Appl. Math. and Comp. Sci., 1995, vol.5, No.2, 391–407

NORMAL FORMS OF KINEMATIC SINGULARITIES OF 3R ROBOT MANIPULATORS

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Using the singularity theory approach we study kinematic singularities of 3R robot manipulators satisfying certain geometric conditions. A classification of singular configurations is set forth and three mathematical models (normal forms) of the kinematics around singular configurations are derived.

1. Introduction

Singular configurations of robot manipulators are commonly defined as these positions of the manipulator's joints at which the end-effector is not capable of moving in certain directions, i.e. the manipulator loses one or more degrees of freedom. The presence of singular configurations (kinematic singularities) is responsible for making ill-conditioned the otherwise effective algorithms of trajectory planning or tracking control of robot manipulators (Nakamura, 1991; Spong and Vidyasagar, 1989; Tchoń, 1993). On the other hand, the existence of kinematic singularities is an intrinsic geometric property of robot manipulators, (Gottlieb, 1986), that cannot be annihilated by any skilful mechanical design.

Within the last decade there has been a growing interest among roboticians directed towards achieving a better understanding of the manipulator's behaviour in a vicinity of singular configurations, motivated both by kinematic and control problems in robotics. We refer the reader to (Tchoń, 1991) and especially to (Kieffer, 1994) for an up-to-date and systematic introduction into this subject as well as for an extensive bibliography.

When looking at the problem of kinematic singularities from the mathematical point of view, it becomes quite natural to approach the problem using the tools of singularity theory. Several publications have recently appeared, situated within or close to this approach (Kieffer, 1992; 1994; 1995; Tchoń, 1991; Tchoń and Urban, 1992; Pai and Leu, 1992). The main advantage of the singularity theory approach is that it can offer mathematically strict local models of manipulator's kinematics, valid in some neighbourhood of singular configurations (Tchoń, 1991). These models are called *normal forms* of kinematics. Their basic idea is to simplify the kinematic map as far as possible while retaining at the same time all qualitative properties of the kinematics. More specifically, if the map

$$y = k(x)$$

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represents the kinematics in some coordinates in the internal and external manifolds (so $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^6$), then a normal form of k around a configuration x_0 is obtained by changing coordinates x around x_0 and y around $k(x_0)$,

$$x \mapsto \varphi(x), \quad y \mapsto \psi(y)$$

in such a way that the map

$$k_0 = \psi \circ k \circ \varphi^{-1}$$

be as simple as possible. In general, transforming the given kinematics to a nice normal form is a difficult problem. Under the assumption of structural stability of kinematics a list of so-called candidate normal forms has been derived in (Tchoń, 1991). A justification of the candidate normal forms in the case of planar kinematics has been provided in (Tchoń and Urban, 1992). However, since manipulator's kinematics are rather specific maps enjoying apparent factorization and symmetry properties, it should not at all be expected that every kinematics will be structurally stable. In this aspect the results of (Tchoń, 1991) need to be complemented. A preliminary step in this direction has already been made in (Tchoń, 1992; 1993), where a structurally unstable normal form of the double pendulum with equal link lengths is derived, and a study of spatial kinematics initiated. In this paper, due to both the improved mathematical analysis as well as intensified use of symbolic computations we have managed to construct a fairly complete collection of normal forms for spatial 3R kinematics with parallel axes of joints 2 and 3, whose Jacobian matrix at singular configurations drops rank exactly by 1. Such singular configurations are usually referred to as *ordinary*. Our main result (Theorem 1) describes the normal forms in detail. There are three normal forms, qualitatively different. The first of them (F1) appears to be *structurally stable* (i.e. insensitive to any, sufficiently small, perturbations of the kinematics), the second form (F2) can be called *geometrically stable* (i.e. within the class of kinematics with preassumed geometry (F2), although not structurally stable, is insensitive with regard to small variations of manipulator's geometric parameters), the third form (F3) is geometrically unstable.

This paper is composed in the following way. Its core section is Section 2. In this section we present the kinematics to be examined, define and classify kinematic singularities, introduce a preliminary normal form and, after some mathematical analysis of this form accompanied by considerable amount of symbolic computations, we state and prove the main result. This result is discussed in Section 3 concluding the paper. Special attention in Section 3 is paid to the presentation of bifurcation diagram of the geometrically stable form (F2).

2. Problem Statement and Main Result

We shall consider a general 3 d.o.f. robot manipulator with unlimited revolute joints. By assigning in the standard way coordinate frames to the base of the manipulator, to its joints and to the end-effector, we are able to characterize the manipulator's kinematics by the following Denavit-Hartenberg data (Paul, 1981; Spong and Vidyasagar, 1989):

joint angle	d	s	α
x_1	d_1	s_1	α_1
x_2	d_2	s_2	α_2
x_3	d_3	s_3	α_3

Hereabove d, s, α denote, respectively, offsets along the z-axis and x-axis, and the axis misalignments. Due to the assumption that the joints are unlimited the joint angles x_1, x_2, x_3 take values in the unit circle S^1, so internal configurations of the manipulator live in the torus T^3 (the internal manifold),

$$x \in S^1 \times S^1 \times S^1 \cong T^3 \tag{1}$$

The kinematics of the manipulator can be reproduced from the Denavit-Hartenberg data above in the straightforward way. In order to do so we first need to introduce the matrices

$$A_i(x_i) = \mathrm{Rot}(z, x_i)\,\mathrm{Trans}(z, d_i)\,\mathrm{Trans}(x, s_i)\,\mathrm{Rot}(x, \alpha_i) \tag{2}$$

for $i = 1, 2, 3$, where Rot and Trans stand for elementary rotations and translations with respect to the axes indicated. Having computed $A_i(x_i)$ we define the kinematics as a map taking its values in the special Euclidean group $SE(3)$ (the external manifold):

$$k(x) = A_1(x_1)A_2(x_2)A_3(x_3) = \begin{bmatrix} R(x) & T(x) \\ 0 & 1 \end{bmatrix} \tag{3}$$

