}_1 - \epsilon \dot{\tilde{q}}_1^T B^T D \dot{\tilde{q}}_1 - \epsilon \tilde{\vartheta}^T C(q_1, \dot{\tilde{q}}_1)\dot{\tilde{q}}_1 \\
& +\epsilon \tilde{\vartheta}^T K_P \tilde{q}_1 + \epsilon \tilde{\vartheta}^T K_D \tilde{\vartheta}
\end{aligned} \tag{21}
$$

By virtue of same properties we can establish the following bounds:

$$
\begin{aligned}
\epsilon \dot{\tilde{q}}_1^T D \dot{\tilde{q}}_1 &\leq \epsilon \bar{\lambda}(D) \|\dot{\tilde{q}}_1\|^2 \\
\epsilon \tilde{q}_1^T C(q_1, \dot{\tilde{q}}_1)\dot{\tilde{q}}_1 &\leq \epsilon k_c \|\tilde{q}_1\| \|\dot{\tilde{q}}_1\|^2 \\
-\epsilon \tilde{q}_1^T K_P \tilde{q}_1 &\leq -\epsilon \underline{\lambda}(K_P) \|\tilde{q}_1\|^2 \\
-\epsilon \tilde{q}_1^T K_D \tilde{\vartheta} &\leq \epsilon \bar{\lambda}(K_D) \|\tilde{q}_1\| \|\tilde{\vartheta}\| \\
\epsilon \tilde{\vartheta}^T A^T D \dot{\tilde{q}}_1 &\leq \epsilon \bar{\lambda}(A) \bar{\lambda}(D) \|\tilde{\vartheta}\| \|\dot{\tilde{q}}_1\| \\
-\epsilon \dot{\tilde{q}}_1^T B^T D \dot{\tilde{q}}_1 &\leq -\epsilon \underline{\lambda}(B) \underline{\lambda}(D) \|\dot{\tilde{q}}_1\|^2 \\
-\epsilon \tilde{\vartheta}^T C(q_1, \dot{\tilde{q}}_1)\dot{\tilde{q}}_1 &\leq \epsilon k_c \|\tilde{\vartheta}\| \|\dot{\tilde{q}}_1\|^2 \\
\epsilon \tilde{\vartheta}^T K_P \tilde{q}_1 &\leq \epsilon \bar{\lambda}(K_P) \|\tilde{\vartheta}\| \|\tilde{q}_1\| \\
\epsilon \tilde{\vartheta}^T K_D \tilde{\vartheta} &\leq \epsilon \bar{\lambda}(K_D) \|\tilde{\vartheta}\|^2
\end{aligned}
$$

To this end, let us define some constants $\beta_i > 0,\ \sum_{i=1}^{3} \beta_i = 1;\ \gamma_i > 0,\ \gamma_1 + \gamma_2 = 1$ and using the previous bounds write (21) in the form

$$
\begin{aligned}
\dot{V} \leq & -\frac{\epsilon}{2} \begin{bmatrix} \|\tilde{q}_1\| \\ \|\tilde{\vartheta}\| \end{bmatrix}^T \overbrace{\begin{bmatrix} 2\underline{\lambda}(K_P) & -\bar{\lambda}(K_P) - \bar{\lambda}(K_D) \\ -\bar{\lambda}(K_P) - \bar{\lambda}(K_D) & \frac{\gamma_1}{\epsilon}\underline{\lambda}(K_D B^{-1} A) \end{bmatrix}}^{Q_1} \begin{bmatrix} \|\tilde{q}_1\| \\ \|\tilde{\vartheta}\| \end{bmatrix} \\
& -\frac{\epsilon}{2} \begin{bmatrix} \|\tilde{\vartheta}\| \\ \|\dot{\tilde{q}}_1\| \end{bmatrix}^T \overbrace{\begin{bmatrix} \frac{\gamma_1}{\epsilon}\underline{\lambda}(K_D B^{-1} A) & -\bar{\lambda}(A)\bar{\lambda}(D) \\ -\bar{\lambda}(A)\bar{\lambda}(D) & 2\beta_1 \underline{\lambda}(B)\underline{\lambda}(D) \end{bmatrix}}^{Q_2} \begin{bmatrix} \|\tilde{\vartheta}\| \\ \|\dot{\tilde{q}}_1\| \end{bmatrix} \\
& -\left[\overbrace{\left[\epsilon\beta_3 \underline{\lambda}(B)\underline{\lambda}(D) - k_c B_d\right]}^{\lambda_1} + \epsilon \overbrace{\left[\beta_2 \underline{\lambda}(B)\underline{\lambda}(D) - \bar{\lambda}(D) - k_c(\|\tilde{\vartheta}\| + \|\tilde{q}_1\|)\right]}^{\lambda_2} \right] \|\dot{\tilde{q}}_1\|^2 \\
& - \overbrace{\left[\gamma_2 \underline{\lambda}(K_D B^{-1} A) - \epsilon \bar{\lambda}(K_D)\right]}^{\lambda_3} \|\tilde{\vartheta}\|^2
\end{aligned} \tag{22}
$$

