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LOGICAL VALUATION OF CONNECTIVES FOR FUZZY CONTROL BY PARTIAL DIFFERENTIAL EQUATIONS

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There exist numerous criteria (axiomatic, progmatic and empirical) to choose from the fuzzy connectives and apply them to an intersection (conjunction) and union (disjunction). So far these valuations have had no direct connection between the fuzzy logic and the control for technical processes. In this paper, a new valuation (logical valuation) of fuzzy connectives is proposed. The dependence of the correcting variable described by the linguistic rules is compared with the dependence of the correcting variable calculated by means of an analytical description of the fuzzy controller.

1. Introduction

By the knowledge-based analysis of fuzzy systems for controlling technical processes an expert specifies his knowledge in the form of linguistic rules. Linguistically generated rules consist mainly of premises and conclusions. Beside the partition of the reference fuzzy sets (Zadeh, 1965) and the method of defuzzification, suitable fuzzy connectives for intersection (conjunction) and union (disjunction) have to be chosen. The essential fuzzy connectives in fuzzy sets theory are triangular norms $\top$ (t-norms for brevity) and triangular conorms $\perp$ (t-conorms (s-norms) for brevity) (Alsina *et al.*, 1983; Gottwald, 1993; Kruse *et al.*, 1994; Weber, 1983). Table 1 shows some t-conorms.

Tab. 1. Some t-conorms.

maximum operator	$\perp_{max}\left(\mu_{X_i},\mu_{Y_j}\right) = \max\left(\mu_{X_i},\mu_{Y_j}\right)$
algebraic sum	$\perp_{as}\left(\mu_{X_i},\mu_{Y_j}\right) = \mu_{X_i} + \mu_{Y_j} - \mu_{X_i}\,\mu_{Y_j}$
drastic sum	$\perp_{ds}\left(\mu_{X_i},\mu_{Y_j}\right) = \begin{cases} \max\left(\mu_{X_i},\mu_{Y_j}\right) & \forall \min\left(\mu_{X_i},\mu_{Y_j}\right) = 0 \\ 1 & \text{otherwise} \end{cases}$

Triangular norms (and their dual conorms) are algebraic operations on $\mathbb{D}_N^+ = [0,1]$ which were suggested by Menger (1942) and proved to be useful in the theory of probabilistic metric spaces (Schweizer and Sklar, 1983). Triangular norms are

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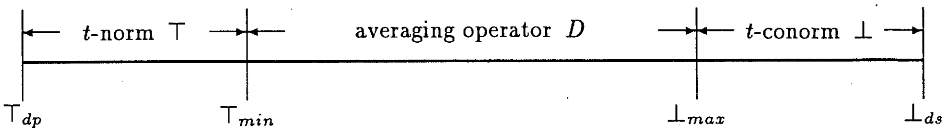

Fig. 1. Relationship between t-norms $\top$, t-conorms $\perp$, and the averaging operator D.

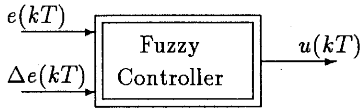

Fig. 2. Block diagram of the fuzzy controller.

generally accepted today (Schweizer and Sklar, 1961) and play a fundamental role in probabilistic metric spaces, probabilistic norms and scalar products, multiple-valued logic, and fuzzy sets theory. Another class of fuzzy connectives are the averaging operators D. The values of the averaging operator D are located between the $\top_{min}$ (t- norm) and $\perp_{max}$ (t-conorm) (see Fig. 1).

In this paper, fuzzy connectives are investigated with respect to the dependence of the correcting variable $u(kT)$ (crisp output). The starting point for our research is a standard fuzzy controller with two inputs, error $e(kT) = w(kT) - y(kT)$ ($w(kT)$ denotes the command variable; $y(kT)$ stands for the control variable), the rate of error $\Delta e(kT) = e(kT) - e((k-1)T)$, and the output $u(kT)$ (Fig. 2). For defuzzification we use the height defuzzification (Driankov *et al.*, 1993) which comprises the operator activation, aggregation and defuzzification. The height defuzzification is both very simple and very quick. The reference fuzzy sets of the output variable of the fuzzy controller are singletons. They are discribed through the modal value m. The fuzzy connective for the union is through the use of the height defuzzification of the sum operator $\xi\left(\mu_{X_i}, \mu_{Y_j}\right) = \mu_{X_i} + \mu_{Y_j}$.

Thus we investigate only the fuzzy connectives for intersection that are t-norms $\top$ and the averaging operators D (Mizumoto, 1989) (Table 2 and 3). Here t-norms $\top$ and averaging operators D are suitable functions $\top : \mathbb{D}_N^+ \times \mathbb{D}_N^+ \to \mathbb{D}_N^+$ and $D : \mathbb{D}_N^+ \times \mathbb{D}_N^+ \to \mathbb{D}_N^+$ whose restrictions to $\mathbb{D}_N^+ \times \mathbb{D}_N^+$ coincide with conjunction, respectively, and which satisfy such general properties as associativity, commutativity and monotonicity. The degrees of fulfilment

$$\alpha_z\Big(e, \Delta e\Big) = \top_z\Big(\mu_{X_i}(e), \mu_{Y_j}(\Delta e)\Big) \tag{1}$$

resp.

$$\alpha_z\left(e, \Delta e\right) = D_z\Big(\mu_{X_i}(e), \mu_{Y_j}(\Delta e)\Big) \tag{2}$$

where X_i (with the degree of possibility $\mu_{X_i}(e)$) and Y_j (with the degree of possibility $\mu_{Y_j}(\Delta e)$), are the reference fuzzy sets

$$\mathbb{X} = \left\{ X_i \mid X_i \rightarrow [0,1] \quad \forall\, i = 1, \ldots, m_E \right\} \tag{3}$$

$$\mathbb{Y} = \left\{ Y_j \mid Y_j \rightarrow [0,1] \quad \forall\, j = 1, \ldots, m_{\Delta E} \right\} \tag{4}$$

respectively, over any space $\mathbb{D}_N = [-1,1]$ for the premise, for each control rule R_z, $z = 1, \ldots, m_E m_{\Delta E}$.

Tab. 2. Investigated t-norms.

algebraic product	$\top_{ap}\left(\mu_{X_i}, \mu_{Y_j}\right) = \mu_{X_i}\, \mu_{Y_j}$
minimum operator	$\top_{min}\left(\mu_{X_i}, \mu_{Y_j}\right) = \min\left(\mu_{X_i}, \mu_{Y_j}\right)$
Lukasiewicz's operator	$\top_{Luk}\left(\mu_{X_i}, \mu_{Y_j}\right) = \max\left(0, \mu_{X_i} + \mu_{Y_j} - 1\right)$
Hamacher's product	$\top_{hp}\left(\mu_{X_i}, \mu_{Y_j}\right) = \dfrac{\mu_{X_i}\, \mu_{Y_j}}{\mu_{X_i} + \mu_{Y_j} - \mu_{X_i}\, \mu_{Y_j}}$
Einstein's product	$\top_{ep}\left(\mu_{X_i}, \mu_{Y_j}\right) = \dfrac{\mu_{X_i}\, \mu_{Y_j}}{2 - \left(\mu_{X_i} + \mu_{Y_j} - \mu_{X_i}\, \mu_{Y_j}\right)}$
drastic product	$\top_{dp}\left(\mu_{X_i}, \mu_{Y_j}\right) = \begin{cases} \min\left(\mu_{X_i}, \mu_{Y_j}\right) & \forall\ \max(\mu_{X_i}, \mu_{Y_j}) = 1 \\ 0 & \text{otherwise} \end{cases}$

