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THEORETICAL AND NUMERICAL ESTIMATES OF INSTANTANEOUS LYAPUNOV EXPONENTS OF QUASI-GEOSTROPHIC OCEAN DYNAMICS

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This paper deals with the problem of predictability of the ocean circulation. Theoretical estimates of instantaneous Lyapunov exponents of a quasi-geostrophic ocean model are developed and compared with calculated values. The comparison shows a rather good correlation of temporal variability of calculated exponents and their estimates.

1. Introduction

It is currently known that non-linear processes can involve instabilities of solutions of dynamical systems, and due to this fact, any error in the initial data increases with time. This leads to the chaotic behaviour of trajectories and limits the time period of the deterministic predictability of the system. This is one of the reasons of rather restricted weather-forecast skills in meteorology, ocean-current predictions in oceanology and the use of predictions in many other fields of science where one deals with non-linear systems.

Attempts to understand principal restrictions of predictability of dynamical systems appear in numerous predictability studies. In particular, this concerns geophysical sciences, because this question is related to the weather prediction, one of crucial problems of meteorology.

Any solution of a forced and dissipative non-linear system reaches asymptotically a bounded region called the attractor. One of the most appealing features of an attractor of a geophysical system is that it has a finite dimension, even though the dynamics is governed by infinite-dimensional systems of partial differential equations (see e.g. Bernier, 1994; Il'in, 1991). However, the behaviour of the system on the attractor could be rather complex. Even in the simplest cases, as e.g. Lorenz' system (Lorenz, 1963) which is composed of three ordinary differential equations, the non-linearity and internal instability of the system lead to the appearance of a fractal attractor of dimension about 2.06.

The predictability of a system is determined by the rate of divergence of initially close trajectories. For infinite time evolution of the system, this rate is given by

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positive Lyapunov exponents which characterize the system from a global point of view, i.e. they concern the whole attractor of the system. However, analyses of simple chaotic models show that the divergence rate of finite segments of trajectories is different for trajectory segments starting from different points on the attractor set (Abarbanel *et al.*, 1991; Dymnikov and Kazantsev, 1993).

The divergence rate of an infinitesimal initial perturbation for finite time evolution of a system can be determined by the singular values of the integral evolution operator of the linearized equation (Lorenz, 1965) integrated over this finite trajectory segment. The logarithms of these values divided by the time interval are called the local Lyapunov exponents (Abarbanel *et al.*, 1991). So, if a trajectory part produced by integration of a dynamical system is known, the predictability of this system on this part of the trajectory can be estimated.

However, local exponents possess several shortcomings. First, they correspond to infinitesimal perturbations of initial conditions only. Moreover, the perturbation must be infinitesimal not only at the initial time, but it is supposed to remain small for all the time period considered. But if we take into account a real system, we have to work with initial data of finite accuracy. Even if we can suppose that the initial error is sufficiently small to be considered as infinitesimal, after some time of integration the error may attain a significant magnitude. In such a case, the discussion of predictability of the system on a particular trajectory has no sense: this trajectory can be realized as well as any other which has a close initial point, but which nevertheless differs a lot.

Furthermore, the estimates of local exponents of a system require the knowledge of a trajectory of this system. Thus, to estimate them, we first have to perform the integration of the model. This means that we can speak of *a posteriori* estimates only. If we are interested in *a priori* estimates, we should use a point on the attractor rather than the whole trajectory segment.

Owing to these two reasons we shall consider instantaneous Lyapunov exponents, which are local Lyapunov exponents with composition length tending to zero. These exponents can be considered as predictability characteristics at a given point on the attractor, i.e. we study the evolution of an error in the initial conditions of a trajectory of infinitesimal length issued at this point.

The paper discusses the instantaneous Lyapunov exponents of quasi-geostrophic ocean dynamics. In Section 2 the model equations and boundary conditions are presented. Section 3 is devoted to the study of the theoretical estimates of instantaneous exponents as eigenvalues of the symmetric part of the linearized operator of the model. In Section 4 the calculation of eigenvalues of finite-element discretization of this operator is performed. These eigenvalues are compared with the estimates in Section 5.

The results of comparison reveal a good coincidence of the temporal variability of calculated exponents and their estimates. This fact gives us a possibility to use these estimates as predictability characteristics of the multi-layer quasi-geostrophic ocean model. Such characteristics are very easy to evaluate.

2. Model Equations

We consider the ocean dynamics in the quasi-geostrophic formulation, i.e. we neglect all the thermodynamic effects. The vertical structure of the ocean is modelled by dividing the domain Ω into K layers of depth thickness H_k. Following (Holland, 1978; Le Provost *et al.*, 1994), the equation of the dynamics can be written as

$$\begin{aligned} &\frac{\partial \theta_k}{\partial t} + J(\psi_k, \theta_k + \beta y) = \mu \Delta^2 \psi_k + F_k^{wind} - D_k^{bottom} \quad \text{in } \Omega, \quad k = 1, \dots, K \\ &\theta = \Delta \psi - \mathcal{W} \psi \quad \text{in } \Omega \end{aligned} \tag{1}$$

