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DISTURBANCE DECOUPLING FOR NON-LINEAR STRUCTURED SYSTEMS

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A structural approach to the disturbance decoupling problem of non-linear systems is introduced. For this purpose, in analogy to linear systems theory, a structural description of non-linear systems is defined. In associating a directed graph to such a system, two important system properties, the differential output rank and the structure at infinity, can be characterized in terms of input-output paths. Based on this concept, the disturbance rejection problem can be easily solved. Moreover, necessary conditions valid for general – non structured – systems can be defined in a graph-theoretic way. The advantages of this approach are demonstrated by illustrative examples.

1. Introduction

Structured systems consider only the existence and respectively non-existence of connections between inputs, states and outputs; explicit functional dependencies are omitted. Though less information is included, a structural approach yields manifold advantages in systems analysis (Reinschke, 1988). Besides a better insight in the physical behaviour, especially for large-scale systems, no numerical problems occur. For non-linear systems this approach is also suitable, because the analysis of such control problems as system rank, input-output decoupling, disturbance decoupling, exact linearization etc. has to be done in a complex mathematical fashion (Schwarz, 1991; Isidori, 1989). The computer-aided determination leads therefore to extensive calculus, which may fail for larger system models due to insufficient memory and a long run time.

A most promising tool for the analysis of structured systems is a graph (Andrásfai, 1991). It consists of vertices and edges, which are associated with the matrix elements of a structured system (Reinschke, 1988). One main advantage of the graph-based approach is the fact that a large number of efficient algorithms for the analysis of graphs are already available. As shown by a great number of authors (Kasinski and Lévine, 1984; D'Andrea and Lévine, 1986; De Luca and Isidori, 1987; Wey *et al.*, 1994; Reinschke, 1994) the use of graphs is successful not only in the analysis but also in the synthesis of non-linear systems.

By extending known results (Commault *et al.*, 1991) it is possible to characterize a non-linear structure at infinity defined by Fliess (1986b) with graph-theoretic methods. More specifically, vertex-disjoint paths between inputs and outputs are examined

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for this purpose. Based on the relation between a graph and the structure at infinity, the necessary and sufficient conditions for disturbance decoupling of non-linear systems can be given in a similar way as for linear systems (Commault *et al.*, 1991). Due to the omitted functional dependencies these conditions are easy to prove. Moreover, utilizing the information contained in the graph explicit feedback laws can be found, which yield disturbance rejection (Wey, 1995).

The methods obtained by the graph-based approach have some important advantages. In particular, the solutions become more obvious in comparsion with other methods. Additionally, the computed results hold for classes of systems with an identical structure.

The paper is organized as follows. First, the algebraic definitions for the rank and the structure at infinity of non-linear systems are shortly introduced. Next, the concept of structured systems is discussed. A graph-theoretic description is connected with such systems allowing a compact representation. In Section 4 the static and dynamic disturbance decoupling problems are defined. Based on this definition, structural properties are given, which characterize the solvability of the disturbance decoupling problem for non-linear systems. Using illustrative examples the advantages of the structural approach are verified.

2. Notation and Preliminaries

Consider a non-linear time-invariant control system $\sum$ in the standard state space form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{a}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{c}(\boldsymbol{x}) = \boldsymbol{C}\boldsymbol{x} \end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$, $\boldsymbol{y}(t) \in \mathbb{R}^p$. Here $\boldsymbol{a}(\cdot)$, columns of $\boldsymbol{B}(\cdot)$ and rows of $\boldsymbol{c}(\cdot)$ are analytic with respect to $\boldsymbol{x}$.

Restricting the class of non-linear systems to a linear output-equation $\boldsymbol{Cx}$ does not mean a loss of generality, since an equivalent system with $n' = n+p$, $m' = m$, $p' = p$ and $\boldsymbol{y} = \boldsymbol{Cx}$ always exists as long as $\boldsymbol{c}(\boldsymbol{x})$ is an analytic function (Schwarz, 1991).

