Appl. Math. and Comp. Sci., 1995, vol.5, No.4, 657–669

THE NECESSARY AND SUFFICIENT CONDITIONS FOR THE INTEGRAL OF A MULTIVALUED MAP TO BE A POLYGON

AHMED BOUSSOUAR*, O. ARINO*, S. GAUTIER*

Let F be a multivalued map from $I := [a,b]$ into $\mathbb{R}^2$ defined by $F(t) = \{h(t), f(t)\}$, where f and h are continuous maps from I into $\mathbb{R}^2$. The aim of this paper is to find the necessary and sufficient conditions on the mappings f and h for an integral of F to be a convex polygon.

1. Introduction

In this paper, we deal with the question: when the integral of a multivalued map is a polyhedron. More precisely, let us consider a measurable multivalued map (from now on, we will abbreviate it as a multi) F from an interval I of $\mathbb{R}$ into $\mathbb{R}^n$ with polyhedral values. The integral $\int_I F(t)\,dt$, as defined by Aumann, is a set consisting of the integrals of integrable selections of F. In general, this integral is not a polyhedron when $F(t)$ is a polyhedron for each $t \in I$. For example, if G is the multi from $[0, 2\pi]$ into $\mathbb{R}^2$ defined by $G(t) = [0,1](\cos(t), \sin(t))$, then the integral $\int_0^{2\pi} G(t)\,dt$ is the ball $2B$, where B is the ball with centre 0 and radius 1 in $\mathbb{R}^2$.

To the best of our knowledge, the problem of determining the geometric shape of the integral has not been considered yet. What we propose here is a first contribution in this direction. We will limit ourselves to the case $n = 2$ and the multi taken to be a segment for each $t \in I$. This situation can be reduced to the case when $F(t) = \{0, f(t)\}$ for each $t \in I$. The integral of F is a convex set defined by

$$C = \left\{ \int_a^b \alpha(t) f(t)\, dt : \alpha \in L^1([a,b], \mathbb{R}),\ 0 \le \alpha(t) \le 1 \right\}$$

(see Aubin and Frankowska, 1990).

Thus, C is the image through the map $\alpha \to \int_a^b \alpha(t) f(t)\, dt$ of a convex set. One can ask about conditions on f under which we can assert that C is a polygon.

* Laboratoire associé C.N.R.S., U.R.A. 1204, Université de Pau, Avenue de l'Université, 64000 Pau, France

We will show that the necessary and sufficient condition for this property to hold is that $\left\{ \frac{f(t)}{\|f(t)\|}, f(t) \neq 0 \right\}$ is a finite set or, in other words, the function f should take only a finite number of directions. A few remarks on the applicability of such results are given in the concluding section of the paper.

2. Preliminaries

We will denote by $\langle \cdot, \cdot \rangle$ the usual inner product in $\mathbb{R}^n$ and by $\|\cdot\|$ the associated norm. Let A be a subset of $\mathbb{R}^n$. We will denote by χ_A the characteristic function of A and by $\sigma(\cdot, A)$ its support function (see Hormander, 1954): $\forall x \in \mathbb{R}^n, \sigma(x, A) = \sup\{\langle x, a \rangle : a \in A\}$.

Definition 1. (Coxeter, 1962; 1963; Valentin, 1992) Let E be a vector space. A *zonotope* Z is the Minkowski sum of a finite number of many segments of E.

Definition 2. (Aubin and Frankowska, 1990; Castaing and Valadier, 1977; Clarke, 1981) Let F be a measurable multi from $I := [a, b] \subset \mathbb{R}$ into the closed subsets of $\mathbb{R}^n$. According to Aumann's definition, the *integral* of F over I is the subset denoted by $\int_I F(t)\,\mathrm{d}t$ and defined as

$$\left\{ \int_I f(t)\,\mathrm{d}t : f \text{ is an integrable selection of } F \right\}$$

We will recall a general result which will play an important role in our later considerations.

Let F be the multi from $I = [a, b] \subset \mathbb{R}$ into the non-empty closed subsets of $\mathbb{R}^n$, measurable and integrably bounded (i.e. there exists an integrable function $k : I \to \mathbb{R}^+$ such that for all $y \in F(t)$ we have $\|y\| \leq k(t)$).

Proposition 1. (Aubin and Frankowska, 1990; Castaing and Valadier, 1977; Clarke, 1981) *Under the above assumptions, we have:*

a) *The integral of F is convex and compact.*

b) *For each $x \in \mathbb{R}^n$, $\sigma\left(x, \int_I F(t)\,\mathrm{d}t\right) = \int_I \sigma(x, F(t))\,\mathrm{d}t$.*

c) $\int_I F(t)\,\mathrm{d}t = \int_I \mathrm{co}\, F(t)\,\mathrm{d}t$, *where* $\mathrm{co}\, F$ *is the convex hull of* F.