In expression (3) $R(x)$ denotes the rotation matrix determining the orientation of the end-effector with respect to the base coordinate frame, similarly $T(x)$ denotes the position of the end-effector. Both the entries of $R(x)$ and components of $T(x)$ depend analytically on x. In what follows we shall examine only the position kinematics $T(x)$ of the manipulator. Furthermore, to make the analysis tractable we shall assume that the second axis misalignment

$$\alpha_2 = 0, \tag{4}$$

i.e. that the axes of joints 2 and 3 are parallel. Under assumption (4) the kinematics $T(x) = (T_1(x), T_2(x), T_3(x))$ can be defined explicitly in the following way:

$$\begin{aligned} T_1(x) = {} & s_1 \cos x_1 + s_2 \cos x_1 \cos x_2 + (d_2 + d_3) \sin\alpha_1 \sin x_1 \\ & -s_2 \cos\alpha_1 \sin x_1 \sin x_2 + s_3 \cos x_3(\cos x_1 \cos x_2 - \cos\alpha_1 \sin x_1 \sin x_2) \\ & -s_3 \sin x_3(\cos x_1 \sin x_2 + \cos\alpha_1 \sin x_1 \cos x_2) \\ \\ T_2(x) = {} & s_1 \sin x_1 + s_2 \sin x_1 \cos x_2 - (d_2 + d_3) \sin\alpha_1 \cos x_1 \\ & +s_2 \cos\alpha_1 \cos x_1 \sin x_2 + s_3 \cos x_3(\sin x_1 \cos x_2 + \cos\alpha_1 \cos x_1 \sin x_2) \\ & -s_3 \sin x_3(\sin x_1 \sin x_2 - \cos\alpha_1 \cos x_1 \cos x_2) \end{aligned} \tag{5}$$

$$T_3(x) = d_1 + (d_2 + d_3)\cos\alpha_1 + s_2 \sin\alpha_1 \sin x_2 \\ + s_3 \sin\alpha_1(\sin x_2 \cos x_3 + \cos x_2 \sin x_3)$$

Now, if we compute the Jacobi matrix for $T(x)$

$$J(x) = \frac{\partial T(x)}{\partial x} = \left[\frac{\partial T_i(x)}{\partial x_j}\right] \tag{6}$$

the singular configurations of (5) will be described by the condition

$$\det J(x) = -s_2 s_3 \sin\alpha_1 \Big(s_1 + s_2 \cos x_2 + s_3 \cos(x_2 + x_3)\Big) \sin x_3 = 0 \tag{7}$$

Clearly, any non-trivial spatial manipulator whose kinematics are defined by (5) should necessarily satisfy

$$s_2 \neq 0, \qquad s_3 \neq 0, \qquad \sin\alpha_1 \neq 0$$

hence the set of singular configurations S consists of two components S_1, S_2 defined as follows

$$S_1 = \Big\{x \in T^3 \mid x_3 = 0, \pm\pi\Big\}$$
$$S_2 = \Big\{x \in T^3 \mid s_1 + s_2 \cos x_2 + s_3 \cos(x_2 + x_3) = 0\Big\} \tag{8}$$
$$S = S_1 \cup S_2$$

It is an immediate consequence of the definition that S_1 is a two-dimensional analytic submanifold of T^3. In the case of S_2 it follows that S_2 is non-empty provided that

$$|s_1| \leq |s_2| + |s_3| \tag{9}$$

while S_2 is a two-dimensional analytic submanifold of T^3, if additionally

$$s_1 \pm s_2 \pm s_3 \neq 0 \tag{10}$$

In order to provide more insight into the two components of the set of singular configurations we shall look more closely at the particular case of a manipulator whose joint axes 1 and 2 are perpendicular, with zero offsets d_1, d_3. For such a manipulator the singular configurations defined by (8) have been represented graphically in Fig. 1.

On the basis of a geometric interpretation of configurations shown in Fig. 1 the following terminology concerned with singular configurations in S_1, S_2, and $S_1 \cap S_2$ will be adopted throughout this paper. First, if $x \in S_1$ and $x_3 = 0$, then x will be called a *fully stretched out* (FSO) configuration. If $x \in S_1$ and $x_3 = \pm\pi$, then x will be referred to as a *fully folded down* (FFD) configuration. Second, for $x \in S_2$ we shall use the name an *almost overhead* (AO) configuration, as along with d_2 tending to 0 the end-effector approaches a position on the z-axis of the base coordinate frame (we should keep $d_2 \neq 0$ as otherwise the manipulator in Fig. 1 is no longer spatial). Eventually, a configuration $x \in S_1 \cap S_2$ with $x_3 = 0$ will be termed

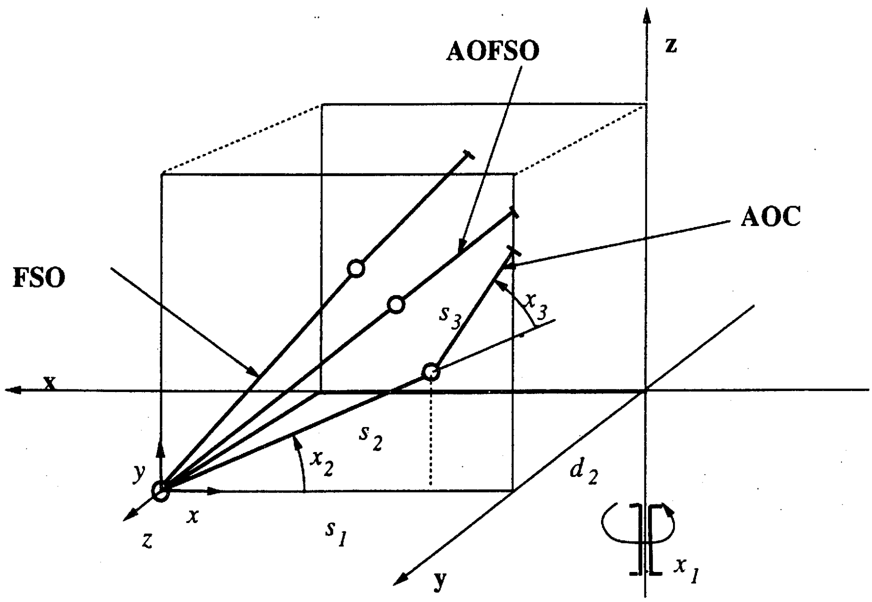

Fig. 1. An illustration of FSO, AO and AOFSO configurations.