2.1. Differential Output Rank and Structure at Infinity

There are several different definitions of rank and structure at infinity for non-linear systems. Here the algebraic definitions of (Moog, 1988) will be used, which show several advantages (Di Benedetto *et al.*, 1989). For an introduction to the algebraic approach consider e.g. (Fliess and Glad, 1993). Due to (Fliess, 1986b) a non-linear system $\sum$ is defined by the fact that the components of the output $\boldsymbol{y}$ are *differentially algebraic* over the field $k\langle \boldsymbol{u} \rangle$. This definition means that the components of $\boldsymbol{u}$ and $\boldsymbol{y}$ are related by a finite number of implicit differential equations with rational coefficients. Based on his description, Fliess (1986b) defines the differential output rank of a system as follows.

Definition 1. (Fliess, 1986b) The *differential output rank* is the differential transcendence degree of the differential field $k\langle \boldsymbol{y} \rangle$ over k:

$$\rho^* = \text{diff.tr.d}^\circ \ k\langle \boldsymbol{y} \rangle / k \ . \tag{2}$$

A statement equivalent to Definition 1 is that ρ^* coincides exactly with the maximum number of independent system outputs y_i (Fliess, 1986b; Wey and Svaricek, 1995).

For computing the differential output rank of a non-linear system, the abstract definition of eqn. (2) is hardly usable. Another definition of ρ^* introduced by Moog (1988) is suited better for this purpose. It is based on the analysis of ordinary vector spaces and closely related to the *structure at infinity* (Di Benedetto *et al.*, 1989; Svaricek and Schwarz, 1993).

First, a chain of vector spaces over the field $\mathcal{K}$ of rational functions in $\boldsymbol{u}, ..., \boldsymbol{u}^{(n-1)}$ with meromorphic coefficients in $\boldsymbol{x}$ is associated to $\sum$. For the elements $\boldsymbol{v} = (v_1, ..., v_j)$ of such a field the derivative operator $\partial/\partial v_i$ acting on a meromorphic function $\eta(\boldsymbol{v}) = p(\boldsymbol{v})/q(\boldsymbol{v})$ is defined as

$$\frac{\partial}{\partial v_i}\frac{p(\boldsymbol{v})}{q(\boldsymbol{v})} := \frac{q(\boldsymbol{v})\frac{\partial}{\partial v_i}p(\boldsymbol{v}) - p(\boldsymbol{v})\frac{\partial}{\partial v_i}q(\boldsymbol{v})}{q^2(\boldsymbol{v})} \tag{3}$$

Then, the differential of $\eta(\boldsymbol{v})$ is given by

$$\mathrm{d}\eta(\boldsymbol{v}) = \sum_{i=1}^{j} \frac{\partial \eta(\boldsymbol{v})}{\partial v_i}\mathrm{d}v_i \tag{4}$$

The time derivatives of the output are given by

$$\begin{aligned} \dot{\boldsymbol{y}}(t) &= \dot{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{u}) = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{y}}(t) &= \ddot{\boldsymbol{y}}(\boldsymbol{x}, \dot{\boldsymbol{u}}) = \frac{\partial \dot{\boldsymbol{y}}}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\partial \dot{\boldsymbol{y}}}{\partial \boldsymbol{u}}\dot{\boldsymbol{u}} \\ &\vdots \\ \boldsymbol{y}^{(k+1)} &= \frac{\partial \boldsymbol{y}^{(k)}}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \sum_{l=0}^{k-1}\frac{\partial \boldsymbol{y}^{(k)}}{\partial \boldsymbol{u}^{(l)}}\boldsymbol{u}^{(l+1)} \end{aligned} \tag{5}$$

Therefore $\boldsymbol{y}, ..., \boldsymbol{y}^{(n)}$ consist of components in the field $\mathcal{K}$. Let $\mathcal{E}$ denote the vector space spanned over $\mathcal{K}$ by the differentials $\{\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{u}, ..., \mathrm{d}\boldsymbol{u}^{(n-1)}\}$. The chain of subspaces $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_n$ of $\mathcal{E}$ is defined by

$$\begin{aligned} \mathcal{E}_0 &:= \operatorname{span}\{\mathrm{d}\boldsymbol{x}\} = \operatorname{span}\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\} \\ &\vdots \\ \mathcal{E}_n &:= \operatorname{span}\{\mathrm{d}\boldsymbol{x}, \mathrm{d}\dot{\boldsymbol{y}}, \ldots, \mathrm{d}\boldsymbol{y}^{(n)}\} \end{aligned} \tag{6}$$

and the associated list of dimensions $\rho_0 \le \cdots \le \rho_n$ by $\rho_k := \dim \mathcal{E}_k$. Using this list of dimensions the infinite zeros (respectively the structure at infinity) can be defined as follows.