Remark 1. Proposition 1 remains true if the Lebesgue measure on I is replaced by a non-atomic finite positive measure space (T, μ). Such a space admits a bounded measurable function $m : T \to \mathbb{R}$ such that, for each t_0 in T, the set $\{t \in T : m(t) = m(t_0)\}$ has μ-measure zero (see Clarke, 1983).

Proposition 2. *Let M and N be two convex subsets of $\mathbb{R}^n$. Then $M + N$ is convex and each extremal point x of $M + N$ can be written as a sum of two extreme* *points of M and N, respectively. Moreover, the decomposition of x as a sum of an element of M and an element of N is unique.*

Proof. The convexity of $M+N$ is obvious. Let x be an extreme point of $M+N$. Let $a \in M$, $b \in N$ be such that $x = a+b$. We show first that a is an extreme point of M; the same will be true for b. If a is not an extreme point, then we can write it in the form $a = \lambda a_1 + (1-\lambda)a_2$, $0 < \lambda < 1, a_1$, $a_2 \in M$. Hence $x = \lambda(a_1+b) + (1-\lambda)(a_2+b)$. As a_1+b, a_2+b are in $M+N$, this yields a contradiction.

We will now show that the decomposition is unique. Suppose there are two decompositions, $x = a+b = a'+b'$. Then, for each $t \in]0,1[$, we have $x = (ta+(1-t)a') + (tb+(1-t)b')$. If $(a,b) \neq (a',b')$, then either $ta+(1-t)a'$ is not an extreme point of M or $tb+(1-t)b'$ is not an extreme point of N, which contradicts what we have just proved. ■

3. The Main Result

Let μ be a non-atomic finite positive measure on I, where $I = [a,b] \subset \mathbb{R}$.

3.1. Sufficient Condition

Proposition 3. *Let $F : I \rightrightarrows \mathbb{R}^2$ be a multi defined by $F(t) = \{h(t), f(t)\}$, where f and h are continuous maps from I into $\mathbb{R}^2$. Suppose that $\mu(E) > 0$, where $E = \{t : f(t) - h(t) \neq 0\}$. Moreover, suppose that there exist a finite family $V = \{e_1, ..., e_n\}$ of $\mathbb{R}^2$ and a denumerable measurable partition (I_j), $j = 1, ..., q$, $1 \leq q \leq \infty$, of E, such that*

$$\forall\, 1 \leq j \leq q,\ \exists e \in V,\ f(t) - h(t) \in \mathbb{R}e \qquad \text{for all} \quad t \in I_j$$

Then $\int_I F(s)\,\mathrm{d}\mu(s)$ *is a polygon.*

Proof. We can assume without loss of generality that $h = 0$, since

$$\int_I \Big\{h(s), f(s)\Big\}\,\mathrm{d}\mu(s) = \int_I \Big\{0, f(s)-h(s)\Big\}\,\mathrm{d}\mu(s) + \int_I h(s)\,\mathrm{d}\mu(s)$$

Let us write $f(t) = g_j(t)e$ for $t \in I_j$ with $g_j(t) \in \mathbb{R}$ and $e \in V$. Then

$$\int_{I_j} F(s)\,\mathrm{d}\mu(s) = \left\{ \int_{I_j} \alpha(s) f(s)\,\mathrm{d}\mu(s) : \alpha \in L^1,\ 0 \leq \alpha(s) \leq 1 \right\}$$

is a segment. Let

$$I_{m_j} = \left\{t \in I : f(t) = g_j(t)e_{m_j}\right\}, \qquad \widetilde{I}_k = \bigcup_{m_j = k} I_{m_j},\ m_j \in \{1, \ldots, n\}$$

Then $\int_{I_{m_j}} F(s)\,\mathrm{d}\mu(s)$ is a segment and $\int_{\widetilde{I_k}} F(s)\,\mathrm{d}\mu(s)$ is a countable sum of segments which have the same direction. Since $\mu\left(\widetilde{I_k}\right) = \sum_{m_j=k} \mu\left(I_{m_j}\right) \leq \mu(I)$ and $\int_I F(s)\,\mathrm{d}\mu(s)$ is integrably bounded, it follows that $\int_{\widetilde{I_k}} F(s)\,\mathrm{d}\mu(s)$ is a segment. Taking into account the fact that the integral $\int_I F(s)\,\mathrm{d}\mu(s)$ is a finite sum of segments $\int_{\widetilde{I_k}} F(s)\,\mathrm{d}\mu(s)$, we conclude from Proposition 2, that the integral $\int_I F(s)\,\mathrm{d}\mu(s)$ is a convex polygon which has at most 2^n vertices, since $k \leq n$. ■