Tab. 3. Investigated averaging operators D.

harmonic mean	$D_{hm}\left(\mu_{X_i}, \mu_{Y_j}\right) = \dfrac{2\, \mu_{X_i}\, \mu_{Y_j}}{\mu_{X_i} + \mu_{Y_j}}$
geometric mean	$D_{gm}\left(\mu_{X_i}, \mu_{Y_j}\right) = \sqrt{\mu_{X_i}\, \mu_{Y_j}}$
arithmetic mean	$D_{am}\left(\mu_{X_i}, \mu_{Y_j}\right) = \dfrac{\mu_{X_i} + \mu_{Y_j}}{2}$

The degree of fulfilment is computed according to the t-norms $\top$ and the averaging operators D. The results of the logical valuation are independent of the membership functions μ if the reference fuzzy sets are convex, orthogonal and normal. The dependence of the correcting variable $u(kT)$ described by the linguistic rules, e.g.

$$u(kT)^R = R\Big(e(kT), \Delta e(kT)\Big) \tag{5}$$

is compared with the dependence of the correcting variable $u(kT)$ calculated by means of the analytical description of the fuzzy controller

$$u(kT)^f = f\Big(e(kT), \Delta e(kT)\Big) \tag{6}$$

The dependence of the correcting variable $u(kT)$ on the error $e(kT)$ and/or the rate of the error $\Delta e(kT)$ are investigated in connection with the linguistic rules. For this purpose, the fuzzy controller is split into partial fuzzy cells $\mathcal{F}(\mathcal{A})$, $\mathcal{F}(\mathcal{B})$, $\mathcal{F}(\mathcal{C})$, and $\mathcal{F}(\mathcal{D})$ (Berger, 1995). The fuzzy cell $\mathcal{F}$ and the partial fuzzy cells result from the partition of the reference fuzzy sets X_i and Y_j with $i = 1, \ldots, m_E$ and $j = 1, \ldots, m_{\Delta E}$ over the space $\mathbb{D}^2$.

2. Analytical Expression

The fuzzy controller can be described by one analytical function for a fuzzy cell $\mathcal{F}$. The fuzzy cell $\mathcal{F}$ results from the partition of the reference fuzzy sets X_i and Y_j (Fig. 3). A maximum of four relation rules (linguistic rules) describes the fuzzy controller in the fuzzy cell $\mathcal{F}$. The partial fuzzy cells $\mathcal{F}(\mathcal{A})$, $\mathcal{F}(\mathcal{B})$, $\mathcal{F}(\mathcal{C})$, and $\mathcal{F}(\mathcal{D})$ consist of one relation rule (Fig. 4) (e.g. the marked square in Fig. 3). For presentation of the fuzzy cell $\mathcal{F}$, the fulfilment $\alpha_z\left(e(kT), \Delta e(kT)\right)$ for $z = 1, \ldots, m_E\, m_{\Delta E}$ together with the membership functions $\mu_{X_i}(e(kT))$ and $\mu_{Y_j}(\Delta e(kT))$ are split into α_a^{RR}, α_b^{RL}, α_c^{LR}, and α_d^{LL}, with $a = \tilde{z}, b = \tilde{z}+1, c = \tilde{z}+m_{\Delta E}$ and $d = \tilde{z}+m_{\Delta E}+1$, where $\tilde{z} \in \{1, \ldots, z - m_{\Delta E} - 1\}$, e.g.

$$\alpha_a^{RR} = \top_a\left(\mu_{X_i}^R, \mu_{Y_j}^R\right), \qquad \alpha_b^{RL} = \top_b\left(\mu_{X_i}^R, \mu_{Y_{j+1}}^L\right)$$

$$\alpha_c^{LR} = \top_c\left(\mu_{X_{i+1}}^L, \mu_{Y_j}^R\right), \qquad \alpha_d^{LL} = \top_d\left(\mu_{X_{i+1}}^L, \mu_{Y_{j+1}}^L\right)$$

resp.

$$\alpha_a^{RR} = D_a\left(\mu_{X_i}^R, \mu_{Y_j}^R\right), \qquad \alpha_b^{RL} = D_b\left(\mu_{X_i}^R, \mu_{Y_{j+1}}^L\right)$$

$$\alpha_c^{LR} = D_c\left(\mu_{X_{i+1}}^L, \mu_{Y_j}^R\right), \qquad \alpha_d^{LL} = D_d\left(\mu_{X_{i+1}}^L, \mu_{Y_{j+1}}^L\right)$$

with the index $z = m_{\Delta E}i - m_{\Delta E} + j$. The superscripts L and R indicate the left and right side of the reference fuzzy sets X_i and Y_j, respectively (Fig. 5).

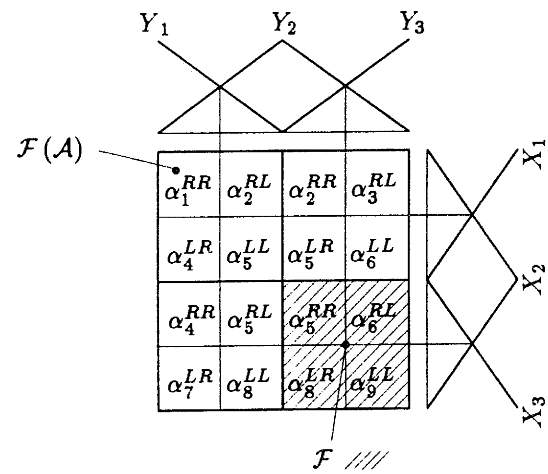

Fig. 3. Fuzzy cells $\mathcal{F}$ with reference fuzzy sets $Y_j\,(j = 1,\ldots,m_{\Delta E})$ and $X_i\,(i = 1,\ldots,m_E)$ for $m_{\Delta E} = 3$ and $m_E = 3$.

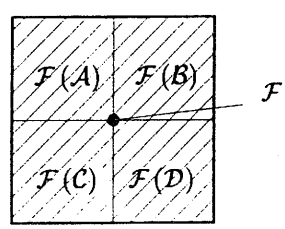

Fig. 4. Fuzzy cell $\mathcal{F}$ with partial fuzzy cells $\mathcal{F}(\mathcal{A})$, $\mathcal{F}(\mathcal{B})$, $\mathcal{F}(\mathcal{C})$, and $\mathcal{F}(\mathcal{D})$.

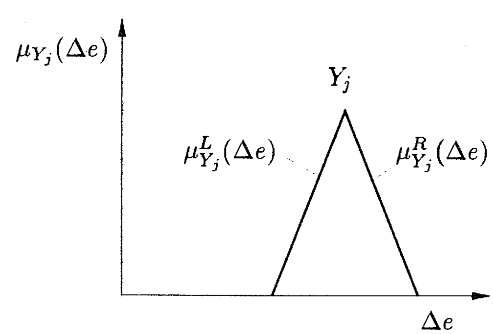

Fig. 5. Separation of the membership function $\mu_{Y_j}(\Delta e)$.