Definition 2. (Moog, 1988) The number σ_k of zeros at infinity of order less than or equal to k, $k \ge 1$, is $\sigma_k = \rho_k - \rho_{k-1}$. The structure at infinity is given by the list $\{\sigma_1, \ldots, \sigma_n\}$ and the orders of zeros at infinity agree with the ordered list

$\{n_1, n_2, \dots, n_{\sigma_n}\}$ of indices, for which the difference $\sigma_k - \sigma_{k-1} \neq 0$. The indices are repeated $\sigma_k - \sigma_{k-1}$ times.

The maximum number σ_n of this list corresponds precisely to the rank ρ^* of the considered non-linear systems (Di Benedetto *et al.*, 1989). For the class of linear systems, Definitions 1 and 2 agree with the usual linear notion of the rank using a transfer matrix approach (Fliess, 1986a).

For a simplified computer-aided calculation of ρ^* the dimensions of the vector spaces $\mathcal{E}_k$ are characterized in terms of rank-conditions of Jacobian matrices (Grizzle *et al.*, 1987):

$$J_k := \frac{\partial(\dot{y}, \dots, y^{(k)})}{\partial(u, \dots, u^{(k-1)})} = \begin{bmatrix} \frac{\partial \dot{y}}{\partial u} & 0 & 0 \\ \vdots & \ddots & 0 \\ \frac{\partial y^{(k)}}{\partial u} & \cdots & \frac{\partial y^{(k)}}{\partial u^{(k-1)}} \end{bmatrix}, \quad k = 1, \dots, n \tag{7}$$

As a function of the ranks $R_k = \text{rank}\, J_k$, the structure at infinity can be established by

$$\sigma_k = R_k - R_{k-1} \tag{8}$$

Then, obviously, for the differential output rank the equation

$$\rho^* = \sigma_n = R_n - R_{n-1} \tag{9}$$

is valid. This is similar to the definition of (Nijmeijer, 1986) which is based on a geometric approach. However, a fundamental difference between these definitions is that the one in (Nijmeijer, 1986) only holds in a local neighbourhood around an initializing point $x_0 = x(t_0)$. On the other hand, ρ^* is independent of x_0, because of the dimensions of $\mathcal{E}_k$ and therefore the ranks R_k are taken with respect to the field $\mathcal{K}$ and *not* to the real numbers (Di Benedetto *et al.*, 1989).

3. Structured Systems and Graphs

Considering eqn. (9) a computation of ρ^* might be very time-extensive, especially in the case of large-scale systems. Not only the computation of the derivatives $\partial y^{(k)}/\partial u^{(l)}$ but also the rank-determination for large matrices J_n and J_{n-1} is difficult. Therefore, another method has to be preferred. One possibility is to use a *structural approach*.

3.1. Structured Systems

In analogy to the theory of linear systems a structured model only considers the existence (respectively non-existence) of dependencies between inputs u, outputs y and states x (Reinschke, 1988). Information about explicit functional connections between variables is omitted. A non-linear structured model is fully described by three matrices $\{\boldsymbol{B}^*, \boldsymbol{C}^*, \boldsymbol{D}^*\}$ which consist of elements 0 and $\{*_h | h \in \mathbb{N}\}$. The relation between a quantitative and a structured non-linear system is defined as follows:

$$\boldsymbol{B}^* = \left.\frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{u}}\right|_{*_h}, \quad \boldsymbol{C}^* = \left.\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right|_{*_h}, \quad \boldsymbol{D}^* = \left.\frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{x}}\right|_{*_h} \tag{10}$$

The operator $|_{*_h}$ means that all non-zero elements of the matrices are replaced by consecutively numbered elements $*_h$. One can think of the structured system as a structural interpretation of a general linear tangent system associated to $\sum$. For further investigations, e.g. the consideration of equilibrium points, it might be useful to examine the quantitative counterparts of eqn. (10):

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{u}}, \ \boldsymbol{C} = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}, \ \boldsymbol{D}(\boldsymbol{x}, \boldsymbol{u}) = \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{x}} \tag{11}$$

For structured systems a *structural rank* is defined as follows.