an *almost overhead fully stretched out* (AOFSO) configuration, while if $x_3 = \pm\pi$ the configuration x will be called an *almost overhead fully folded down* (AOFFD) configuration. Observe that AO configurations exist provided that

$$|s_1| \leq |s_2| + |s_3|$$

Furthermore, to admit an AOFSO configuration the kinematics should additionally satisfy

$$|s_1| \leq |s_2 + s_3| \tag{11}$$

whereas AOFFD configurations are possible whenever

$$|s_1| \leq |s_2 - s_3| \tag{12}$$

The purpose of this paper is to find out local normal forms of the kinematics (5) around singular configurations of all aforementioned types. In order to do so, let us define in T^3 an open subset

$$U = \left\{x \in T^3 \mid x_2 + x_3 \neq \pm\pi/2 \text{ and } T_2(x) \neq 0\right\} \tag{13}$$

and restrict to singular configurations belonging to $S \cap U$. Indeed, all such configurations are ordinary, i.e. at any $x \in S \cap U$

$$\operatorname{rank} J(x) = 2$$

because the matrix

$$M(x) = \begin{bmatrix} \dfrac{\partial T_1(x)}{\partial x_1} & \dfrac{\partial T_1(x)}{\partial x_3} \\ \dfrac{\partial T_3(x)}{\partial x_1} & \dfrac{\partial T_3(x)}{\partial x_3} \end{bmatrix} \tag{14}$$

has a determinant

$$\det M(x) = -s_3 \sin\alpha_1 \cos(x_2 + x_3) T_2(x) = -T_2(x) \frac{\partial T_3(x)}{\partial x_3} \tag{15}$$

non-vanishing at $S \cap U$ just by definition of U.

Since, by assumption, $s_3 \neq 0$, $\sin\alpha_1 \neq 0$, around any point $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in U$, the following map

$$\begin{aligned} &\xi = \varphi(x) = \big(\varphi_1(x), \varphi_2(x), \varphi_3(x)\big) : \\ &\xi_1 = T_1(x) - T_1(\bar{x}), \qquad \xi_2 = x_2 - \bar{x}_2, \qquad \xi_3 = T_3(x) - T_3(\bar{x}) \end{aligned} \tag{16}$$

is a local diffeomorphism of a neighbourhood of $\bar{x}$ onto a neighbourhood of $0 \in \mathbb{R}^3$, that will be used to accomplish the first step of construction of the normal form. The diffeomorphism (16) is well defined only within U, if $\bar{x}$ is not in U, another diffeomorphism, based on a suitable non-singular 2×2 submatrix of $J(x)$, should be chosen instead of (16). Such a diffeomorphism exists as far as the singular configurations are ordinary. In the sequel we shall restrict only to $x \in U$ though.

On application of (16) at some fixed $\bar{x} \in U \cap S$ followed by standard shifts in the first and in the third coordinate of the external manifold, we transform kinematics (5) to a preliminary normal form

$$\big(\xi_1, K(\xi), \xi_3\big) \tag{17}$$

where the function $K(\xi)$ is analytic in some neighbourhood of $0 \in \mathbb{R}^3$ and satisfies

$$K \circ \varphi(x) = T_2(x), \qquad K \circ \varphi(\bar{x}) = K(0) = T_2(\bar{x}) \tag{18}$$

for x in some neighbourhood of $\bar{x}$. Further simplifications of the form (17) will depend on the behaviour of $K(\xi)$, more specifically on properties of partial derivatives of $K(\xi)$ at the singular configuration $\bar{\xi} = \varphi(\bar{x}) = 0$. Thus to proceed we need to compute some partial derivatives of the function $K(\xi)$ which is not known explicitly. The computations will be carried out on the basis of formula (18). By differentiating both sides of (18) with respect to x_i, $i = 1, 2, 3$, and by making suitable substitutions from (16) we obtain the following system of linear equations for unknown derivatives $\frac{\partial K}{\partial \xi_1}, \frac{\partial K}{\partial \xi_2}, \frac{\partial K}{\partial \xi_3}$:

$$\begin{aligned}
\frac{\partial K\circ\varphi(x)}{\partial \xi_1}\frac{\partial T_1(x)}{\partial x_1}+\frac{\partial K\circ\varphi(x)}{\partial \xi_2}\frac{\partial \varphi_2(x)}{\partial x_1}+\frac{\partial K\circ\varphi(x)}{\partial \xi_3}\frac{\partial T_3(x)}{\partial x_1}&=\frac{\partial T_2(x)}{\partial x_1}\\
\frac{\partial K\circ\varphi(x)}{\partial \xi_1}\frac{\partial T_1(x)}{\partial x_2}+\frac{\partial K\circ\varphi(x)}{\partial \xi_2}\frac{\partial \varphi_2(x)}{\partial x_2}+\frac{\partial K\circ\varphi(x)}{\partial \xi_3}\frac{\partial T_3(x)}{\partial x_2}&=\frac{\partial T_2(x)}{\partial x_2}\\
\frac{\partial K\circ\varphi(x)}{\partial \xi_1}\frac{\partial T_1(x)}{\partial x_3}+\frac{\partial K\circ\varphi(x)}{\partial \xi_2}\frac{\partial \varphi_2(x)}{\partial x_3}+\frac{\partial K\circ\varphi(x)}{\partial \xi_3}\frac{\partial T_3(x)}{\partial x_3}&=\frac{\partial T_2(x)}{\partial x_3}
\end{aligned}\tag{19}$$