Theorem 1. *Let $F : I \rightrightarrows \mathbb{R}^2$ be a multi defined by $F(t) = \sum_{i=1}^{p} F_i(t)$, where $F_i(t) = [0, f_i(t)]$ for all $t \in I$ and $f_i : I \to \mathbb{R}^2$ is a continuous map for each i. Suppose that there exist a finite family $V = \{e_1, ..., e_n\} \subset \mathbb{R}^2$ and a denumerable measurable partition (I_j), $j = 1, ..., q$, $1 \leq q \leq \infty$, of I, such that:*

$$\forall i = 1, ..., p,\ \forall 1 \leq j \leq q,\ \exists e \in V,\ \forall t \in I_j,\ f_i(t) \in \mathbb{R}e$$

Then the integral $\int_I F(t)\,\mathrm{d}\mu(t)$ is a zonotope.

Proof. We apply Proposition 3 to each of the multis F_i. We thus get that $\int_I F_i(s)\,\mathrm{d}\mu(s)$ is a polygon. But in the proof of the same proposition, it is shown that each integral $\int_I F_i(s)\,\mathrm{d}\mu(s)$ is a finite sum of segments, that is, a zonotope. Hence $\int_I F(s)\,\mathrm{d}\mu(s)$ is a zonotope, as a finite sum of zonotopes. ■

Remark 2. The conclusion of Theorem 1 holds in a more general situation. Namely, we may consider any finite measured space $(T, \mathcal{T}, \mu)$ instead of I and any Banach space instead of $\mathbb{R}^2$, assuming that the mappings f_i are μ- integrable and q is finite.

4. The Necessary Condition

Definition 3. Let P be a convex polygon of $\mathbb{R}^2$ with n vertices. By the *ordered pair* for P we mean any family $\left((z_i)_{1\leq i\leq n+1}, (e_i)_{1\leq i\leq n+1}\right)$ such that z_1 is one of arbitrarily chosen vertices of P, z_{i+1} is the vertex which follows z_i when moving along the polygon counterclockwise, with $z_{n+1} = z_1$, and e_i is the unit vector perpendicular to the vector $\overrightarrow{z_i z_{i+1}}$, i.e. Angle $\left(e_i, \overrightarrow{z_i z_{i+1}}\right) = \dfrac{\pi}{2}$.

Theorem 2. *Let $F : I \rightrightarrows \mathbb{R}^2$ be a multi defined by $F(t) = \{h(t), f(t)\}$, where f and h are continuous maps from I into $\mathbb{R}^2$. Suppose that $\mu(E) > 0$, where*

$E = \{t : f(t) - h(t) \neq 0\}$ and $\int_I F(s)\, \mathrm{d}\mu(s)$ is a polygon with n vertices. Then there exist a finite family $V = \{a_i : 1 \leq i \leq p\}$, with $p \leq n$, of $\mathbb{R}^2$ and a denumerable measurable partition (I_j), $j = 1, \ldots, q$, $1 \leq q \leq \infty$, of E, such that

$$\forall 1 \leq j \leq q,\ \exists e \in V,\ f(t) - h(t) \in \mathbb{R}e \qquad \textit{for all} \quad t \in I_j$$

Theorem 2 will be a consequence of the proposition and lemma which follow.

Proposition 4. *Let $F : I \rightrightarrows \mathbb{R}^2$ be a multi defined by $F(t) = \{0, f(t)\}$, where f is a continuous map from I into $\mathbb{R}^2$. Suppose that the integral $P = \int_I F(s)\, \mathrm{d}\mu(s)$ is a polygon with n vertices. Let $((z_i), (e_i))$ be an ordered pair for P. Then*

$$\forall t \in I,\ \langle f(t), e_i \rangle \langle f(t), e_{i+1} \rangle \geq 0, \qquad i = 1, \ldots, n$$

Proof. Let us denote by D_i, $i = 1, \ldots, n$, the line segment from z_i to z_{i+1} $(z_{n+1} = z_1)$. Thus

$$z_i \in D_{i-1} \cap D_i \qquad (D_0 = D_n) \tag{1}$$

Then the equation of D_i is $\langle e_i,\ z \rangle = \sigma(e_i, P), z \in \mathbb{R}^2$. We conclude from Proposition 1 that

$$\sigma(e_i, P) = \sigma\Big(e_i, \int_I F(t)\, \mathrm{d}\mu(t)\Big) = \int_I \sigma\Big(e_i, F(t)\Big)\, \mathrm{d}\mu(t)$$

Having in mind the fact that z_i is a vertex of P, we have

$$z_i \in \int_I F(t)\, \mathrm{d}\mu(t), \qquad i = 1, \ldots, n$$

Hence, by Proposition 1,

$$z_i \in \int_I F(t)\, \mathrm{d}\mu(t)$$

From Definition 2, we obtain

$$z_i = \int_I g_i(t)\, \mathrm{d}\mu(t) \tag{2}$$

for a function g_i which is an integrable selection of $t \rightrightarrows \mathrm{co}\, F(t)$, i.e. $g_i(t) \in \mathrm{co}\, F(t)$ for a.e. $t \in I$. Then we can write $g_i(t) = \alpha_i(t) f(t)$ for all $t \in I$ where α_i is a measurable function such that $0 \leq \alpha_i(t) \leq 1$.