Definition 3. The structural rank ρ^*_{gen} of a non-linear system $\sum$ is given by

$$\rho^*_{\text{gen}} = R^*_n - R^*_{n-1}, \qquad R^*_i = \max_{*_h}\{\text{rank } \boldsymbol{J}^*_i\} \tag{12}$$

The Jacobian matrices $\boldsymbol{J}^*_i$ are computed according to eqn. (7), considering only structural information $\{\boldsymbol{B}^*, \boldsymbol{C}^*, \boldsymbol{D}^*\}$.

The differential output rank of a quantitatively given non-linear state space model is always limited by the structural rank of the corresponding structured model, which is itself less than or equal to the minimum of m and p:

$$\rho^* \leq \rho^*_{\text{gen}} \leq \min(m, p) \tag{13}$$

It turns out that the structural matrices $\boldsymbol{J}^*_k$ can be simplified in such a way that an easy graph-theoretic description is possible. For this purpose, the Toeplitz-matrix

$$\boldsymbol{H}^*_k = \begin{bmatrix} \boldsymbol{E}^*_2 & & & 0 \\ \boldsymbol{E}^*_3 & \boldsymbol{E}^*_2 & & \\ \vdots & \ddots & \ddots & \\ \boldsymbol{E}^*_{k+1} & \cdots & \boldsymbol{E}^*_3 & \boldsymbol{E}^*_2 \end{bmatrix}, \quad \boldsymbol{E}^*_l = \boldsymbol{C}^* \boldsymbol{D}^{*^{l-2}} \boldsymbol{B}^*, \quad \boldsymbol{H}^*_0 = \boldsymbol{0} \tag{14}$$

has to be considered. The following equality holds (Wey *et al.*, 1994):

$$\rho^*_k = \max_{*_h}\{\text{rank } \boldsymbol{J}^*_k\} + n = \max_{*_h}\{\text{rank } \boldsymbol{H}^*_k\} + n, \quad k = 1, \ldots, n \tag{15}$$

Hence, the structural rank can be evaluated by

$$\rho^*_{\text{gen}} = \max_{*_h}\{\text{rank } \boldsymbol{H}^*_n\} - \max_{*_h}\{\text{rank } \boldsymbol{H}^*_{n-1}\} \tag{16}$$

In a similar way, it is possible to define the structural orders at infinity. The analogy to (Moog, 1988) produces the following result.

Theorem 1. *For a structured system the number* σ_k^* *of zeros at infinity of order less than equal to* k, $k \geq 1$ *corresponds to the difference* $\max_{*_h}\{\boldsymbol{H}_k^*\} - \max_{*_h}\{rank\ \boldsymbol{H}_{k-1}^*\}$. *The orders at infinity are given by the ordered list* $\{n_1^*, n_2^*, \ldots, n_{\sigma_n^*}^*\}$ *of indices, for which* $\sigma_k^* - \sigma_{k-1}^* \neq 0$. *The indices are repeated* $\sigma_k^* - \sigma_{k-1}^*$ *times.* ∎

If a non-linear system has a structural behaviour, i.e. all the elements of $\{\boldsymbol{B}(\boldsymbol{x}), \boldsymbol{C}, \boldsymbol{D}(\boldsymbol{x}, \boldsymbol{u})\}$ are independent of one another, then the list $\{n_1^*, n_2^*, \ldots, n_{\sigma_n^*}^*\}$ coincides with the orders given by Definition 2.

3.2. Graphs

With the given non-linear system $\sum$ a weighted directed graph (weighted digraph) $\mathcal{G}$ defined by a vertex-set and an edge-set is associated as follows (Reinschke, 1988).