Taking into account the property that

$$\frac{\partial \varphi_2}{\partial x_1}=\frac{\partial \varphi_2}{\partial x_3}=\frac{\partial T_3}{\partial x_1}=0, \qquad \frac{\partial \varphi_2}{\partial x_2}=1$$

and having deduced from (5)

$$\frac{\partial T_1}{\partial x_1}=-T_2, \qquad \frac{\partial T_2}{\partial x_1}=T_1, \qquad \frac{\partial T_3(x)}{\partial x_3}=s_3\sin\alpha_1\cos(x_2+x_3) \tag{20}$$

we conclude using (13) that for any $x\in U$ system (19) is solvable. Its solution can be written down as follows:

$$\begin{aligned}
X_{100}&=\frac{\partial K\circ\varphi}{\partial \xi_1}=-T_2^{-1}\frac{\partial T_2}{\partial x_1}=-\frac{T_1}{T_2}\\
X_{001}&=\frac{\partial K\circ\varphi}{\partial \xi_3}=\left(\frac{\partial T_3}{\partial x_3}\right)^{-1}\left(\frac{\partial T_2}{\partial x_3}-X_{100}\frac{\partial T_1}{\partial x_3}\right)\\
&=\frac{T_1\frac{\partial T_1}{\partial x_3}+T_2\frac{\partial T_2}{\partial x_3}}{T_2\frac{\partial T_3}{\partial x_3}}=-\frac{T_1\frac{\partial T_1}{\partial x_3}+T_2\frac{\partial T_2}{\partial x_3}}{\det M}\\
X_{010}&=\frac{\partial K\circ\varphi}{\partial \xi_2}=\frac{\partial T_2}{\partial x_2}-X_{100}\frac{\partial T_1}{\partial x_2}-X_{001}\frac{\partial T_3}{\partial x_2}=-\frac{\det J}{T_2\frac{\partial T_3}{\partial x_3}}=\frac{\det J}{\det M}
\end{aligned}\tag{21}$$

where J denotes the Jacobi matrix (6), M – submatrix (14), and for simplicity of notation the argument x has been omitted. The formula for X_{010} yields immediately that at any singular configuration $\bar{x}\in U$

$$X_{010}(\bar{x})=\frac{\partial K(0)}{\partial \xi_2}=0 \tag{22}$$

as expected since $\bar{x}$ was singular. Also, it is worth noticing that the property of T_1 and T_2 mentioned in (20) results from the apparent invariance of the square distance $T_1^2 + T_2^2$ with respect to rotations around the z-axis of the base coordinate frame.

Having concluded (22) we want to compute further derivatives of $K(\xi)$. By a simple induction formulae (21) can be reliably generalized to arbitrary derivatives leading for any

$$X_{\alpha_1\alpha_2\alpha_3} = \frac{\partial^{|\alpha|} K \circ \varphi}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}\partial\xi_3^{\alpha_3}}, \qquad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \tag{23}$$

to the following equations:

$$\begin{aligned} X_{\alpha_1+1\alpha_2\alpha_3} &= -T_2^{-1}\frac{\partial X_{\alpha_1\alpha_2\alpha_3}}{\partial x_1} \\ X_{\alpha_1\alpha_2\alpha_3+1} &= \frac{\partial T_3}{\partial x_3}^{-1}\left(\frac{\partial X_{\alpha_1\alpha_2\alpha_3}}{\partial x_3} - X_{\alpha_1+1\alpha_2\alpha_3}\frac{\partial T_1}{\partial x_3}\right) \\ X_{\alpha_1\alpha_2+1\alpha_3} &= \frac{\partial X_{\alpha_1\alpha_2\alpha_3}}{\partial x_2} - X_{\alpha_1+1\alpha_2\alpha_3}\frac{\partial T_1}{\partial x_2} - X_{\alpha_1\alpha_2\alpha_3+1}\frac{\partial T_3}{\partial x_2} \end{aligned} \tag{24}$$

It turns out that (24) produces (21) under the substitution

$$X_{000} = T_2$$

From (24) we deduce in particular that at any singular configuration $\bar{x} \in U$

$$X_{020}(\bar{x}) = \frac{\partial^2 K(0)}{\partial\xi_2^2} = -\frac{\frac{\partial}{\partial x_2}\det J(\bar{x})}{T_2(\bar{x})\frac{\partial T_3(\bar{x})}{\partial x_3}} + \frac{\frac{\partial T_3(\bar{x})}{\partial x_2}\frac{\partial}{\partial x_3}\det J(\bar{x})}{T_2(\bar{x})\left(\frac{\partial T_3(\bar{x})}{\partial x_3}\right)^2} \tag{25}$$

After computing derivatives of determinant (7) we discover that for $\bar{x} \in U \cap S_1 \cap S_2$

$$X_{020}(\bar{x}) = 0. \tag{26}$$

Next, with a little assistance of symbolic computations it can be established that for a singular configuration $\bar{x} \in U \setminus (S_1 \cap S_2)$, $X_{020}(\bar{x})$ takes one of the following forms:

i) $\bar{x} \in S_1$ and $\bar{x}_3 = 0$:

$$X_{020}(\bar{x}) = -\frac{s_2 s_3 (s_2 + s_3)(s_1 + (s_2 + s_3)\cos\bar{x}_2)\sin^2\alpha_1}{T_2(\bar{x})\left(\frac{\partial T_3(\bar{x})}{\partial x_3}\right)^2}$$

ii) $\bar{x} \in S_1$ and $\bar{x}_3 = \pm\pi$:

$$X_{020}(\bar{x}) = \frac{s_2 s_3 (s_2 - s_3)(s_1 + (s_2 - s_3)\cos\bar{x}_2)\sin^2\alpha_1}{T_2(\bar{x})\left(\frac{\partial T_3(\bar{x})}{\partial x_3}\right)^2} \tag{27}$$

iii) $\bar{x} \in S_2$:

$$X_{020}(\bar{x}) = \frac{s_2^2 s_3^2 \sin^2\bar{x}_3 \sin^2\alpha_1}{T_2(\bar{x})\left(\frac{\partial T_3(\bar{x})}{\partial x_3}\right)^2}$$