The correcting variable is computed by means of the height defuzzification

$$u(kT) = \frac{\sum_{k=1}^{m_U} \int_{\mathbb{D}} \tilde{\mu}_{\tilde{U},k}(u)\tilde{u}\,\mathrm{d}u}{\sum_{k=1}^{m_U} \int_{\mathbb{D}} \tilde{\mu}_{\tilde{U},k}(u)\,\mathrm{d}u} = \frac{\sum_{k=1}^{m_U} \int_{\mathbb{D}} \mu_{\tilde{U},k}(u) \sum_{z=1}^{m_E m_{\Delta E}} \alpha_{k,z}\tilde{u}\,\mathrm{d}u}{\sum_{k=1}^{m_U} \int_{\mathbb{D}} \mu_{\tilde{U},k}(u) \sum_{z=1}^{m_E m_{\Delta E}} \alpha_{k,z}\,\mathrm{d}u}$$

$$= \frac{\sum_{k=1}^{m_U} \left(\sum_{z=1}^{m_E m_{\Delta E}} \alpha_{k,z} \right) A_k m_k}{\sum_{k=1}^{m_U} \left(\sum_{z=1}^{m_E m_{\Delta E}} \alpha_{k,z} \right) A_k} = \frac{\sum_{k=1}^{m_U} \left(\sum_{z=1}^{m_E m_{\Delta E}} \mathrm{T}^{k,z}(\mu_{\mathbb{X}}, \mu_{\mathbb{Y}}) \right) A_k m_k}{\sum_{k=1}^{m_U} \left(\sum_{z=1}^{m_E m_{\Delta E}} \mathrm{T}^{k,z}(\mu_{\mathbb{X}}, \mu_{\mathbb{Y}}) \right) A_k} \tag{7}$$

resp.

$$u(kT) = \frac{\sum_{k=1}^{m_U} \left(\sum_{z=1}^{m_E m_{\Delta E}} D^{k,z}(\mu_{\mathbb{X}}, \mu_{\mathbb{Y}}) \right) A_k m_k}{\sum_{k=1}^{m_U} \left(\sum_{z=1}^{m_E m_{\Delta E}} D^{k,z}(\mu_{\mathbb{X}}, \mu_{\mathbb{Y}}) \right) A_k} \tag{8}$$

where $\tilde{\mu}_{\tilde{U},k}$ is the k-th membership function of the reference fuzzy set $\tilde{\tilde{U}}_k$ of the output u, $k = 1, \ldots, m_U$, $\mu_{\tilde{U},k}$ denotes the k-th membership function of the reference fuzzy set $\tilde{U}_k$, the result of all rules z for the reference fuzzy set U_k, A_k stands for the k-th surface under the reference fuzzy set $\tilde{U}_k$ (for singletons $A_k = 1\ \forall\, k$), m_k is the k-th modal value of the reference fuzzy set $\tilde{U}_k$, $\mathrm{T}^{k,z}$ is the k-th t-norm of the z-th rule, $z = 1, \ldots, m_E\, m_{\Delta E}$, $D^{k,z}$ denotes the k-th averaging operator D of the z-th rule, $\alpha^{k,z}$ is the k-th degree of fulfilment of the z-th rule.

Equation (7), resp. (8), can be rewritten as

$$u(kT) = fbf^a m_A + fbf^b m_B + fbf^c m_C + fbf^d m_D \tag{9}$$

where fbf^a is the fuzzy basis function (Wang, 1994) in the partial fuzzy cell $\mathcal{F}(\mathcal{A})$ for the control rules R_a. The quantities fbf^b, fbf^c, and fbf^d are the other fuzzy basis functions for the partial fuzzy cells $\mathcal{F}(\mathcal{B})$, $\mathcal{F}(\mathcal{C})$, and $\mathcal{F}(\mathcal{D})$ as well as m is the modal value (Pedrycz, 1993) of the reference fuzzy sets $U_A, U_B, U_C, U_D \in \mathbb{U}$, with

$$\mathbb{U} = \left\{ U_k \mid U_k \to [0,1] \quad \forall\, k = 1, \ldots, m_U \right\} \tag{10}$$

With the use of eqn. (9) we computed an analytical description $u(kT)^f$ for the logical valuation.

3. Logical Valuation

Definition 1. A function $\top : \mathbb{D}_N^+ \times \mathbb{D}_N^+ \to \mathbb{D}_N^+$ is called the t-norm if for any $\mu_{X_i}, \mu_{Y_j}, \mu_{Z_k} \in \mathbb{D}_N^+$ the following conditions are safisfied:

(I) $\top(\mu_{X_i}, 1) = \mu_{X_i}$ (the existence of the unit element)

(II) $\mu_{X_i} \leq \mu_{Y_j} \Longrightarrow \top(\mu_{X_i}, \mu_{Z_k}) \leq \top(\mu_{Y_j}, \mu_{Z_k})$ (the monotonic property)

(III) $\top(\mu_{X_i}, \mu_{Y_j}) = \top(\mu_{Y_j}, \mu_{X_i})$ (the commutative property)

(IV) $\top(\mu_{X_i}, \top(\mu_{Y_j}, \mu_{Z_k})) = \top(\top(\mu_{X_i}, \mu_{Y_j}), \mu_{Z_k})$ (the associative property)

Definition 2. The averaging operator (or a mean) is a function $D : \mathbb{D}_N^+ \times \mathbb{D}_N^+ \to \mathbb{D}_N^+$ which satisfies the conditions:

(I) $\min(\mu_{X_i}, \mu_{Y_j}) \leq D(\mu_{X_i}, \mu_{Y_j}) \leq \max(\mu_{X_i}, \mu_{Y_j})$ and $D \notin \{\min, \max\}$

(II) $D(\mu_{X_i}, \mu_{Y_j}) = D(\mu_{Y_j}, \mu_{X_i})$ (commutativity)

(III) D is increasing and continuous

(IV) $D(0,0) = 0, \quad D(1,1) = 1$

(V) $D(\mu_{X_i}, \mu_{X_i}) = \mu_{X_i}$ (idempotency)

The logical valuation checks the applicability of t-norms $\top$ (Definition 1) and averaging operators D (Definition 2) to the control of technical processes. The active rules in the fuzzy cell $\mathcal{F}$ are investigated for different arrangements for the conclusion, with reference fuzzy sets U_A and U_B for the correcting variable $u(kT)$. An arrangement for the conclusion (shown in Fig. 6, the 1st situation) can be described by the following four relation rules (cf. Fig. 3):

$$\text{IF } (E \text{ Is } X_2) \text{ AND } (\Delta E \text{ Is } Y_2) \text{ THEN } (U \text{ Is } U_A) \tag{11}$$

$$\text{IF } (E \text{ Is } X_2) \text{ AND } (\Delta E \text{ Is } Y_3) \text{ THEN } (U \text{ Is } U_A) \tag{12}$$

$$\text{IF } (E \text{ Is } X_3) \text{ AND } (\Delta E \text{ Is } Y_2) \text{ THEN } (U \text{ Is } U_B) \tag{13}$$

$$\text{IF } (E \text{ Is } X_3) \text{ AND } (\Delta E \text{ Is } Y_3) \text{ THEN } (U \text{ Is } U_B) \tag{14}$$

U_A	U_A
U_B	U_B

Fig. 6. Fuzzy cell $\mathcal{F}$ with reference fuzzy set U_A in the partial fuzzy cells $\mathcal{F}(\mathcal{A})$, $\mathcal{F}(\mathcal{B})$ and reference fuzzy set U_B in the partial fuzzy cells $\mathcal{F}(\mathcal{C})$, $\mathcal{F}(\mathcal{D})$ (the 1st situation).