The vertex-set is given by m input vertices denoted by $u_1, u_2, \ldots, u_m$, by n state vertices denoted by $1, 2, \ldots, n$ and by p output vertices denoted by $y_1, y_2, \ldots, y_p$. The edge-set results from the following rules:

- If the state variable x_j really occurs in $a_i(\boldsymbol{x}) + \boldsymbol{b}_i(\boldsymbol{x})\boldsymbol{u}$, i.e. $\partial(a_i + \boldsymbol{b}_i\boldsymbol{u})/\partial x_j \neq 0$, then there exists an edge from vertex j to vertex i with the edge weight $\partial(a_i + \boldsymbol{b}_i\boldsymbol{u})/\partial x_j$.
- If $\partial \boldsymbol{b}_i\boldsymbol{u}/\partial u_j = b_{ij} \neq 0$, then there exists an edge from input vertex u_j to state vertex i with the edge weight b_{ij}.
- If $\partial \boldsymbol{c}_i\boldsymbol{x}/\partial x_j = c_{ij} \neq 0$, then there exists an edge from state vertex j to output vertex y_i with the edge weight c_{ij}.

An example illustrating the construction of graphs is given in Section 5. For the analysis of directed graphs an additional definition is required.

Definition 4. A (directed) *path* is a sequence of edges $\{e_1, e_2, \ldots\}$ such that the initial vertex of the succeeding edge is the final vertex of the preceding edge. The edges occurring in the sequence $\{e_1, e_2, \ldots\}$ are not necessarily distinct. The number of edges contained in the sequence $\{e_1, e_2, \ldots\}$ is called the *length* of the path.

A compact representation of the paths is the so-called *adjacency matrix* (Reinschke, 1988). Using this tool all information about the paths between inputs u_j and outputs y_i of a system $\sum$ can be written through simple matrices $\boldsymbol{E}_l$

$$\boldsymbol{E}_l = \boldsymbol{C}\boldsymbol{D}^{l-2}(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{B}(\boldsymbol{x}), \quad \boldsymbol{D}(\boldsymbol{x}, \boldsymbol{u}) = \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{x}} \tag{17}$$

The element $e_{l_{ij}}$ of $\boldsymbol{E}_l$ is equal to a weighted sum of all the paths between u_j and y_i with length l. Using the matrices $\{\boldsymbol{B}^*, \boldsymbol{C}^*, \boldsymbol{D}^*\}$ structural adjacency matrices $\boldsymbol{E}_l^*$ can be constructed which omit the path-weights but characterize the existence (respectively non-existence) of input-output paths.

The previously introduced properties of structural systems can be easily translated into a graph-based description. The orders at infinity are characterized by the following dependencies:

$$\begin{aligned} n_1^* &= L_1 - 1 \\ n_2^* &= L_2 - n_1^* - 2 \\ &\vdots \\ n_k^* &= L_k - k - \sum_{j=1}^{k-1} n_j^*, \qquad k = 2, \dots, \sigma_n^* \end{aligned} \tag{18}$$

where L_k is the minimum sum of k vertex disjoint input-output path lengths. The total number of orders is equal to the maximum number of vertex disjoint paths in $\mathcal{G}$ (Van der Woude, 1991; Commault *et al.*, 1991).

4. Dynamic Disturbance Decoupling

Consider a system $\sum$ of the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{a}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{P}(\boldsymbol{x})\boldsymbol{w} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} \end{aligned} \tag{19}$$

with $\boldsymbol{x}(t) \in \mathbb{R}^n; \boldsymbol{u}(t) \in \mathbb{R}^m; \boldsymbol{y}(t) \in \mathbb{R}^p; \boldsymbol{w}(t) \in \mathbb{R}^q$ and analytic matrix elements. Disturbance decoupling in the sense given by Isidori (1989) corresponds to the existence of a regular static state feedback law

$$\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{v} \tag{20}$$

with new m-dimensional control $\boldsymbol{v}$ and $\boldsymbol{G}(\boldsymbol{x})$ non-singular for all $\boldsymbol{x}$, which renders the output $\boldsymbol{y}$ independent of the disturbance $\boldsymbol{w}$. In the first step two conditions have to be met for solvability of the non-linear disturbance decoupling problem (DDP).