From (27) it follows that if, respectively, $s_2 + s_3 \neq 0$ or $s_2 - s_3 \neq 0$, then for $\bar{x} \in U \setminus (S_1 \cap S_2)$

$$X_{020}(\bar{x}) \neq 0 \tag{28}$$

Now let us concentrate on singular configurations $\bar{x} \in U \cap S_1 \cap S_2$. We have already established that for such configurations

$$X_{010}(\bar{x}) = 0 \quad \text{and} \quad X_{020}(\bar{x}) = 0 \tag{29}$$

By symbolic computations based on expressions (24) we obtain that

$$X_{030}(\bar{x}) = \frac{\partial^3 K(0)}{\partial \xi_2^3} = 0 \tag{30}$$

and

$$X_{040} = \frac{\partial^4 K(0)}{\partial \xi_2^4} = \frac{3 s_2^2 (s_2 \pm s_3)^4}{s_1^2 s_3^2 T_2(\bar{x})} \tag{31}$$

where $+$ sign refers to $\bar{x}_3 = 0$, $-$ sign to $\bar{x}_3 = \pm\pi$.

Eventually, we should consider two very specific geometries of the manipulator assuming either an FSO or FFD configuration. More specifically, we let $\bar{x} \in (S_1 \setminus S_2) \cap U$, and either $\bar{x}_3 = \pm\pi$ along with $s_2 - s_3 = 0$ or $\bar{x} = 0$ and $s_2 + s_3 = 0$. Observe that, due to the fact that geometrically the translation along the x-axis by s_3 followed by the zero rotation around the z-axis is equivalent to the translation by $-s_3$ followed by the rotation by $\pm\pi$, it sufficies to analyse in detail only the former case. Thus, assuming that $\bar{x}_3 = \pm\pi$ and $s_2 = s_3$, we obtain from (27) the case (ii)

$$X_{020}(\bar{x}) = 0 \tag{32}$$

In order to compute higher order derivatives of $K(\xi)$ we first deduce from (5)

$$\frac{\partial^3 T_1}{\partial x_2^3} = -\frac{\partial T_1}{\partial x_2}, \quad \frac{\partial^3 T_3}{\partial x_2^3} = -\frac{\partial T_3}{\partial x_2}$$

$$\frac{\partial T_1(\bar{x})}{\partial x_2} = \frac{\partial T_3(\bar{x})}{\partial x_2} = \frac{\partial^2 T_1(\bar{x})}{\partial x_2^2} = \frac{\partial^2 T_3(\bar{x})}{\partial x_2^2} = 0$$

so, in conclusion, for every $k \geq 1$

$$\frac{\partial^k T_1(\bar{x})}{\partial x_2^k} = \frac{\partial^k T_3(\bar{x})}{\partial x_2^k} = 0 \tag{33}$$

Due to (33) an inductive reasoning applied to (24) allows us to prove that for any $k \geq 1$

$$X_{0k+10}(\bar{x}) = \frac{\partial}{\partial x_2} X_{0k0}(\bar{x}) = \frac{\partial^k}{\partial x_2^k} X_{010}(\bar{x}) \tag{34}$$

or, equivalently, that

$$X_{0k+10}(\bar{x}) = \left(T_2(\bar{x})\frac{\partial T_3(\bar{x})}{\partial x_3}\right)^{-1}\left(-\frac{\partial^k \det J(\bar{x})}{\partial x_2^k} + \sum_{j=1}^{k} \alpha_j(\bar{x}) X_{0j0}(\bar{x})\right) \tag{35}$$

for certain functions α_j. Therefore, in order to compute an arbitrary derivative $X_{0k0}(\bar{x})$, it is necessary first to find derivatives of determinant (7) with respect to x_2. But, clearly,

$$\begin{aligned} \frac{\partial}{\partial x_2} \det J(x) &= s_2 s_3 \sin\alpha_1 \Big(s_2 \sin x_2 + s_3 \sin(x_2 + x_3)\Big) \sin x_3 \\ \frac{\partial^2}{\partial x_2^2} \det J(x) &= s_2 s_3 \sin\alpha_1 \Big(s_2 \cos x_2 + s_3 \cos(x_2 + x_3)\Big) \sin x_3 \\ \frac{\partial^3}{\partial x_2^3} \det J(x) &= -\frac{\partial}{\partial x_2} \det J(x) \end{aligned} \tag{36}$$

Calculated at $\bar{x}$ ($\bar{x}_3 = \pm\pi, \ s_2 = s_3$) all the derivatives vanish:

$$\frac{\partial^k}{\partial x_2^k} \det J(\bar{x}) = 0 \tag{37}$$

Consequently, since $X_{010}(\bar{x}) = 0$, we derive from (35) that for every $k \geq 1$

$$X_{0k0}(\bar{x}) = \frac{\partial^k K(0)}{\partial \xi_2^k} = 0 \tag{38}$$

Having established that all partial derivatives of $K(\xi)$ with respect to ξ_2 vanish at 0 we want to look at other derivatives. By (21) we obtain the following relationships

$$X_{100}(\bar{x}) = \frac{\partial K(0)}{\partial \xi_1} = -\frac{s_1 \cos\bar{x}_1 + (d_2 + d_3)\sin\alpha_1 \sin\bar{x}_1}{T_2(\bar{x})}$$

$$X_{001}(\bar{x}) = \frac{\partial K(0)}{\partial \xi_3} = \frac{s_3\Big((d_2 + d_3)\sin\alpha_1\cos\alpha_1\cos\bar{x}_2 + s_1\sin\bar{x}_2\Big)}{T_2(\bar{x})\dfrac{\partial T_3(\bar{x})}{\partial x_3}} \tag{39}$$

$$X_{011}(\bar{x}) = \frac{\partial^2 K(0)}{\partial \xi_2 \partial \xi_3} = s_1 s_3^2 \sin\alpha_1$$

The data (26)–(39) are sufficient for performing the final reduction of kinematics (17) to a normal form. To this aim we shall apply some basic results from the singularity theory (Golubitsky and Guillemin, 1973; Martinet, 1982).