The relation rules show that the conclusion $(U \text{ Is } U_A)$ (resp. $(U \text{ Is } U_B)$) is independent of which value of membership functions $\mu_{Y_2}(\Delta e(kT))$ and $\mu_{Y_3}(\Delta e(kT))$

is projected on the rate of error $\Delta e(kT)$ on the reference fuzzy sets Y_2 and Y_3. This does not depend on the fuzzy rate of error ΔE, but depends on the fuzzy error E. The relation rules describe the correcting variable $u(kT)$ independently of the rate of error $\Delta e(kT)$. The other arrangement for the conclusion (shown in Fig. 7, the 2nd situation) can be described as the following four relation rules (Fig. 3):

$$\text{IF } (E \text{ IS } X_2) \text{ AND } (\Delta E \text{ IS } Y_2) \text{ THEN } (U \text{ IS } U_A) \tag{15}$$

$$\text{IF } (E \text{ IS } X_2) \text{ AND } (\Delta E \text{ IS } Y_3) \text{ THEN } (U \text{ IS } U_B) \tag{16}$$

$$\text{IF } (E \text{ IS } X_3) \text{ AND } (\Delta E \text{ IS } Y_2) \text{ THEN } (U \text{ IS } U_A) \tag{17}$$

$$\text{IF } (E \text{ IS } X_3) \text{ AND } (\Delta E \text{ IS } Y_3) \text{ THEN } (U \text{ IS } U_B) \tag{18}$$

U_A	U_B
U_A	U_B

Fig. 7. Fuzzy cell $\mathcal{F}$ with reference fuzzy set U_A in the partial fuzzy cells $\mathcal{F}(\mathcal{A})$, $\mathcal{F}(\mathcal{C})$ and reference fuzzy set U_B in the partial fuzzy cells $\mathcal{F}(\mathcal{B})$, $\mathcal{F}(\mathcal{D})$ (the 2nd situation).

The relation rules show that the conclusion $(U$ Is $U_A)$ (resp. $(U$ Is $U_B))$ is independent of which value of membership functions $\mu_{X_2}(e(kT))$ and $\mu_{X_3}(e(kT))$ is projected on the rate of error $e(kT)$ on the reference fuzzy sets X_2 and X_3. This does not depend on the fuzzy error E, but depends on the fuzzy rate of error ΔE. The relation rules describe the correcting variable $u(kT)$ independently of the error $e(kT)$. The fuzzy connectives are investigated for two different arrangements of the conclusion (situations) in the fuzzy cell $\mathcal{F}$. The two situations and the form of the correcting variable u^R are given in Table 4. The two situations of the different arrangements of the conclusion in the fuzzy cell $\mathcal{F}$ remain true in the whole input space $\mathbb{D}^2$.

Tab. 4. Dependence forms of the correcting variable u^R in connection with the arrangement of the reference fuzzy sets U_A and U_B in the fuzzy cell $\mathcal{F}$.

Arrangement of the fuzzy sets	Dependence form of the correcting variable u^R
U_A U_A U_B U_B	dependence on the error e (1st situation)
U_A U_B U_A U_B	dependence on the rate of error Δe (2nd situation)

The dependence form of the correcting variable is compared with the dependence form of the linguistic rules $u^R(kT) = R(e(kT),\ \Delta e(kT))$ (Table 3) and the dependence form of the analytical expression $u^f(kT) = f(e(kT), \Delta e(kT))$ from the fuzzy controller. If

$$u^R = u^f \tag{19}$$

then the fuzzy controller (according to the fuzzy connectives for the intersection) approximates the linguistic rules. Equation (19) is the valuation for the t-norms $\top$ and averaging operators D for fuzzy control.

4. Algebraic Product $\top_{ap}$

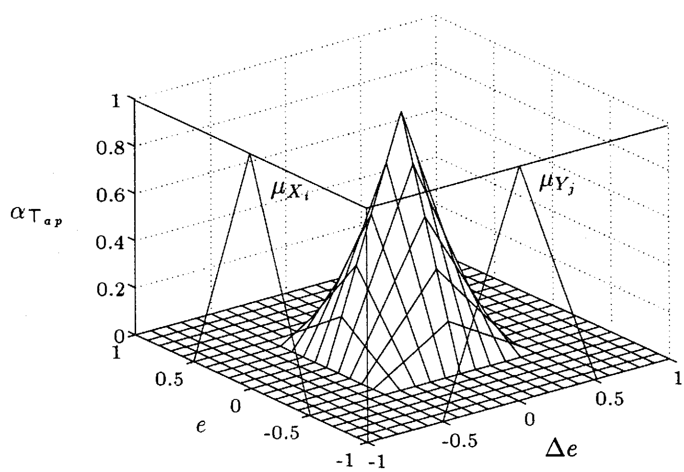

Fig. 8. Binary relation of the algebraic product $\top_{ap}$, $R = \{((e, \Delta e), \alpha(e, \Delta e)) \mid e \in \mathbb{X},\ \Delta e \in \mathbb{Y}\}$, with triangular membership functions μ_{X_i} and μ_{Y_j}.

The algebraic product $\top_{ap}$ is interactive, sensitive, and strict (Fig. 8). A smoother defuzzified output is obtained, and it is an Archimedean t-norm with the additive generator $f(x) = -\log x$ (Mizumoto, 1989). From the mathematical point of view (algebraic properties), the algebraic product $\top_{ap}$ has e.g. commutative and associative properties (Gupta and Qi, 1991), and belongs to the axiomatic class (Kacprzyk, 1983). The degree of fulfilment is

$$\alpha_z\Big(\mu_{X_i}(e), \mu_{Y_j}(\Delta e)\Big) = \mu_{X_i}(e)\ \mu_{Y_j}(\Delta e) \tag{20}$$

The computation of the fulfilment for the partial fuzzy cells $\mathcal{F}(\mathcal{A})$, $\mathcal{F}(\mathcal{B})$, $\mathcal{F}(\mathcal{C})$, and $\mathcal{F}(\mathcal{D})$ can be carried out e.g. according to the formula $\alpha_a^{RR} = \mu_{X_i}^R\ \mu_{Y_j}^R$. By means of convex (Definition 3), normalized ($\max_{x \in \mathbb{D}} \mu(x) = 1$), and orthogonal reference fuzzy sets X_i and Y_j (see Definition 4) (here $1 = \mu^R_{X_i} + \mu^L_{X_{i+1}}$ and $1 = \mu^R_{Y_j} + \mu^L_{Y_{j+1}}$) we obtain

$$\sum_{z=1}^{m_E m_{\Delta E}} \alpha_z = 1 \tag{21}$$

Definition 3. (Lowen, 1980) A fuzzy subset $\mu : \mathbb{D} \rightarrow \mathbb{D}^+_N$ is a convex fuzzy set if for all $x, y \in \mathbb{D}$ and $a \in \mathbb{D}^+_N$

$$\mu\Big(a\,x + (1-a)\,y\Big) \geq \min\Big(\mu(x), \mu(y)\Big)$$

Definition 4. (Berger, 1994) A family $\mathbb{A} = \{A^{(n)}_{i_n}\}_{i=1,\dots,K}$ of fuzzy sets $A^{(n)}_{i_n} = \{(x^{(n)}, \mu^{(n)}_{A^{(n)}_{i_n}}(x^{(n)})) \mid x^{(n)} \in \mathbb{D}_N\}$ is called orthogonal about $\mathbb{D}_N$ if

$$\sum_{i=1}^{K} \mu^{(n)}_{A^{(n)}_{i_n}}(x^{(n)}) = 1 \qquad \forall\, x^{(n)} \in \mathbb{D}_N$$

where $x^{(n)}$ is the n-th input variable of the fuzzy controller.