We derive now sufficient conditions for $\dot{V}$ to be *locally* negative definite. First, considering (6) and (7), the matrix Q_1 is positive definite if

$$\frac{2\gamma_1 \underline{\lambda}(K_P)\underline{\lambda}(K_D B^{-1} A)}{\left[\bar{\lambda}(K_P) + \bar{\lambda}(K_D)\right]^2} > \epsilon \tag{23}$$

In a similar way, $Q_2 > 0$ if

$$\frac{2\gamma_1 \beta_1 \underline{\lambda}(K_D AB)\underline{\lambda}(D)}{\bar{\lambda}(B)[\bar{\lambda}(A)\bar{\lambda}(D)]^2} > \epsilon \tag{24}$$

Let us note that the positivity of constant λ_1 imposes a lower bound on ϵ, i.e.

$$\frac{k_c B_d}{\beta_3 \underline{\lambda}(B)\underline{\lambda}(D)} \leq \epsilon \tag{25}$$

while $\lambda_2 \geq 0$ if

$$\frac{1}{2k_c}\left[\beta_2 \underline{\lambda}(B)\underline{\lambda}(D) - \bar{\lambda}(D)\right] > ||x|| \tag{26}$$

Hence, condition (7) results. Finally, λ_3 is positive if

$$\frac{\underline{\lambda}(K_D A)}{2\bar{\lambda}(K_D B)} > \epsilon \tag{27}$$

Conditions (19), (20), (23), (24) and (27) are satisfied for ϵ sufficiently small, while (25) is satisfied for a B suficiently large. Thus, it is always posible to find some controller gains depending on initial conditions and the desired trajectories to insure that all the above-written inequalities hold. Therefore, (21) is *locally* negative definite and the equilibrium is exponentially stable in the sense of Lyapunov.

Remark 4. Finally, let us note that in contrast to (Berghuis *et al.*, 1992) the constant ϵ is not used in the controller but only for stability proof purposes. For an interesting historical review of this technique see (Koditschek, 1989). Some other interesting papers using cross terms in the Lyapunov function are (Kelly, 1993) and (Tomei, 1991a).

4.1.4. Domain of Attraction

In this section we define the domain of attraction and we prove that it can be enlarged by increasing the controller gains. For this, we will first find some positive constants α_1, α_2 such that

$$\alpha_1 ||x(t)||^2 \leq V(x(t), t) \leq \alpha_2 ||x(t)||^2 \tag{28}$$

Let us notice that from (19) and (20) we have

$$V \geq W_2 \geq \frac{1}{4}\left[\underline{\lambda}(K_P)||\tilde{q}_1||^2 + \underline{\lambda}(K_D B^{-1})||\tilde{\vartheta}||^2 + \underline{\lambda}(D)||\dot{q}||^2\right]$$

so we define α_1 as

$$\alpha_1 := \frac{1}{4}\min\{\underline{\lambda}(K_P),\ \underline{\lambda}(K_D B^{-1}),\ \underline{\lambda}(D)\}$$

In a similar manner, an upper bound on (15) is

$$V \le \left[\frac{\epsilon}{2}\bar{\lambda}(D) + \frac{1}{2}\bar{\lambda}(K_P)\right]||\tilde{q}_1||^2 + \left[(\epsilon + \frac{1}{2})\bar{\lambda}(D)\right]||\dot{\tilde{q}}_1||^2 + \frac{1}{2}\left[\epsilon\bar{\lambda}(D) + \bar{\lambda}(K_D B^{-1})\right]||\tilde{\vartheta}||^2$$

so we define

$$\alpha_2 := \max\left\{\left[\frac{\epsilon}{2}\bar{\lambda}(D) + \frac{1}{2}\bar{\lambda}(K_P)\right],\ \left[(\epsilon + \frac{1}{2})\bar{\lambda}(D)\right],\ \frac{1}{2}\left[\epsilon\bar{\lambda}(D) + \bar{\lambda}(K_D B^{-1})\right]\right\}$$

From (26) and (28) we conclude that the domain of attraction contains the set

$$||x|| \le c_1 = \frac{1}{2k_c}\left[\beta_2\underline{\lambda}(B)\underline{\lambda}(D) - \bar{\lambda}(D)\right]\sqrt{\frac{\alpha_1}{\alpha_2}}$$

4.1.5. Semiglobal Stability

To establish *semiglobal stability*, we must prove that, with a suitable choice of the controller gains, we can arbitrarily enlarge the domain of attraction. To this end, we propose to increase $\underline{\lambda}(B)$. Thus, the proof is completed checking that $\lim_{\underline{\lambda}(B)\to\infty} c_1 = \infty$.