Let $A_i(t) = \langle e_i, f(t) \rangle$ for all $t \in I$, $i = 1, \ldots, n$. Set $I_i^+ = \{t \in I : A_i(t) > 0\}$, $I_i^0 = \{t \in I : A_i(t) = 0\}$, $I_i^- = \{t \in I : A_i(t) < 0\}$. We will prove that

$$\alpha_i(t) = \begin{cases} 0 & \text{a.e. if } \ t \in I_i^- \\ 1 & \text{a.e. if } \ t \in I_i^+ \end{cases}$$

Since $z_i \in D_i$, it follows that $\langle e_i, z_i\rangle = \sigma(e_i, P)$. Thus

$$\left\langle e_i, \int_I g_i(t)\,\mathrm{d}\mu(t)\right\rangle = \int_I \sigma\Big(e_i, F(t)\Big)\,\mathrm{d}\mu(t) = \int_I \max\Big\{0, \langle e_i, f(t)\rangle\Big\}\,\mathrm{d}\mu(t)$$

This yields

$$\int_I \Big(\max\{0, \langle e_i, f(t)\rangle\} - \langle e_i, g_i(t)\rangle\Big)\,\mathrm{d}\mu(t) = 0 \tag{3}$$

We know that $g_i(t) = \alpha_i(t)f(t)$. Substituting this into (3) we obtain

$$\int_I \Big(\max\{A_i(t), 0\} - \alpha_i(t)A_i(t)\Big)\,\mathrm{d}\mu(t) = 0$$

The above relation is equivalent to

$$\int_I \alpha_i(t)A_i(t)\,\mathrm{d}\mu(t) = \int_I \max\Big\{A_i(t), 0\Big\}\,\mathrm{d}\mu(t) = \int_{I_i^+} A_i(t)\,\mathrm{d}\mu(t)$$

which is the same as

$$\int_{I_i^+} \alpha_i(t)A_i(t)\,\mathrm{d}\mu(t) + \int_{I_i^-} \alpha_i(t)A_i(t)\,\mathrm{d}\mu(t) = \int_{I_i^+} A_i(t)\,\mathrm{d}\mu(t)$$

Consequently,

$$\int_{I_i^+} \Big(1 - \alpha_i(t)\Big)A_i(t)\,\mathrm{d}\mu(t) = \int_{I_i^-} \alpha_i(t)A_i(t)\,\mathrm{d}\mu(t)$$

In this equality, the right-hand side is non-positive while the other is non-negative. Hence both are equal to zero. Therefore we obtain $\int_{I_i^-} (1 - \alpha_i(t))\,A_i(t)\,\mathrm{d}\mu(t) = 0$ and $\int_{I_i^-} \alpha_i(t)A_i(t)\,\mathrm{d}\mu(t) = 0$, from which we deduce that

$$\begin{aligned} &\Big(1 - \alpha_i(t)\Big)A_i(t) = 0 && \text{a.e. on } I_i^+ \\ &\alpha_i(t)A_i(t) = 0 && \text{a.e. on } I_i^- \end{aligned} \tag{4}$$

Consequently,

$$\alpha_i(t) = \begin{cases} 1 & \text{a.e. on } I_i^+ \\ 0 & \text{a.e. on } I_i^- \end{cases} \tag{5}$$

From (1) we have $z_i \in D_{i-1}$, and so $\langle e_{i-1}, z_i\rangle = \sigma(e_i, P)$. We may now continue in the same fashion to obtain

$$\begin{aligned} &\Big(1 - \alpha_i(t)\Big)A_{i-1}(t) = 0 && \text{a.e. on } I_{i-1}^+ \\ &\alpha_i(t)A_{i-1}(t) = 0 && \text{a.e. on } I_{i-1}^- \end{aligned}$$

We observe that the sets I_i^+, I_i^-, I_i^0 are disjoint and their union is the whole interval I. By using Proposition 4, we get $\alpha_i(t) = \begin{cases} 1 & \text{a.e. on } I_i^+ \\ 0 & \text{a.e. on } I_i^- \end{cases}$. Thus,

$$A_{i-1}(t) = 0 \text{ a.e. on } I_{i-1}^- \cap I_i^+$$
$$A_{i-1}(t) = 0 \text{ a.e. on } I_{i-1}^+ \cap I_i^-$$

from which it may be concluded that

$$A_i(t)A_{i-1}(t) \geq 0 \quad \text{a.e. on} \quad I, \qquad i = 1, \ldots, n \tag{6}$$