The fuzzy basis function fbf is computed in the partial fuzzy cell, e.g. $\mathcal{F}(\mathcal{A})$ as $fbf^a = \alpha^{RR}_a$. The crisp output can be then computed as the following analytical expression:

$$\begin{aligned} u(kT) = {} & \underbrace{\Big(1 - \mu^L_{X_{i+1}} - \mu^L_{Y_{j+1}} + \mu^L_{X_{i+1}}\,\mu^L_{Y_{j+1}}\Big)\,m_A}_{\mathcal{F}(\mathcal{A})} + \underbrace{\Big(\mu^L_{Y_{j+1}} - \mu^L_{X_{i+1}}\,\mu^L_{Y_{j+1}}\Big)\,m_B}_{\mathcal{F}(\mathcal{B})} \\ & + \underbrace{\Big(\mu^L_{X_{i+1}} - \mu^L_{X_{i+1}}\mu^L_{Y_{j+1}}\Big)\,m_C}_{\mathcal{F}(\mathcal{C})} + \underbrace{\mu^L_{Y_{j+1}}\,\mu^L_{X_{i+1}}\,m_D}_{\mathcal{F}(\mathcal{D})} \end{aligned} \tag{22}$$

Situation 1. From eqn. (22), after the arrangement of the conclusions in the partial fuzzy cells (partial fuzzy cells $\mathcal{F}(\mathcal{A}), \mathcal{F}(\mathcal{B})$ with $m_\mathcal{A} = m_A = m_\mathcal{B}$, and $\mathcal{F}(\mathcal{C}), \mathcal{F}(\mathcal{D})$ with $m_\mathcal{B} = m_C = m_D$), we get the crisp output as

$$\begin{aligned} u(kT) &= \underbrace{\Big(1 - \mu^L_{X_{i+1}}\Big)\,m_\mathcal{A}}_{\mathcal{F}(\mathcal{A}),\mathcal{F}(\mathcal{B})} + \underbrace{\mu^L_{X_{i+1}}\,m_\mathcal{B}}_{\mathcal{F}(\mathcal{C}),\mathcal{F}(\mathcal{D})} \\ u^f(kT) &= f\Big(e(kT)\Big) \end{aligned} \tag{23}$$

Situation 2. From eqn. (22), after the arrangement of the conclusions in the partial fuzzy cells (partial fuzzy cells $\mathcal{F}(\mathcal{A}), \mathcal{F}(\mathcal{C})$ with $m_\mathcal{A} = m_A = m_C$, and $\mathcal{F}(\mathcal{B}), \mathcal{F}(\mathcal{D})$

with $m_{\mathcal{B}} = m_B = m_D$), we get the crisp output as

$$u(kT) = \underbrace{\left(1 - \mu^L_{Y_{j+1}}\right) m_{\mathcal{A}}}_{\mathcal{F}(\mathcal{A}),\mathcal{F}(\mathcal{C})} + \underbrace{\mu^L_{Y_{j+1}} m_{\mathcal{B}}}_{\mathcal{F}(\mathcal{B}),\mathcal{F}(\mathcal{D})}$$
$$u^f(kT) = f\Big(\Delta e(kT)\Big) \tag{24}$$

The two situations show that the t-norm $\top_{ap}$ approximates the linguistic variable. In the first situation, where the linguistic rules need the generation of the correcting variable $u(kT)$ according to the rate of error $\Delta e(kT)$ (by means of the arrangement of the conclusion in the partial fuzzy cells $\mathcal{F}(\mathcal{A})$, $\mathcal{F}(\mathcal{B})$, $\mathcal{F}(\mathcal{C})$ and $\mathcal{F}(\mathcal{D})$), this can be documented by the analytical expression of the fuzzy controller $u^f(kT) = f(\Delta e(kT))$. In the second situation, where the linguistic rules need the generation of the correcting variable $u(kT)$ according to the error $e(kT)$, this can be documented by the analytical expression of the fuzzy controller $u^f(kT) = f(e(kT))$. The results can also be documented by the plots of the fuzzy basis functions fbf. Figures 9 and 10 show the fuzzy basis functions fbf^a and fbf^b of the partial fuzzy cells $\mathcal{F}(\mathcal{A})$ and $\mathcal{F}(\mathcal{B})$, respectively. Figure 11 shows the addition of the fuzzy basis functions fbf^a and fbf^b. It can be seen that the result is independent of the rate of error Δe.

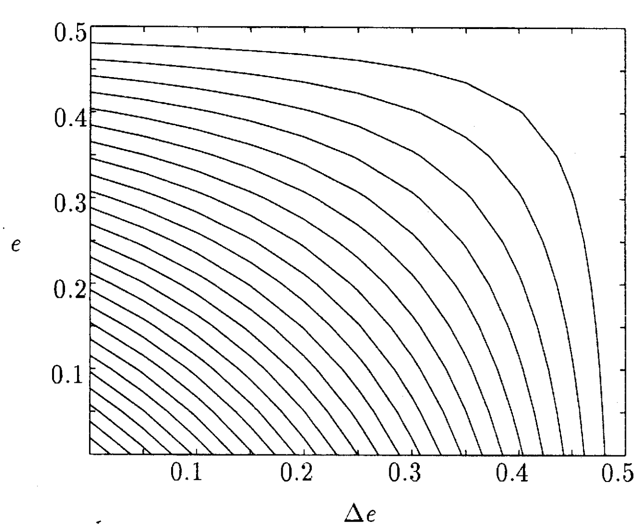

Fig. 9. Fuzzy basis function $fbf^a = \alpha_a^{RR}$ for the partial fuzzy cell $\mathcal{F}(\mathcal{A})$ (partial contour plot of Fig. 8).

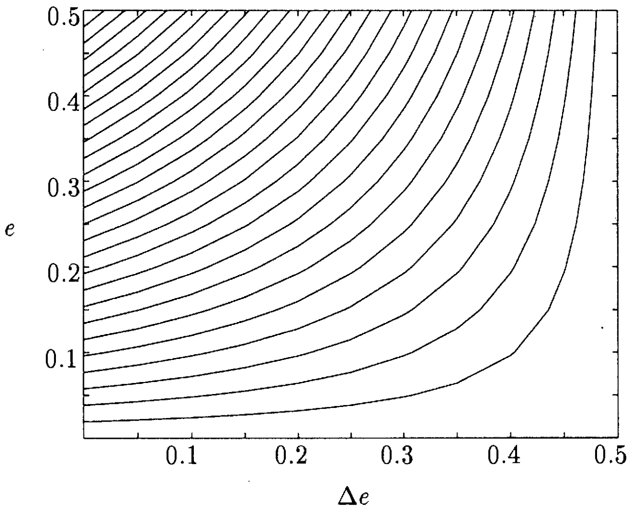

Fig. 10. Fuzzy basis function $fbf^b = \alpha_b^{RL}$ for the partial fuzzy cell $\mathcal{F}(\mathcal{B})$ (partial contour plot of Fig. 8).