To this end, let us notice that

$$\lim_{\underline{\lambda}(B)\to\infty} \alpha_1 = c_3\bar{\lambda}^{-1}(B), \qquad \lim_{\underline{\lambda}(B)\to\infty} \alpha_2 = c_4$$

where c_3, c_4 are constants independent of B. Consequently,

$$\lim_{\underline{\lambda}(B)\to\infty} c_1 = \lim_{\underline{\lambda}(B)\to\infty} c_5\frac{\underline{\lambda}(B)}{\sqrt{\bar{\lambda}(B)}} = \infty$$

where c_3 is also independent of B and to get the last identity we have used the fact that $|\bar{\lambda}(B) - \underline{\lambda}(B)|$ is uniformly bounded.

4.2. Proof of Proposition 2

In a similar way as in the previous proof, the error equation of (1) and (9) is

$$\begin{aligned} &\mathcal{D}\ddot{\tilde{q}} + (\mathcal{C} + \mathcal{C}_d)\dot{\tilde{q}} + \mathcal{K_P}\tilde{q} + \mathcal{K_D}\tilde{\vartheta} = 0 \\ &\dot{\tilde{\vartheta}} = -\mathcal{A}\tilde{\vartheta} + \mathcal{B}\dot{\tilde{q}} \end{aligned} \tag{29}$$

where for simplicity we have omitted the arguments and $\tilde{q} = [\tilde{q}_1^T, \tilde{q}_2^T]^T$, $\tilde{\vartheta} = [\tilde{\vartheta}_1^T, \tilde{\vartheta}_2^T]^T$, $\mathcal{K_D} = \text{blockdiag}\{K_{D1}, K_{D2}\}$, $\mathcal{A} = \text{blockdiag}\{A_1, A_2\}$, $\mathcal{B} = \text{blockdiag}\{B_1, B_2\}$, and

$$\mathcal{C} = \begin{bmatrix} C(q_1, \dot{q}_1) & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C}_d = \begin{bmatrix} C(q_1, \dot{q}_{1d}) & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{K_P} = \begin{bmatrix} K_{P1} + K & -K \\ -K & K_{P2} + K \end{bmatrix}$$

Properties P1 and P2 hold for $\mathcal{D}$, $\mathcal{C}$, as well as for $\mathcal{C}_d$. It is obvious that $\mathcal{K_P} > 0$ if $K_{P1} > 0$ and $K_{P2} > 0$. Let us notice that, as before, a solution of (29) is $\tilde{x} = [\tilde{q}^T, \dot{\tilde{q}}^T, \tilde{\vartheta}^T]^T = [0\ 0\ 0]^T$. Now, consider the Lyapunov candidate function

$$V(x(t), t) = \frac{1}{2}\dot{\tilde{q}}^T \mathcal{D} \dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T \mathcal{K_P} \tilde{q} + \frac{1}{2}\tilde{\vartheta}^T \mathcal{K_D} B^{-1} \tilde{\vartheta} + \epsilon \tilde{q}^T \mathcal{D} \dot{\tilde{q}} - \epsilon \tilde{\vartheta}^T \mathcal{D} \dot{\tilde{q}} \quad (30)$$

and note that (29) is similar to (14), as well as (30) is similar to (15). The main difference between this equations is that $\mathcal{K_P}$ is no longer diagonal, yet, it is positive definite so all the conditions (19), (20), (23), (24), (25) and (27) apply *mutatis mutandi* to this case. Thus, the rest of the proof is based on similar arguments to those of the previous section.

5. Concluding Remarks

An alternative solution to the tracking control problem for rigid joint robots *without velocity measurements* has been given. The main contribution of our approach is to show that by simply replacing the velocity by its approximate derivative in a computed torque based scheme it is still possible to reach semiglobal exponential stability, thus obviating the necessity for observers.

In the case when flexibility in the joints cannot be neglected, we have used a similar computed torque based controller and we have proven that semiglobal exponential stability is still reachable by using the approximate derivatives. Its main feature is that it does not require calculation of jerk.