Theorem 2. (Isidori, 1989) *There exists a feedback of the form* $\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{v}$ *defined in a neighbourhood* $\mathcal{U}$ *of* $\boldsymbol{x}_0$ *which renders the output* $\boldsymbol{y}$ *independent of the disturbance* $\boldsymbol{w}$ *if and only if*

$$L_{\boldsymbol{P}} L_{\boldsymbol{a}}^k c_i(\boldsymbol{x}) = 0 \qquad \textit{for all} \;\; 0 \le k \le r_i - 1, \quad 1 \le i \le m \tag{21}$$

There exists a feedback of the form $\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{v} + \boldsymbol{H}(\boldsymbol{x})\boldsymbol{w}$ *which renders the output* $\boldsymbol{y}$ *independent of the disturbance* $\boldsymbol{w}$ *if and only if*

$$L_{\boldsymbol{P}} L_{\boldsymbol{a}}^k c_i(\boldsymbol{x}) = 0 \qquad \textit{for all} \;\; 0 \le k \le r_i - 2, \quad 1 \le i \le m \tag{22}$$

■

In other words, it is necessary for the existence of a non-linear static state feedback rejecting the disturbance $\boldsymbol{w}$ that all the outputs fulfil the condition

$$\begin{aligned} y_i^{(k)} &= \frac{\partial^k y_i}{\partial t^k} \neq f(\boldsymbol{w}), \qquad k = 1, \dots, r_i - 1 \\ y_i^{(r_i)} &= \frac{\partial^{r_i} y_i}{\partial t^{r_i}} = f(\boldsymbol{w}) \end{aligned} \tag{23}$$

(in the case of non-measurable perturbation). In a more down-to-earth language, the solution is possible if "the control inputs reach the outputs more quickly than the disturbances" (Commault *et al.*, 1991). A similar but weaker result can be achieved when disturbance measurement is considered (DDPdm).

An extension to the static case is to allow dynamic feedback of the form

$$\begin{aligned} \dot{\boldsymbol{z}} &= \boldsymbol{m}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) + \boldsymbol{N}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w})\boldsymbol{v} \\ \boldsymbol{u} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) + \boldsymbol{G}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w})\boldsymbol{v} \end{aligned} \tag{24}$$

which solves the dynamic disturbance decoupling problem with disturbance measurement (DDDPdm), respectively DDDP, if $\boldsymbol{w}$ is omitted (Huijberts *et al.*, 1992). In what follows, the existence of such feedback is determined by considering the structure at infinity. Let Σ_0 denote the system Σ with $\boldsymbol{w} = \boldsymbol{0}$ and Σ_w be an abbreviation for the system Σ where the disturbances are assumed to be an extra set of inputs. In this case one can conclude that the following result holds.

Theorem 3. (Huijberts *et al.*, 1992) *Consider an analytic system Σ and assume that the conditions $p = m$ (square system) and $\rho^* = m$ (invertible system) are satisfied. Let $\boldsymbol{x}_0$ be a strongly regular point for Σ. Then the DDDPdm is locally solvable around $\boldsymbol{x}_0$ if and only if Σ_w and Σ_0 have the same algebraic structure at infinity.* ■

For the DDDP a similar one can be found, if an auxiliary system with additional integrators in front of $\boldsymbol{u}$ is introduced (Huijberts *et al.*, 1992).

Now consider structured systems with an associated graph $\mathcal{G}$. Then a necessary condition for DDDPdm can be characterized in a graph-theoretic way.

Theorem 4. *The DDDPdm defined above is solvable for a non-linear system if its associated graph meets the demands:*

- *the maximum number of vertex disjoint input-output paths is the same for $\mathcal{G}_0$ and $\mathcal{G}_w$ and is equal to ρ^*,*
- *the minimum sum of ρ^* vertex disjoint input-output path lengths is the same for $\mathcal{G}_0$ and $\mathcal{G}_w$.*

■

Proof. A structured system is a linear one with parameters equal to zero or independent of one another. Because of its generic behaviour, the following result is valid.

Definition 5. (Van der Woude, 1991) The maximum number of vertex-disjoint input-/output-paths in $\mathcal{G}_{\text{struc}}$ is equal to the generic rank ρ^*_{gen} of the transfer matrix $\boldsymbol{F}^*(s) = \boldsymbol{C}^*[\boldsymbol{I}s - \boldsymbol{D}^*]^{-1}\boldsymbol{B}^*$.