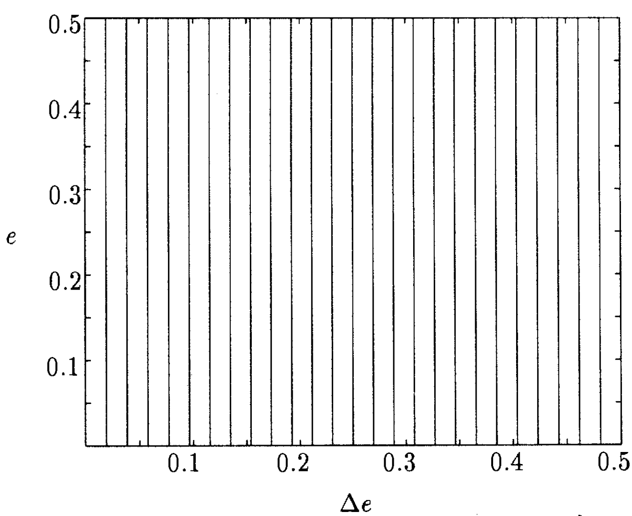

Fig. 11. Sum of the fuzzy basis function fbf^a and fbf^b (contour plot).

5. Valuation with Partial Differential Equations

The logical valuation can also be described as the partial derivative of the fuzzy connectives. The partial derivative of the output of the fuzzy controller (with height defuzzification) is computed as:

$$\frac{\partial u}{\partial \boldsymbol{x}} = \frac{\left(\sum_{k=1}^{m_U}\left(\sum_{z=1}^{m_E m_{\Delta E}} \frac{\partial \top^{k,z}(\mu_{\mathbb{X}},\mu_{\mathbb{Y}})}{\partial \boldsymbol{x}}\right) A_k m_k\right)\left(\sum_{k=1}^{m_U}\left(\sum_{z=1}^{m_E m_{\Delta E}} \top^{k,z}(\mu_{\mathbb{X}},\mu_{\mathbb{Y}})\right) A_k\right)}{\left(\sum_{k=1}^{m_U}\left(\sum_{z=1}^{m_E m_{\Delta E}} \top^{k,z}(\mu_{\mathbb{X}},\mu_{\mathbb{Y}})\right) A_k\right)^2}$$
$$- \frac{\left(\sum_{k=1}^{m_U}\left(\sum_{z=1}^{m_E m_{\Delta E}} \top^{k,z}(\mu_{\mathbb{X}},\mu_{\mathbb{Y}})\right) A_k m_k\right)\left(\sum_{k=1}^{m_U}\left(\sum_{z=1}^{m_E m_{\Delta E}} \frac{\partial \top^{k,z}(\mu_{\mathbb{X}},\mu_{\mathbb{Y}})}{\partial \boldsymbol{x}}\right) A_k\right)}{\left(\sum_{k=1}^{m_U}\left(\sum_{z=1}^{m_E m_{\Delta E}} \top^{k,z}(\mu_{\mathbb{X}},\mu_{\mathbb{Y}})\right) A_k\right)^2} \tag{25}$$

where $\boldsymbol{x} = [e(kT), \Delta e(kT)]^T$. The right-hand side of eqn. (25) can be split (for the fuzzy cell $\mathcal{F}$, with partial fuzzy cells $\mathcal{F}(\mathcal{A})$, $\mathcal{F}(\mathcal{B})$, $\mathcal{F}(\mathcal{C})$, and $\mathcal{F}(\mathcal{D})$) into four terms, as follows:

$$\frac{\partial u}{\partial \boldsymbol{x}} = \frac{\partial fbf^a m_A}{\partial \boldsymbol{x}} + \frac{\partial fbf^b m_B}{\partial \boldsymbol{x}} + \frac{\partial fbf^c m_C}{\partial \boldsymbol{x}} + \frac{\partial fbf^d m_D}{\partial \boldsymbol{x}} \tag{26}$$

A fuzzy connective approximates the linguistic rules (according to the arrangement of the conclusion in the partial fuzzy cells $\mathcal{F}(\mathcal{A})$, $\mathcal{F}(\mathcal{B})$, $\mathcal{F}(\mathcal{C})$, and $\mathcal{F}(\mathcal{D})$) if

$$\frac{\partial u}{\partial \Delta e} = 0 \tag{27}$$

$$\frac{\partial u}{\partial e} = 0 \tag{28}$$

On this assumption about the partial fuzzy cells, using eqns. (27) and (28), we obtain for Situation 1

$$0 = \underbrace{\underbrace{\frac{\partial\left(fbf^a + fbf^b\right)}{\partial \Delta e}}_{=0} m_{\mathcal{A}}}_{\mathcal{F}(\mathcal{A}),\mathcal{F}(\mathcal{B})} + \underbrace{\underbrace{\frac{\partial\left(fbf^c + fbf^d\right)}{\partial \Delta e}}_{=0} m_{\mathcal{B}}}_{\mathcal{F}(\mathcal{C}),\mathcal{F}(\mathcal{D})} \tag{29}$$

and for Situation 2

$$0 = \underbrace{\underbrace{\frac{\partial\left(fbf^a + fbf^c\right)}{\partial e}}_{=0} m_{\mathcal{A}}}_{\mathcal{F}(\mathcal{A}),\mathcal{F}(\mathcal{C})} + \underbrace{\underbrace{\frac{\partial\left(fbf^b + fbf^d\right)}{\partial e}}_{=0} m_{\mathcal{B}}}_{\mathcal{F}(\mathcal{B}),\mathcal{F}(\mathcal{D})} \tag{30}$$

Now, we can describe the logical valuation by means of the valuation equation (i.e. a partial differential equation) in Tables 5 and 6.

Tab. 5. Valuation with valuation equation Situation 1.

Valuation equation	Statement about the fuzzy connectives
$\frac{\frac{\partial\left(\alpha_a^{RR}+\alpha_b^{RL}\right)}{\partial\Delta e}}{\alpha_a^{RR}+\alpha_b^{RL}} = \frac{\frac{\partial\left(\alpha_c^{LR}+\alpha_d^{LL}\right)}{\partial\Delta e}}{\alpha_c^{LR}+\alpha_d^{LL}}$	linguistic rules are approximated
$\frac{\frac{\partial\left(\alpha_a^{RR}+\alpha_b^{RL}\right)}{\partial\Delta e}}{\alpha_a^{RR}+\alpha_b^{RL}} \neq \frac{\frac{\partial\left(\alpha_c^{LR}+\alpha_d^{LL}\right)}{\partial\Delta e}}{\alpha_c^{LR}+\alpha_d^{LL}}$	linguistic rules are not approximated

Tab. 6. Valuation with valuation equation Situation 2.