Since the A_i's are continuous, property (6) holds everywhere on I. We deduce that

$$\forall t \in I, \langle f(t), e_i \rangle \langle f(t), e_{i-1} \rangle \geq 0, \qquad i = 1, \ldots, n \tag{7}$$

Lemma 1. *Let $F : I = [a,b] \rightrightarrows \mathbb{R}^2$ be a multi defined by $F(t) = \{0, f(t)\}$, where f is continuous. Suppose that the integral $P = \int_I F(s)\, d\mu(s)$ is a polygon with n vertices. Let $((z_i), (e_i))$ be an ordered pair for P. Let J be a subinterval of I. Assume that $f(t) \neq 0$ for each $t \in J$. Then there exists a vector $e_{i_0} \in \{e_i : i = 1, \ldots, n\}$ such that $\langle f(t), e_{i_0} \rangle = 0$ for every $t \in J$.*

Proof. Let P_1 be a half-plane limited by a straight line Δ perpendicular to the vector $f(t)$, which contains $f(t)$. Let Q_i, $i = 1, \ldots, 4$, be the octants determined by $f(t)$ and Δ (see Fig. 1.)

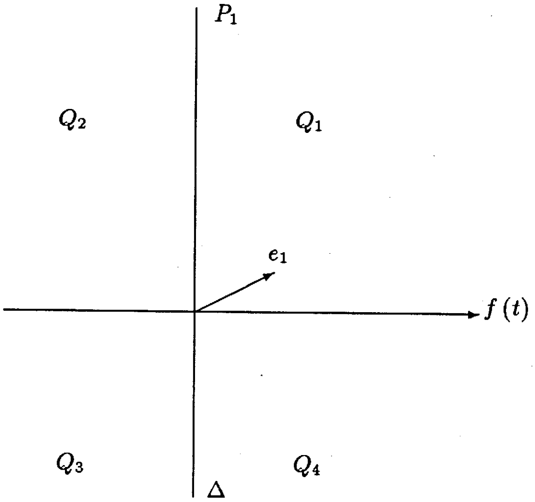

Fig. 1.

Suppose that none of the vectors of the family (e_i) is perpendicular to $f(t)$. We choose the ordered pair so that e_1 is the first vector following the vector $f(t)$ when moving counterclockwise. Then e_1 belongs either to Q_1 or to Q_2, i.e. e_1 cannot live neither in Q_3 nor in Q_4. Otherwise, the angle (e_n, e_1) between e_n and e_1, counted in the trigonometric orientation, would be larger than π, which is impossible in the case of a convex polygon.

We consider each of these two situations separately:

i) $e_1 \in Q_1$.

Suppose that $e_1, \dots, e_{m-1} \in Q_1$, $1 \leq m < n-1$. Then either $e_m \in Q_2$ or $e_m \in Q_3$ because the angle (e_{m-1}, e_m) is less than π (since the polygon is convex). It follows that $\langle f(t), e_{m-1}\rangle$ and $\langle f(t), e_m\rangle$ have opposite signs and we obtain $\langle f(t), e_{m-1}\rangle\langle f(t), e_m\rangle < 0$. This contradicts (7) and, in consequence, there exists an i_0 such that $\langle f(t), e_{i_0}\rangle = 0$.

ii) $e_1 \in Q_2$.

In this case, Q_1 contains a vector e_i, and so $e_n \in Q_4$. Therefore $\langle f(t), e_1\rangle$ and $\langle f(t), e_n\rangle$ are of opposite signs, i.e. $\langle f(t), e_n\rangle\langle f(t), e_1\rangle < 0$. This contradicts (7) and consequently there exists an i_0 such that $\langle f(t), e_{i_0}\rangle = 0$.

It remains to prove that e_{i_0} does not depend on t. Let us take $J_i = \{t \in J : \langle f(t), e_i\rangle = 0\}$, $i = 1, \dots, n$. The sets J_i are closed and yield a finite partition of the interval J. On account of the fact that J is a connected set, we have $J_i = J$ or $J_i = \emptyset$. Consequently, there exists an i_0 such that $J_{i_0} = J$. Then $\langle f(t), e_{i_0}\rangle = 0$ for each $t \in J$. ■

Proof of Theorem 2. We can suppose without loss of generality that $h = 0$, since

$$\int_I \left\{h(s), f(s)\right\} \mathrm{d}\mu(s) = \int_I \left\{0, f(s) - h(s)\right\} \mathrm{d}\mu(s) + \int_I h(s)\, \mathrm{d}\mu(s)$$

Let $P = \displaystyle\int_I F(s)\, \mathrm{d}\mu(s)$ be a polygon with n vertices. Let $((z_i), (e_i))$ be an ordered pair for P. Write $B = (e_i)_{1 \leq i \leq n}$.