Theorem 1. *Consider the position kinematics (5), and let $\bar{x}$ denote a singular configuration that belongs to the set U defined by (13). Then, in some neighbourhood of $\bar{x}$, the kinematics can be transformed by local coordinate changes in the internal and external manifolds to one from among the following normal forms.*

- *if $\bar{x}$ is either an FSO configuration of the kinematics whose geometric parameters satisfy $s_2 + s_3 \neq 0$ or an FFD configuration of the kinematics satisfying $s_2 - s_3 \neq 0$ or any AO configuration, then the normal form is quadratic*

$$(x_1, x_2^2, x_3) \tag{F1}$$

- *if $s_1 \neq 0$ and $\bar{x}$ is either an AOFSO configuration, or an AOFFD configuration then the normal form is*

$$\Big(x_1, x_2^4 + a(x_1, x_3)x_2^2 + b(x_1, x_3)x_2, x_3\Big) \tag{F2}$$

for certain functions a, b.

- *if $s_1 \neq 0$ and $\bar{x}$ is either an FSO configuration of the kinematics whose geometric parameters satisfy $s_2 + s_3 = 0$ or an FFD configuration such that $s_2 - s_3 = 0$, then the normal form is*

$$\Big(x_1, x_2x_3 + x_1c(x), x_3\Big) \tag{F3}$$

for a certain function c.

Proof. Item 1: From (22) and (i) or (ii) of (27) it follows that

$$\frac{\partial K(0)}{\partial \xi_2} = 0 \quad \text{and} \quad \frac{\partial^2 K(0)}{\partial \xi_2^2} \neq 0$$

Therefore, the preliminary normal form (17) can be transformed further to a Morin canonical form (Golubitsky and Guillemin, 1973; Martinet, 1982)

$$(x_1, x_2^2, x_3)$$

Item 2: Since $s_1 \neq 0$, we deduce from the existence conditions (11), (12) of AOFSO and AOFFD configurations that, respectively,

$$|s_2 \pm s_3| > 0$$

The above inequality allows us to conclude from (29), (30) and (31) that

$$\frac{\partial K(0)}{\partial \xi_2} = \frac{\partial^2 K}{\partial \xi_2^2} = \frac{\partial^3 K(0)}{\partial \xi_2^3} = 0, \quad \text{while} \quad \frac{\partial^4 K(0)}{\partial \xi_2^4} \neq 0$$

Now we can apply to the form (17) the universal unfolding theorem that yields a "pre-Morin" normal form (Martinet, 1982)

$$\left(x_1, x_2^4 + a(x_1, x_3)x_2^2 + b(x_1, x_3)x_2, x_3\right)$$

Item 3: By (38) all derivatives $\dfrac{\partial^k K(0)}{\partial \xi_2^k}$ vanish. Consequently, since $K(\xi)$ is analytic around $0 \in \mathbb{R}^3$, its Taylor expansion at 0 will not contain any monomial depending only on ξ_2. This means that $K(\xi)$ assumes the form

$$K(\xi) = K(0) + \xi_1\alpha(\xi) + \xi_3\beta(\xi)$$

for certain analytic functions α, β. From (39) we infer that in most cases $\alpha(0)$ and $\beta(0)$ are non-zero and, more importantly, that $s_1 \neq 0$ implies

$$\frac{\partial^2 K(0)}{\partial \xi_2 \partial \xi_3} = \frac{\partial \beta(0)}{\partial \xi_2} \neq 0$$

The last observation allows us to introduce a local diffeomorphism

$$\varphi(\xi) = \left(\xi_1, \beta(\xi) - \beta(0), \xi_3\right)$$

which, acompanied by a suitable shift in the second external coordinate, transforms (17) to

$$\left(\xi_1, \xi_1 c(\xi) + \beta(0)\xi_3 + \xi_2\xi_3, \xi_3\right)$$

Finally, by changing external coordinates in accordance with the formula

$$(y_1, y_2, y_3) \mapsto \Big(y_1, y_2 + \beta(0)y_3, y_3\Big)$$

we remove the term $\beta(0)\xi_3$ and arrive at the normal form

$$\Big(x_1, x_2x_3 + x_1c(x), x_3\Big)$$

By construction the unknown function $c(x)$ is analytic. ■

Remark 1. The normal form (F3) proposed in item 3 of Theorem 1 can be regarded as a generalization of the normal form of the so-called cyclic double pendulum introduced in (Tchoń, 1992).

Remark 2. Assumptions made in Theorem 1 exclude AOFSO or AOFFD configurations of kinematics whose offset equals $s_1 = 0$. Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be such a configuration. Thus we have $\bar{x}_3 = 0, \pm\pi$ and $(s_2 \pm s_3)\cos\bar{x}_2 = 0$. Clearly, the last equality requires that $\bar{x}_2 = \pm\pi/2$ or $s_2 \pm s_3 = 0$. In the former case we have $\bar{x} \notin U$, hence we cannot use the preliminary normal form (17). In the latter case, according to (10), the component S_2 of S is no longer an analytic submanifold of T^3.