On the one hand, the simplicity of this controller makes it attractive for industrial aplications and on the other hand, from the theoretical point of view, even though the rate of convergence depends on the initial conditions, it has been shown that it is always possible to tune the filter gain in such a way that $\tilde{x} \underset{t\to\infty}{\longrightarrow} 0$ exponentially.

Appendix

Simulation Results

In this section we present some simulation results. We used the two link robot arm model of (Berghuis and Nijmeijer, 1992). Figure 1 shows the first link trajectory as well as the reference. In this case we fixed the controller gains to K_P =diag([5000 6000]) and K_D=diag([7000 8800]) and the filter parameters to A =diag([1000 1000]) and B =diag([1000 1000]). The reference followed in both cases is $q_{1d} = [\frac{1}{10\pi}\sin(10\pi t)\ ;\ \frac{1}{10\pi}\sin(10\pi t)]$. Figure 2 shows the response for the case when flexibility is not neglected thus set to K =diag([10000 10000]). In this

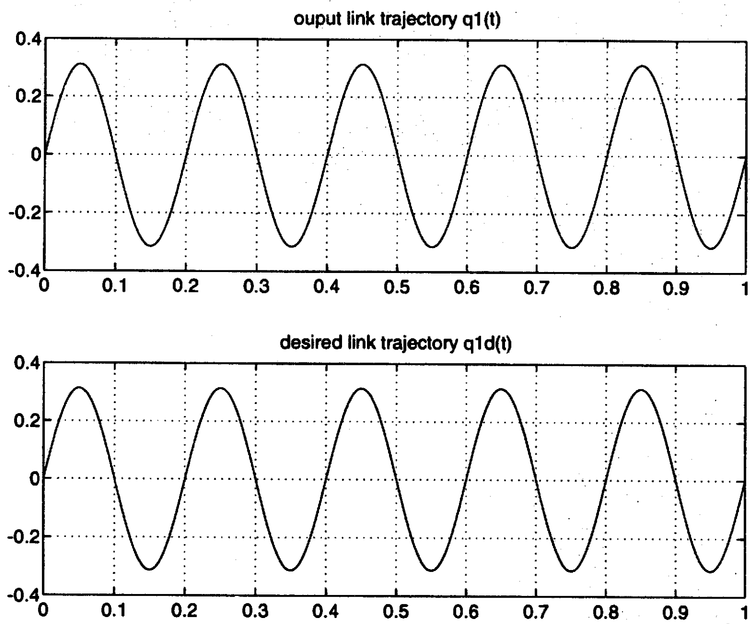

Fig. 1. Control law (4) without velocity measurement. Rigid Joints.

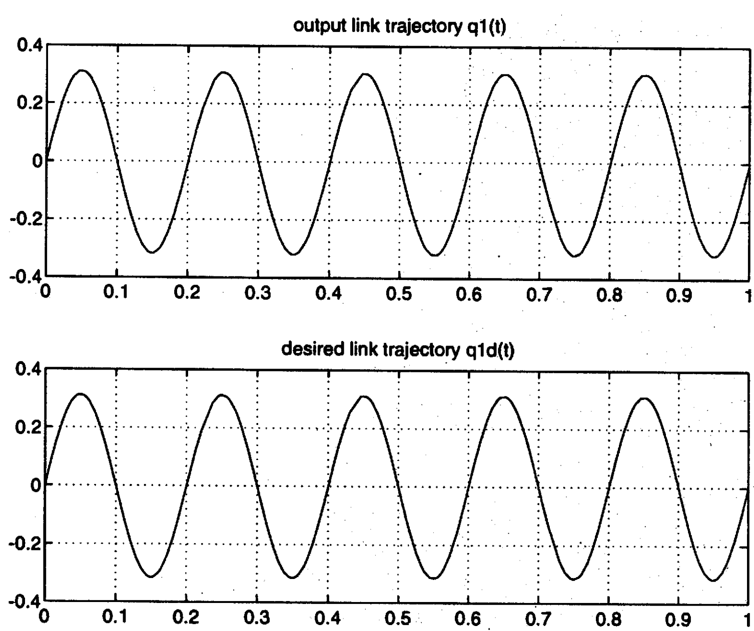

Fig. 2. Control law (8)–(10). Full state feedback. Flexible Joints.

case we have set K_{Pj} =diag([10000 10000]), K_{Dj}=diag([7000 8800]) and we have considered actuators inertias to be J =diag([0.10 0.10]).

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