Let us denote by E a set of points where f is not equal to zero. Then E is an open set and we can write it as a union of at most denumerably many open intervals (I_J). In view of Lemma 1 we have $f(t) \in \mathbb{R}a_j$ for each $t \in I_j$, where a_j is a vector in $\mathbb{R}^2$ perpendicular to one of the vectors of B. The family of a_j can be reduced to at most n distinct vectors. ■

Theorem 3. *Let $F : I \rightrightarrows \mathbb{R}^2$ be a multi defined by $F(t) = \sum_{i=1}^{p} F_i(t)$, where $F_i(t) = [0, f_i(t)]$ for all $t \in I$ and $f_i : I \to \mathbb{R}^2$ is a continuous map for each i. Suppose that $\int_I F(t)\, \mathrm{d}t$ is a zonotope. Then there exists a finite family $V = \{e_1, \dots, e_n\}$ of $\mathbb{R}^2$ and a denumerable measurable partition (I_j), $j = 1, \dots, q$, $1 \leq q \leq \infty$, of I, such that*

$$\forall i = 1, \dots, p,\ \forall 1 \leq j \leq q,\ \exists e \in V,\ \forall t \in I_j,\ f_i(t) \in \mathbb{R}e$$

Lemma 2. *Let A and B be two convex compact subsets of $\mathbb{R}^n$. If $C = A + B$ is a polyhedron, then A and B are polyhedrons.*

Proof. Let A and B be two convex, compact subsets of $\mathbb{R}^n$. Assume that C is a polyhedron with n vertices z_i, $1 \le i \le n$. According to Proposition 2, each vertex $z_i \in C$ can be written in a unique way as a sum $a_i + b_i$, where a_i (resp. b_i) is an extreme point of A (resp. B). Denoting by P_A (resp. P_B) closed convex hull of the points $\{a_i : 1 \le i \le n\}$ (resp. $\{b_i : 1 \le i \le n\}$), we have $C = A + B$. We also have $C = P_A + B = A + P_B$. We claim that $B = P_B$ and $A = P_A$. Assuming that this is not true, say $B \ne P_B$, we select $x \in B \setminus P_B$. For each $a \in P_A$, we have $x + a \in C$. Therefore, for each $1 \le j \le n$, there exits a family $(\lambda_{i,j})$, $\lambda_{i,j} \ge 0$, $\sum_{i=1}^{n} \lambda_{i,j} = 1$ such that $x + a_j = \sum_{i=1}^{n} \lambda_{i,j} (a_i + b_i)$.

For each family $(\mu_j)_{1 \le j \le n}$, $\mu_j \ge 0$, $\sum_{j=1}^{n} \mu_j = 1$, we have

$$x + \sum_{j=1}^{n} \mu_j a_j = \sum_{j=1}^{n} \sum_{i=1}^{n} \mu_j \lambda_{i,j} a_i + \sum_{j=1}^{n} \sum_{i=1}^{n} \mu_j \lambda_{i,j} b_i$$

We will obtain a contradiction if we can determine (μ_j) so that, for each i,

$$\mu_i = \sum_{j=1}^{n} \mu_j \lambda_{i,j}$$

In fact, with such a choice, we have $x \in P_B$.

The existence of $\mu = (\mu_j)$ follows as a consequence of the Perron-Frobenius theorem (Horn and Johnson, 1985; Th.8.3.1, p.503) on positive matrices. The matrix $\Lambda = (\lambda_{i,j})$ is non-negative and $e = (1, \dots, 1)$ is a left fixed point of Λ. Therefore the spectral radius of Λ, $r = r(\Lambda)$, is greater than or equal to one and is associated with a positive eigenvector μ. Multiplying both sides of the equality $\Lambda\mu = r\mu$ by e, we obtain $e\mu = re\mu$. From $e\mu > 0$, we conclude that $r = 1$, and so $\mu = \Lambda\mu$, which is the desired property. ■

Proof of Theorem 3. For each $i = 1, ..., p$, $\int_I F_i(s)\, d\mu(s)$ is a convex compact subset and, since by assumption $\int_I F(s)\, d\mu(s) = \sum_{i=1}^{p} \int_I F_i(s)\, d\mu(s)$ is a polygon, in view of Lemma 2 each $\int_I F_i(s)\, d\mu(s)$ is a polygon. Therefore, for each i, there exists a finite family V_i of vectors and a denumerable measurable partition $(J_{i,k})$ such that, for each $k \in \mathbb{N}$, there exists an $e \in V_i$ with the property that $f_i(t) \in \mathbb{R}e$ for each $t \in J_{i,j}$.