3. Conclusions

We have proposed a fairly complete description of kinematic behaviour of 3R manipulators with parallel axes of joints 2 and 3, whose singular configurations are ordinary. Three normal forms (F1), (F2), (F3) have been introduced. The form (F1) is structurally stable and has already been present on the list of candidate normal forms of the robot kinematics derived in (Tchoń, 1991). The other two normal forms (F2), (F3) are not structurally stable; in fact, they depend on some unknown functions. However, from the robotic point of view the two unstable normal forms differ substantially with regard to their sensitivity to variations of geometric parameters of the kinematics. Namely, if the parameters change (although in a way preserving $\alpha_2 = 0$), the form (F2) will survive, while the form (F3) will not. This constatation makes sense while speaking of *geometric stability* of (F2) and *geometric instability* of (F3). It is reasonable to expect that from the viewpoint of the design of robot manipulators geometrically unstable kinematics have little practical significance.

The structurally stable form (F1) belongs to quadratic or Morse normal forms of kinematics and is well understood (Tchoń, 1991). This being so, and in view of what we have said above about the form (F3), in conclusion of this paper we wish to examine in more detail the geometrically stable form (F2), paying special attention to its bifurcation diagram. Let us recall that by the bifurcation diagram of a normal form $k(x)$ of kinematics we mean the set-valued map

$$\alpha \mapsto X_\alpha(k) = k^{-1}(\alpha) \tag{40}$$

where α travels through the external manifold of the kinematics. In the case of (F2) we set

$$k(x) = \Big(x_1, x_2^4 + a(x_1,x_3)x_2^2 + b(x_1,x_3)x_2, x_3\Big) \tag{41}$$

Since the functions a, b do not depend on x_2, the bifurcation diagram of k is in a sense "embedded" into the bifurcation diagram of the normal form $j4$ introduced in (Tchoń, 1991). Both bifurcation diagrams coincide, if functions a, b are independent. This normal form can be represented as

$$j4(x) = (x_1, x_2^4 + x_3x_2^2 + x_1x_2, x_3) \tag{42}$$

For a fixed $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ the bifurcation diagram of $j4(x)$ takes the (set-) value

$$(j4)^{-1}(\alpha) = \Big\{(x_1,x_2,x_3) \mid x_1 = \alpha_1,\ x_3 = \alpha_3, x_2^4+\alpha_3x_2^2+\alpha_1x_2 = \alpha_2\Big\} \tag{43}$$

so the shape of the bifurcation diagram is in turn determined by an object that is well known in catastrophe theory, namely by the catastrophe manifold of the swallow's tail catastrophe (Poston and Stewart, 1978)

$$M_4 = \Big\{(x,a,b,c) \in \mathbb{R}^4 \mid x^4 + ax^2 + bx + c = 0\Big\} \tag{44}$$

Several sections of the manifold M_4 have been plotted in Figs. 2–5 below.

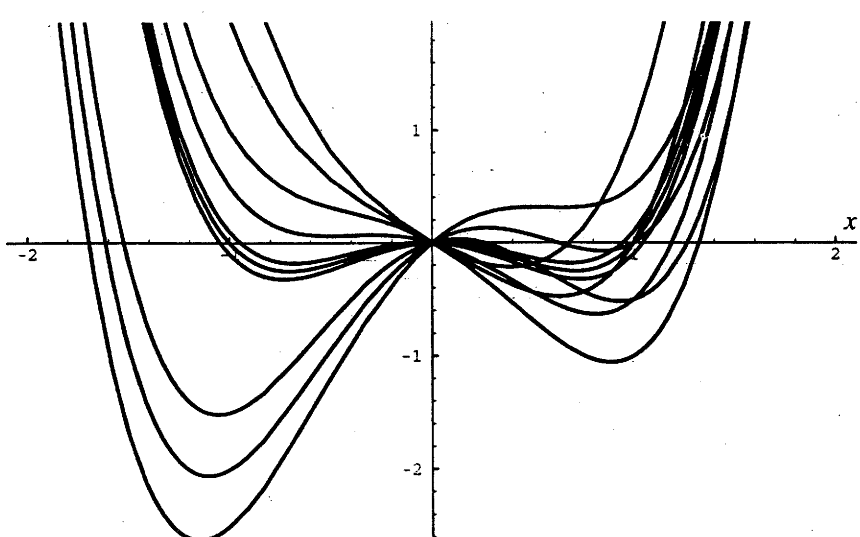

Fig. 2. Catastrophe manifold M_4; sections parameterized by (a, b).

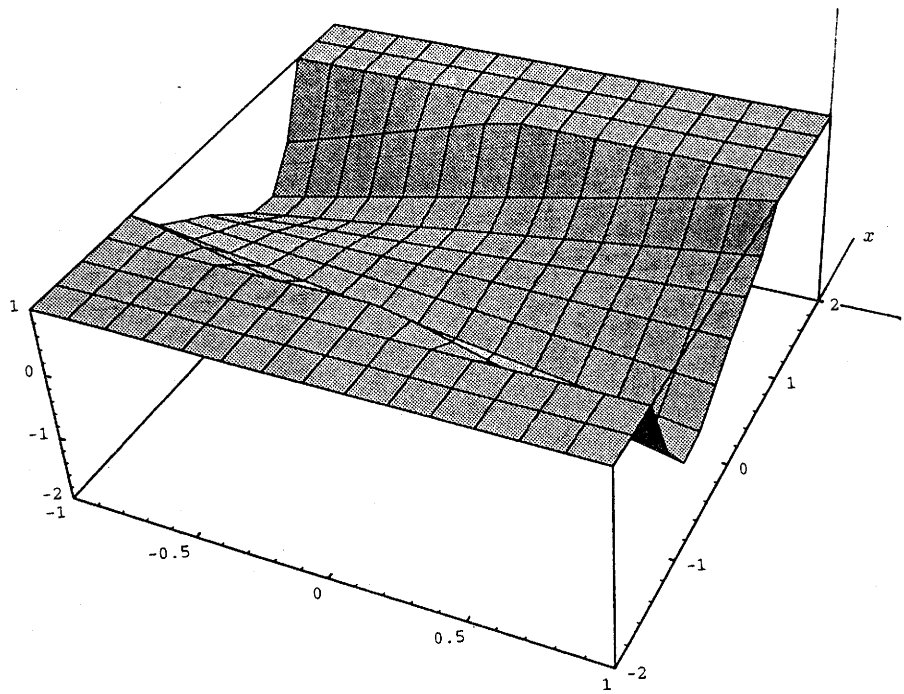

Fig. 3. Catastrophe manifold M_4; a section at $a = -0.5$.