Valuation equation	Results of the fuzzy connectives
$\frac{\frac{\partial\left(\alpha_a^{RR}+\alpha_c^{LR}\right)}{\partial e}}{\alpha_a^{RR}+\alpha_c^{LR}} = \frac{\frac{\partial\left(\alpha_b^{RL}+\alpha_d^{LL}\right)}{\partial e}}{\alpha_b^{RL}+\alpha_d^{LL}}$	linguistic rules are approximated
$\frac{\frac{\partial\left(\alpha_a^{RR}+\alpha_c^{LR}\right)}{\partial e}}{\alpha_a^{RR}+\alpha_c^{LR}} \neq \frac{\frac{\partial\left(\alpha_b^{RL}+\alpha_d^{LL}\right)}{\partial e}}{\alpha_b^{RL}+\alpha_d^{LL}}$	linguistic rules are not approximated

The following equations show the results of the logical valuation (Situation 1) for any t-norms $\top$ and averaging operators D:

Algebraic product $\top_{ap}$:

$$\frac{\partial\left(\alpha_{\tilde{z}}^{RR}+\alpha_{\tilde{z}+1}^{RL}\right)}{\partial\Delta e} = -\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}\left(1-\mu_{X_{i+1}}^{L}\right)+\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}-\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}\mu_{X_{i+1}}^{L}=0$$

$$\frac{\partial\left(\alpha_{\tilde{z}+m_{\Delta E}}^{LR}+\alpha_{\tilde{z}+m_{\Delta E}+1}^{LL}\right)}{\partial\Delta e} = -\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}\mu_{X_{i+1}}^{L}+\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}\mu_{X_{i+1}}^{L}=0$$

$$\Longrightarrow \frac{\frac{\partial\left(\alpha_{\tilde{z}}^{RR}+\alpha_{\tilde{z}+1}^{RL}\right)}{\partial\Delta e}}{\alpha_{\tilde{z}}^{RR}+\alpha_{\tilde{z}+1}^{RL}} = \frac{\frac{\partial\left(\alpha_{\tilde{z}+m_{\Delta E}}^{LR}+\alpha_{\tilde{z}+m_{\Delta E}+1}^{LL}\right)}{\partial\Delta e}}{\alpha_{\tilde{z}+m_{\Delta E}}^{LR}+\alpha_{\tilde{z}+m_{\Delta E}+1}^{LL}} \qquad (31)$$

Einstein's product $\top_{ep}$:

$$\frac{\partial\left(\alpha_{\bar{z}}^{RR}+\alpha_{\bar{z}+1}^{RL}\right)}{\partial\Delta e}=\frac{\mu_{X_{i+1}}^{L}\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}}{2-\mu_{X_{i+1}}^{L}-\mu_{Y_{j+1}}^{L}+\mu_{X_{i+1}}^{L}\mu_{Y_{j+1}}^{L}}$$
$$-\frac{\mu_{X_{i+1}}^{L}\mu_{Y_{j+1}}^{L}\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}\left(\mu_{X_{i+1}}^{L}-1\right)}{\left(2-\mu_{X_{i+1}}^{L}-\mu_{Y_{j+1}}^{L}+\mu_{X_{i+1}}^{L}\mu_{Y_{j+1}}^{L}\right)^{2}}$$
$$-\frac{\mu_{X_{i+1}}^{L}\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}}{2-\mu_{X_{i+1}}^{L}-\mu_{Y_{j+1}}^{L}+\mu_{X_{i+1}}^{L}\mu_{Y_{j+1}}^{L}}-\frac{\mu_{X_{i+1}}^{L}\left(1-\mu_{Y_{j+1}}^{L}\right)\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}\left(\mu_{X_{i+1}}^{L}-1\right)}{\left(1+\mu_{Y_{j+1}}^{L}+\mu_{X_{i+1}}^{L}\mu_{Y_{j+1}}^{L}\right)^{2}}.$$

$$\frac{\partial\left(\alpha_{\bar{z}+m_{\Delta E}}^{LR}+\alpha_{\bar{z}+m_{\Delta E}+1}^{LL}\right)}{\partial\Delta e}=\frac{\mu_{X_{i+1}}^{L}\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}}{2-\mu_{X_{i+1}}^{L}-\mu_{Y_{j+1}}^{L}+\mu_{X_{i+1}}^{L}\mu_{Y_{j+1}}^{L}}$$
$$-\frac{\mu_{X_{i+1}}^{L}\mu_{Y_{j+1}}^{L}\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}\left(\mu_{X_{i+1}}^{L}-1\right)}{\left(2-\mu_{X_{i+1}}^{L}-\mu_{Y_{j+1}}^{L}+\mu_{X_{i+1}}^{L}\mu_{Y_{j+1}}^{L}\right)^{2}}-\frac{\mu_{X_{i+1}}^{L}\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}}{2-\mu_{X_{i+1}}^{L}-\mu_{Y_{j+1}}^{L}+\mu_{X_{i+1}}^{L}\mu_{Y_{j+1}}^{L}}$$
$$-\frac{\mu_{X_{i+1}}^{L}\left(1-\mu_{Y_{j+1}}^{L}\right)\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}\left(\mu_{X_{i+1}}^{L}-1\right)}{\left(1+\mu_{Y_{j+1}}^{L}+\mu_{X_{i+1}}^{L}\mu_{Y_{j+1}}^{L}\right)^{2}}$$
$$\Longrightarrow\frac{\frac{\partial\left(\alpha_{\bar{z}}^{RR}+\alpha_{\bar{z}+1}^{RL}\right)}{\partial\Delta e}}{\alpha_{\bar{z}}^{RR}+\alpha_{\bar{z}+1}^{RL}}\neq\frac{\frac{\partial\left(\alpha_{\bar{z}+m_{\Delta E}}^{LR}+\alpha_{\bar{z}+m_{\Delta E}+1}^{LL}\right)}{\partial\Delta e}}{\alpha_{\bar{z}+m_{\Delta E}}^{LR}+\alpha_{\bar{z}+m_{\Delta E}+1}^{LL}} \tag{32}$$

Harmonic mean D_{hm}:

$$\frac{\partial\left(\alpha_{\bar{z}}^{RR}+\alpha_{\bar{z}+1}^{RL}\right)}{\partial\Delta e}=-2\left[\frac{\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}(1-\mu_{X_{i+1}}^{L})}{2-\mu_{X_{i+1}}^{L}-\mu_{Y_{j+1}}^{L}}-\frac{(1-\mu_{X_{i+1}}^{L})(1-\mu_{Y_{j+1}}^{L})\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}}{\left(2-\mu_{X_{i+1}}^{L}-\mu_{Y_{j+1}}^{L}\right)^{2}}\right]$$
$$+2\left[\frac{\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}(1-\mu_{X_{i+1}}^{L})}{1-\mu_{X_{i+1}}^{L}+\mu_{Y_{j+1}}^{L}}-\frac{\frac{\mathrm{d}\mu_{Y_{j+1}}^{L}}{\mathrm{d}\Delta e}\mu_{X_{i+1}}^{L}(1-\mu_{X_{i+1}}^{L})}{\left(1-\mu_{X_{i+1}}^{L}+\mu_{Y_{j+1}}^{L}\right)^{2}}\right]$$

$$\frac{\partial\left(\alpha^{LR}_{\tilde z+m_{\Delta E}}+\alpha^{LL}_{\tilde z+m_{\Delta E}+1}\right)}{\partial\Delta e}=-2\left[\frac{\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}\mu^L_{X_{i+1}}}{\mu^L_{X_{i+1}}-1-\mu^L_{Y_{j+1}}}+\frac{\mu^L_{X_{i+1}}(1-\mu^L_{Y_{j+1}})\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}}{\left(\mu^L_{X_{i+1}}-1-\mu^L_{Y_{j+1}}\right)^2}\right]$$

$$+2\left[\frac{\mu^L_{X_{i+1}}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}}{\mu^L_{X_{i+1}}+\mu^L_{Y_{j+1}}}-\frac{\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}\mu^L_{Y_{j+1}}\mu^L_{X_{i+1}}}{\left(\mu^L_{X_{i+1}}+\mu^L_{Y_{j+1}}\right)^2}\right] \tag{33}$$

$$\Longrightarrow\quad\frac{\frac{\partial\left(\alpha^{RR}_{\tilde z}+\alpha^{RL}_{\tilde z+1}\right)}{\partial\Delta e}}{\alpha^{RR}_{\tilde z}+\alpha^{RL}_{\tilde z+1}}\neq\frac{\frac{\partial\left(\alpha^{LR}_{\tilde z+m_{\Delta E}}+\alpha^{LL}_{\tilde z+m_{\Delta E}+1}\right)}{\partial\Delta e}}{\alpha^{LR}_{\tilde z+m_{\Delta E}}+\alpha^{LL}_{\tilde z+m_{\Delta E}+1}}$$