Define $V = \bigcup_{i=1}^{p} V_i$ and consider the partition (I_l) of I obtained by taking all the possible subsets of the form $\bigcap_{\substack{i=1 \\ k\in\mathbb{N}}}^{p} J_{i,k}$. Then we obtain

$$\forall i,\ i = 1, \cdots, p,\ l \in \mathbb{N},\ \exists e \in V,\ \forall t \in I_l,\ f_i(t) \in \mathbb{R}e$$

■

5. Numerical Examples

We will now illustrate our results with a few examples. We choose the same interval $I = [0, 1.8]$, the step size for integration 0.02 and a multi of the form

$$F(t) = \{h(t), f(t)\} := \{(0,0), (1, g(t))\}$$

where $g : I \to \mathbb{R}$. As usual, the notation $[\cdot]$ corresponds to the integer part and $\chi_A(\cdot)$ corresponds to the characteristic function of the set A.

Example 1. (Fig. 2) $g(t) = [t/2]$. Then $f(t) - h(t) = (1, 0)$ for each $t \in [0, 1.8]$ and there is only one direction. In this case, the integral is a segment.

Example 2. (Fig. 3) $g(t) = [t]$. Then

$$f(t) - h(t) = \chi_{[0,1[}(t)(1,0) + \chi_{[1,1.8]}(t)(1,1)$$

The multi takes two independent directions. Therefore the integral is a parallelogram.

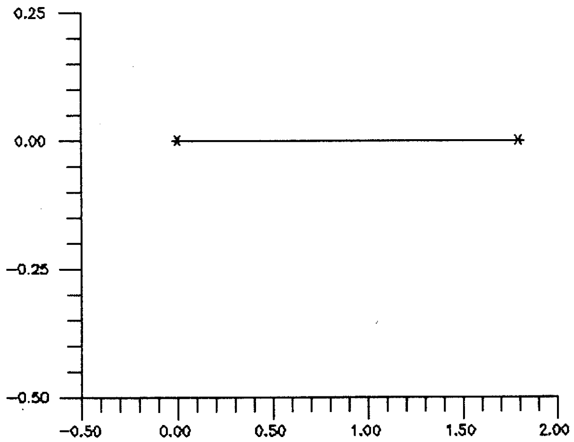

Fig. 2.

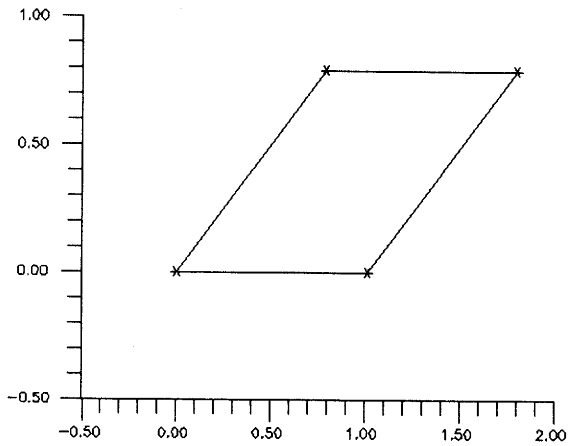

Fig. 3.

Example 3. (Fig. 4) $g(t) = [3t/2]$. Then

$$f(t) - h(t) = \chi_{[0,\frac{2}{3}[}(t)(1,0) + \chi_{[\frac{2}{3},\frac{4}{3}[}(t)(1,1) + \chi_{[\frac{4}{3},1.8]}(t)(1,2)$$

In this case, there are three directions. The integral is a polygon with eight vertices.

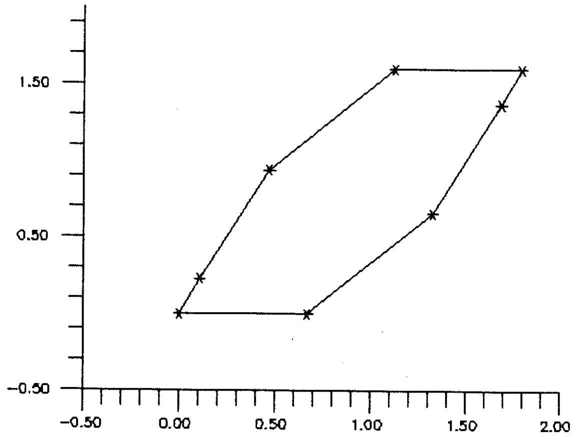

Fig. 4.

Example 4. (Fig. 5) $g(t) = [7t]$. Then

$$f(t) - h(t) = \sum_{i=0}^{11} \chi_{[i,\frac{i+1}{7}[}(t)(1,i) + \chi_{[\frac{12}{7},1.8]}(t)(1,12)$$

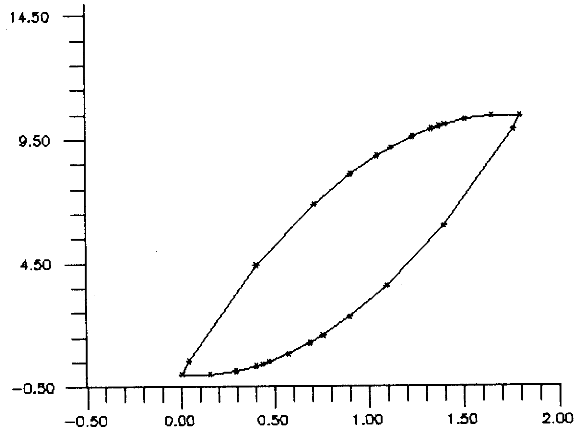

Fig. 5.