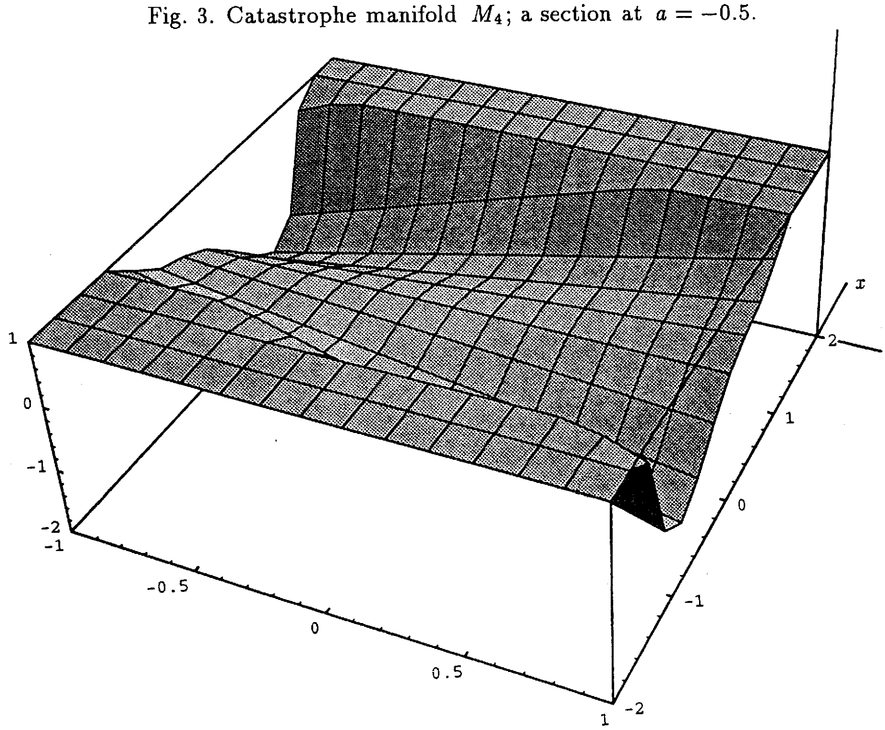

Fig. 4. Catastrophe manifold M_4; a section at $a = -1$.

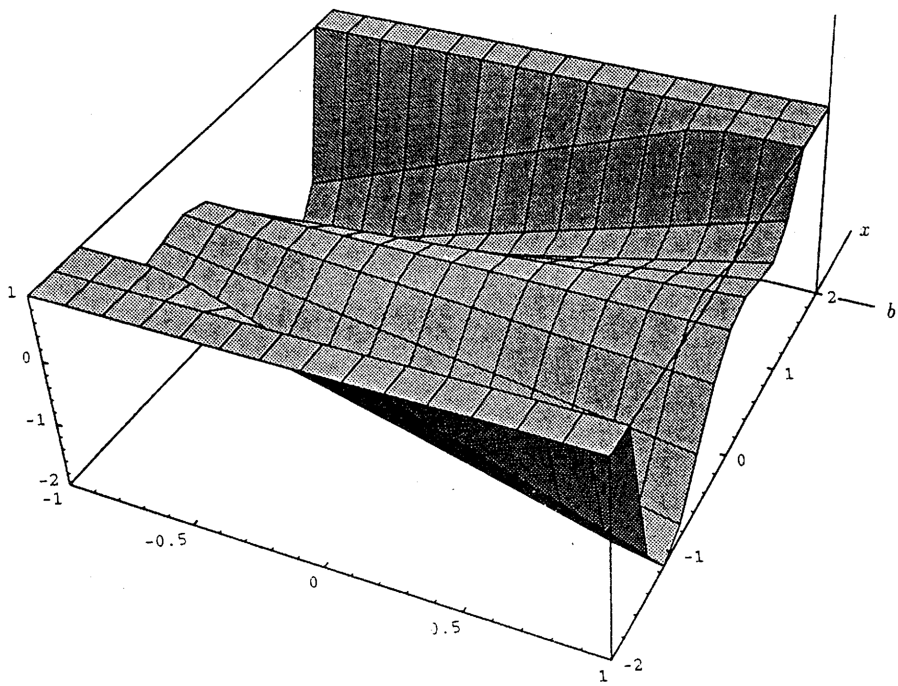

Fig. 5. Catastrophe manifold M_4; a section at $a = -2$.

In robotic terms these plots tell us a.o. that around AOFSO or AOFFD configurations we can expect zero, two or four solutions to the inverse kinematic problem. However, since the form $k(x)$ is somewhat more restrictive than $j4(x)$ (parameters a, b in (44) can be adjusted completely freely, whereas in (41) only through unknown functions $a(x_1, x_3)$, $b(x_1, x_3)$), the "true" bifurcation diagram of (41) is in general a subdiagram of the diagram derived for (43).

Acknowledgements

This research contributes to the research project *Differential Topological Methods in Robotics* sponsored by the Polish State Committee of Scientific Research. It was done during the author's visit to The Systems and Control Group, Faculty of Applied Mathematics, University of Twente, The Netherlands. The author is indebted to Dr. Henk Nijmeijer for arranging the visit and having made it scientifically productive. The author has also pleasure to thank Prof. A. Bagchi who has encouraged him to exploit symbolic computation facilities of the Faculty.

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Received: January 15, 1995