Due to Theorem 3 both the base system and the disturbed one must have the same structure at infinity. Obviously, the rank has to be identical, too. But due to the assumption Σ_0 is invertible and therefore the rank is equal to m. Hence, eqn. (13) leads to

$$m = \rho^* \leq \rho^*_{\text{gen}} \leq \min(m,p) = m \Rightarrow \rho^*_{\text{gen}} = m \tag{25}$$

This completes the first part of the proof.

The second part is a direct consequence of the identity statement in Theorem 3. As connected with the graph-theoretic determination of the orders at infinity (eqn. (18)), the identity of the path lengths L_k is immediate. ■

The application of Theorem 4 leads to an efficient test of solvability before any analytical terms are examined. A large number of fast algorithms developed for solving transportation problems (Gondran and Minoux, 1986) can be used for this purpose.

When no disturbance measurement is available, the DDDP instead of the DDDPdm must be considered. In this case, one edge is added before each control input in $\boldsymbol{u}$ and Theorem 3 can be applied without changes.

5. Examples

The theory developed in the previous sections will be illustrated by means of two examples. First, consider a non-linear system of the form (19) with (Huijberts *et al.*, 1991)

$$\boldsymbol{a}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ x_3 \\ 0 \end{bmatrix}, \ \boldsymbol{B}(\boldsymbol{x}) = \begin{bmatrix} x_2 \\ 0 \\ x_1 \end{bmatrix}, \ \boldsymbol{P}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{26}$$

Applying the graph-theoretic approach yields the non-weighted directed graph in Fig. 1, where the disturbance w is marked using bold edges. Neglecting w, the

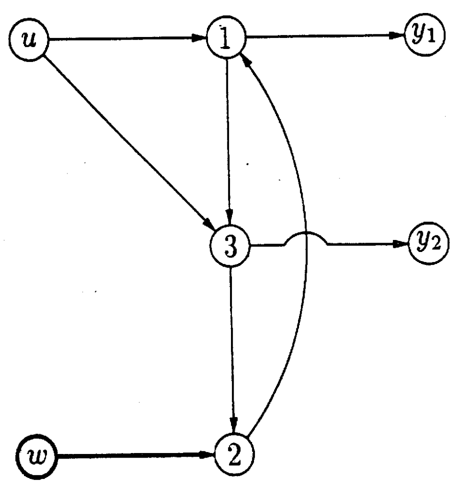

Fig. 1. Directed graph $\mathcal{G}$ associated to system (26).

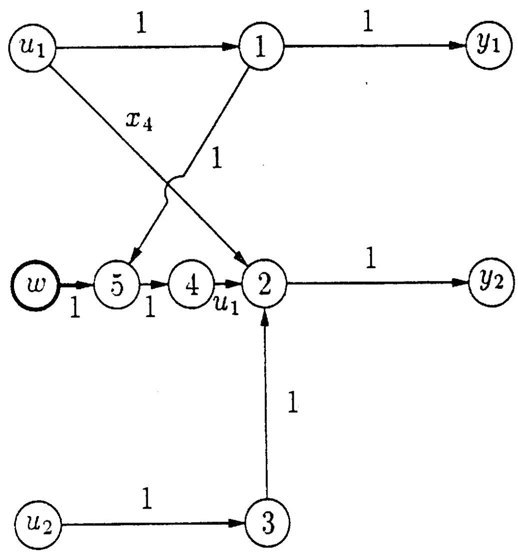

Fig. 2. Weighted directed graph $\mathcal{G}$ associated to system (28).

maximum number of vertex disjoint input-output paths is equal to one. But taking w into consideration two vertex disjoint paths in $\mathcal{G}_w$ exist:

$$u \longrightarrow 3 \longrightarrow y_2 \qquad \text{and} \qquad w \longrightarrow 2 \longrightarrow 1 \longrightarrow y_1 \tag{27}$$

Since the genericity is guaranteed for all non-linear systems characterized by $\mathcal{G}_0$ and $\mathcal{G}_w$, the DDDPdm is not solvable. Adding an edge in front of u does not change this behaviour. Hence the DDDP is not solvable, either.