In this case, there are thirteen directions. The integral is a polygon which has at most 2^{13} vertices.

6. Conclusion

The problem we considered here belongs to the category of inverse ones, that is, based on some known properties of the solutions to an equation, the objective is to reconstruct its coefficients. Obviously, the result obtained in this paper is partial. We want to give two examples which illustrate potential interest in our result.

1) Let $F(t) = \{0, f(t)\}$ be a multi. Then the integral of F is a convex subset which may have nearly any shape. It may be desirable to approximate this integral. Polygonal approximations are the most natural. For such approximations to be useful, they should correspond to approximations of the multi. Our result shows that the only multis whose integrals constitute polygons are of the type $f(t) = \sum_{i=1}^{m} g_i(t)e_i$, where $g_i : I \to \mathbb{R}$, and in fact it also shows how to derive the multi from the integral.

2) We consider a control system described by the vector differential equation (in $\mathbb{R}^2$)
$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + B(t)u(t) \tag{8}$$
with fixed initial data $x(0) = x_0$. The vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ will always be two-dimensional, A and B are 2×2-matrix-valued continuous functions components, and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a measurable vector-valued function with values $u(t)$ constrained to lie in the set $U = \{u \in \mathbb{R}^2 : 0 \leq u_i \leq 1,\ i = 1, 2\}$. Let $X(t)$ be a

fundamental matrix solution of the homogeneous system $\dot{x}(t) = A(t)\,x(t)$. Then, for any admissible control u, the solution of (8) is given by

$$x(t;u) = X(t)\,x_0 + X(t)\int_0^t X^{-1}(s)\,B(s)u(s)\,\mathrm{d}s$$

Define $Y(s) = X^{-1}(s)\,B(s) := \begin{pmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{pmatrix}$ and $f_1(s) = \begin{pmatrix} y_{11}(s) \\ y_{21}(s) \end{pmatrix}$, $f_2(s) = \begin{pmatrix} y_{12}(s) \\ y_{22}(s) \end{pmatrix}$. Then

$$x(t;u) = X(t)\left(x_0 + \int_0^t u_1(s)\,f_1(s)\,\mathrm{d}s + \int_0^t u_2(s)\,f_2(s)\,\mathrm{d}s\right)$$

Let the multis F_1 and F_2 be defined by $F_1(t) = \{0, f_1(t)\}$ and $F_2(t) = \{0, f_2(t)\}$. Then, for each $T > 0$, the attainable set $A(T) = \{x(T,u) : u \in U\}$ is given by

$$A(T) = X(t)\left(x_0 + \int_0^T F_1(s)\,\mathrm{d}s + \int_0^T F_2(s)\,\mathrm{d}s\right)$$

The necessary and sufficient condition for $A(T)$ to be a polygon is that f_1 and f_2 verify the conditions of Theorem 1.

References

Aubin J.P. and Frankowska H. (1990): *Set-Valued Analysis.* — Boston: Birkhäuser.

Castaing C. and Valadier M. (1977): *Convex Analysis and Mesurable Multifunctions.* — Lectures Notes in Mathematics 580, Berlin: Springer-Verlag.

Clarke F.H. (1981): *A variational proof of Aumann's theorem.* — App. Math. Optim., v.7, No.4, pp.373–378.

Clarke F.H. (1983): *Optimization and Nonsmooth Analysis.* — New York: Wiley.

Coxeter H.S.M. (1962): *The classification of zonohedra by means of projective diagrams.* — J. Math. Pures Appl., v.9, No.41, pp.137–156.

Coxeter H.S.M. (1963): *Regular Polytopes.* — New-York: Mac Millan, (2-nd ed.).

Hormander L. (1954): *Sur la fonction d'appui des ensembles convexes d'un espace localement convexe.* — Arkiv. För Math., v.3, No.12, pp.182–186.

Horn B.A. and Johnson C.R. (1985): *Matrix Analysis.* — Cambridge: University Press.

Valentin P. (1992): *Theory of Zonoids: I: A Mathematical Summary.* — NATO Institute of Advanced Studies, Advances in Theory of Chromatography, Ferrare, Aug. 1991, (Dondi F. and Guiochon G., Eds.), Dordrecht: Kluwer.

Received: June 20, 1994
Revised: January 30, 1995