Geometric mean D_{gm}**:**

$$\frac{\partial\left(\alpha^{RR}_{\tilde z}+\alpha^{RL}_{\tilde z+1}\right)}{\partial\Delta e}=\frac{1}{2}\left[\frac{\sqrt{1-\mu^L_{X_{i+1}}}}{\sqrt{\mu^L_{Y_{j+1}}}}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}-\frac{\sqrt{1-\mu^L_{X_{i+1}}}}{\sqrt{1-\mu^L_{Y_{j+1}}}}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}\right]$$

$$\frac{\frac{\partial\left(\alpha^{RR}_{\tilde z}+\alpha^{RL}_{\tilde z+1}\right)}{\partial\Delta e}}{\alpha^{RR}_{\tilde z}+\alpha^{RL}_{\tilde z+1}}=\frac{\frac{1}{2}\left[\frac{\sqrt{1-\mu^L_{X_{i+1}}}}{\sqrt{\mu^L_{Y_{j+1}}}}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}-\frac{\sqrt{1-\mu^L_{X_{i+1}}}}{\sqrt{1-\mu^L_{Y_{j+1}}}}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}\right]}{\sqrt{1-\mu^L_{X_{i+1}}}\sqrt{\mu^L_{Y_{j+1}}}+\sqrt{1-\mu^L_{X_{i+1}}}\sqrt{1-\mu^L_{Y_{j+1}}}}$$

$$=\frac{1}{2}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}\frac{\frac{1}{\sqrt{\mu^L_{Y_{j+1}}}}-\frac{1}{\sqrt{1-\mu^L_{Y_{j+1}}}}}{\sqrt{\mu^L_{Y_{j+1}}}+\sqrt{1-\mu^L_{Y_{j+1}}}}$$

$$\frac{\partial\left(\alpha^{LR}_{\tilde z+m_{\Delta E}}+\alpha^{LL}_{\tilde z+m_{\Delta E}+1}\right)}{\partial\Delta e}=\frac{1}{2}\left[\frac{\sqrt{\mu^L_{X_{i+1}}}}{\sqrt{\mu^L_{Y_{j+1}}}}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}-\frac{\sqrt{\mu^L_{X_{i+1}}}}{\sqrt{1-\mu^L_{Y_{j+1}}}}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}\right]$$

$$\frac{\frac{\partial\left(\alpha^{LR}_{\tilde z+m_{\Delta E}}+\alpha^{LL}_{\tilde z+m_{\Delta E}+1}\right)}{\partial\Delta e}}{\alpha^{LR}_{\tilde z+m_{\Delta E}}+\alpha^{LL}_{\tilde z+m_{\Delta E}+1}}=\frac{\frac{1}{2}\left[\frac{\sqrt{\mu^L_{X_{i+1}}}}{\sqrt{\mu^L_{Y_{j+1}}}}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}-\frac{\sqrt{\mu^L_{X_{i+1}}}}{\sqrt{1-\mu^L_{Y_{j+1}}}}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}\right]}{\sqrt{\mu^L_{X_{i+1}}}\sqrt{1-\mu^L_{Y_{j+1}}}+\sqrt{\mu^L_{X_{i+1}}}\sqrt{\mu^L_{Y_{j+1}}}}$$

$$=\frac{1}{2}\frac{\mathrm{d}\mu^L_{Y_{j+1}}}{\mathrm{d}\Delta e}\frac{\frac{1}{\sqrt{\mu^L_{Y_{j+1}}}}-\frac{1}{\sqrt{1-\mu^L_{Y_{j+1}}}}}{\sqrt{\mu^L_{Y_{j+1}}}+\sqrt{1-\mu^L_{Y_{j+1}}}} \tag{34}$$

$$\Longrightarrow\frac{\frac{\partial\left(\alpha^{RR}_{\tilde z}+\alpha^{RL}_{\tilde z+1}\right)}{\partial\Delta e}}{\alpha^{RR}_{\tilde z}+\alpha^{RL}_{\tilde z+1}}=\frac{\frac{\partial\left(\alpha^{LR}_{\tilde z+m_{\Delta E}}+\alpha^{LL}_{\tilde z+m_{\Delta E}+1}\right)}{\partial\Delta e}}{\alpha^{LR}_{\tilde z+m_{\Delta E}}+\alpha^{LL}_{\tilde z+m_{\Delta E}+1}}$$

Table 7 shows the results of the new valuation for fuzzy connectives.

Tab. 7. Results of the new valuation.

fuzzy connectives	fitted	not fitted
algebraic product $\top_{ap}$	×	
minimum operator $\top_{min}$		×
Łukasiewicz's operator $\top_{Łuk}$		×
Hamacher's product $\top_{hp}$		×
Einstein's product $\top_{ep}$		×
drastic product $\top_{dp}$		×
harmonic mean D_{hm}		×
arithmetic mean D_{am}		×
geometric mean D_{gm}	×	

6. Conclusion

In this paper, a new valuation of fuzzy connectives for fuzzy control was proposed. The dependence of the correcting variable described by the linguistic rules u^R was compared with the dependence of the correcting variable calculated by means of an analytical description u^f of the fuzzy controller.

The fuzzy controller can be described by one analytical function for a fuzzy cell $\mathcal{F}$. The fuzzy cell $\mathcal{F}$ results from the partition of the reference fuzzy sets X_i and Y_j. The reference fuzzy sets U_A and U_B in the partial fuzzy cells $\mathcal{F}(\mathcal{A})$, $\mathcal{F}(\mathcal{B})$, $\mathcal{F}(\mathcal{C})$, and $\mathcal{F}(\mathcal{D})$ are split into two situations. In the first situation, the correcting variable depends on the rate of error, whereas in the other the correcting variable depends on the error. A fuzzy connective for the intersection (in this paper we have investigated the t-norms and the averaging operator) approximates the linguistic rules if the dependence of the linguistic rules u^R is equivalent to the dependence of the analytical expression of the fuzzy controller u^f.

The research showed that only the algebraic product and the geometric mean approximate the linguistic rules regarding the logical valuation. We have investigated the algebraic product, minimum operator, Łukasiewicz's operator, Hamacher's product, Einstein's product, drastic product, harmonic mean, geometric mean and the arithmetic mean (Berger, 1995).

The logical valuation is an additional valuation of fuzzy connectives e.g. in the control of technical processes. Beside the usual valuation (axiomatic, pragmatic), an expert has an additional valuation with a direct connection between the fuzzy logic and the fuzzy control.

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