The second example is given by the system matrices (Respondek, 1991):

$$\begin{aligned} \boldsymbol{a}(\boldsymbol{x}) &= \begin{bmatrix} 0 \\ x_3 \\ 0 \\ x_5 \\ x_1 \end{bmatrix}, \quad \boldsymbol{B}(\boldsymbol{x}) = \begin{bmatrix} 1 & 0 \\ x_4 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \boldsymbol{P}(\boldsymbol{x}) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{28}$$

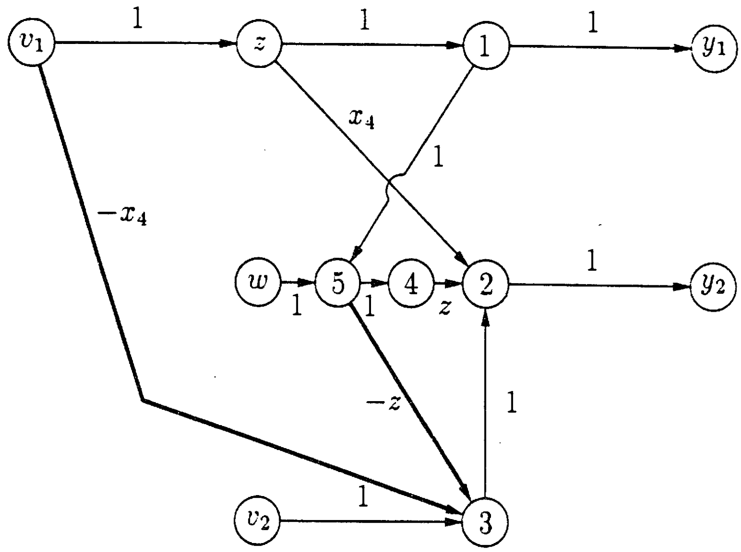

Fig. 3. Weighted directed graph associated to the dynamically disturbance decoupled system (28).

Analysing the corresponding graph for both $\mathcal{G}_0$ and $\mathcal{G}_w$, the number of vertex disjoint input-output paths is equal to two. Moreover, the disturbance does not affect the minimum path length of the disjoint paths

$$u_1 \to 1 \to y_1 \qquad \text{and} \qquad u_2 \to 3 \to 2 \to y_2 \tag{29}$$

Therefore, by Theorem 4, the DDDPdm and the DDDP are solvable. Furthermore, in this example the explicit feedback law needed for disturbance rejection can be determined by use of path weights only. The idea is to break several edges in the graph such that w cannot influence the outputs any more. Respondek (1991) has shown that the DDP is not solvable. Hence an integrator z (a vertex in the graph) is added in front of u_1 and the new inputs v_1, v_2 are introduced. The output y_1 and its time derivatives do not depend on the disturbance, but for y_2 there exists a connecting path

$$w \to 5 \to 4 \to 2 \to y_2 \tag{30}$$

u_2 affects 3, therefore a compensating path can be created by an additional edge between 5 and 3 with weight $-z$ (cf. Fig. 3: bold edges). In a second step the weight x_4 of $z \to 2$ is eliminated by adding an edge $v_1 \to 3$. Transferring the graph-theoretic results back into an analytic form yields the dynamic feedback law

$$\begin{aligned} \dot{z} &= v_1 \\ u_1 &= z \\ u_2 &= -x_4 v_1 - z x_5 + v_2 \end{aligned} \tag{31}$$

which solves the DDDP.

6. Concluding Remarks

In this paper, a structural approach for the solution of the disturbance decoupling problem for non-linear systems is presented. It turns out that the necessary conditions for the solvability of the DDDP (respectively DDDPdm) can be given in terms of graph-theoretic properties. For this aim a directed graph is associated to a non-linear system, which characterizes the dependencies between inputs, states and outputs. Structural results are valid for classes of systems with identical structure and can be evaluated by efficient graph-theoretic algorithms. In general, structural system properties coincide with the original ones whenever the non-linear system behaves like a generic one, that is to say the system parameters are independent of one another. Nevertheless, non-generic systems can be analysed, too. Besides the analysis, it is also possible to find explicit feedback laws using graphical methods. To this end, path weights, calculated by partial differentiations, are taken into consideration. In further activities it must be clarified for which classes of non-linear systems such an approach is useful.

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