where $\psi = \psi_k(x, y)$, $k = 1, \dots, K$ is the quasi-geostrophic stream function of the k-th layer. We suppose that Ω is a bounded open subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. The k-th layer is characterized by its thickness H_k, its reduced gravity g_k and its streamfunction ψ_k. The forcing F_k^{wind} influences only the upper layer: $F_k^{wind} = 0$ for $k \neq 1$, and F_1^{wind} is equal to the curl of the wind stress on the surface. The bottom drag is neglected: $D_k^{bottom} = 0$. The parameter of Coriolis and its meridional gradient in the middle of the basin are f_0 and β, respectively. We consider a linear approximation of the Coriolis parameter $f = f_0 + \beta y$. The term $\mathcal{W}$ is the $K \times K$ tridiagonal matrix defined by

$$\mathcal{W} = \begin{pmatrix} R_1 & -R_1 & \cdots & 0 \\ -R'_2 & R_2 + R'_2 & -R_2 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & -R'_K & R'_K \end{pmatrix} \quad \text{with} \quad \begin{cases} R_k = \dfrac{f_0^2 \rho_0}{g H_k(\rho_{k+1} - \rho_k)} \\ R'_k = \dfrac{f_0^2 \rho_0}{g H_k(\rho_k - \rho_{k-1})} \end{cases} \tag{2}$$

where ρ_0 is the mean density of water, ρ_k stands for the mean density in the k-th layer and g is the gravity acceleration.

We remark that the matrix $\mathcal{W}$ has a complete system of eigenvectors. So we introduce Λ as the diagonal matrix of the eigenvalues of $\mathcal{W}$, and define the matrix of passage to its basis of eigenfunctions P so that

$$\Lambda = P^{-1} \mathcal{W} P \tag{3}$$

We shall indicate each vector in the eigenbasis of $\mathcal{W}$ by a star: $\psi^* = P^{-1}\psi$, $\theta^* = P^{-1}\theta$, etc.

The boundary conditions and initial data for eqn. (1) are formulated as

$$\begin{cases} \Delta \psi_k(x, y, t) = 0 & \text{on } \partial\Omega \times]0, T[, \quad k = 1, \dots, K \\ \psi_k^*(x, y, t) = C_k^*(t) & \text{on } \partial\Omega \times]0, T[, \quad k = 1, \dots, K \\ \displaystyle\int_\Omega \psi_k^* \, d\Omega = 0, & k = 2, \dots, K, \quad C_1^* = 0 \\ \theta_k(x, y, 0) = \theta_{0,k}(x, y) & \text{in } \Omega, \quad k = 1, \dots, K \end{cases} \tag{4}$$

where $C_k^*(t)$ are determined by the condition $\int_\Omega \psi_k^* \, d\Omega = 0$.

3. Theoretical Estimates of Instantaneous Lyapunov Exponents

3.1. Existence and Uniqueness of the Solution, Existence of Attractor

We denote by H^{-1} the space $(H^{-1}(\Omega))^N$, by L^2 the space $(L^2(\Omega))^N$, etc. The forcing F_1^{wind} is supposed to be independent of time and it belongs to $L^2(\Omega)$. We recall some results from (Bernier, 1994; 1995) concerning solvability of the system (1). We do not give here the proofs, since they are beyond the scope of this paper.

Theorem 1. *(Bernier, 1994). For θ_0 in H^{-1}, the system (1) admits a unique solution $\theta(t,x)$ in $C([0,T],H^{-1})\cap L^2(0,T,L^2)\cap L^2_{loc}(0,T,H^1_0)$. The semigroup $G(t)$ from H^{-1} to H^{-1}, $G(t)\theta_0(x)=\theta(t,x)$, associated with these equations, is such that there exists a maximal attractor A which is bounded in L^2, compact and connected in H^{-1}, and whose basin of attraction is the whole space H^{-1}. This attractor has finite Hausdorff and fractal dimensions. For θ_0 in H^1, the system (1) admits a unique solution $\theta(t,x)$ in $C([0,T],H^1)\cap L^2(0,T,H^2)$.*

To study the evolution of a small initial error θ we consider the system linearized around the solution $\bar\theta$ of (1) on the attractor:

$$\begin{aligned} &\frac{\partial\theta_k}{\partial t}+J(\bar\psi_k,\theta_k)+J(\psi_k,\bar\theta_k+\beta y)=\mu\Delta^2\psi_k+F_k^{wind} && \text{in } \Omega\times]0,T[,\\ & && k=1,\dots,K \\ &\theta=\Delta\psi-\mathcal{W}\psi && \text{in } \Omega\times]0,T[\end{aligned} \tag{5}$$

The boundary conditions for the perturbation ψ are the same as the boundary conditions for the solution $\bar\psi$ of (4):

$$\begin{cases} \Delta\psi_k=0 & \text{on } \partial\Omega\times]0,T[\quad \text{for } k=1,2,\dots,K\\ \psi_1^*=0 & \text{on } \partial\Omega\times]0,T[\\ \psi_k^*=C_k^*(t) & \text{on } \partial\Omega\times]0,T[\quad \text{for } k=2,3,\dots,K\\ \int_\Omega\psi_k^*\,d\Omega=\int_\Omega\psi_k^*\,|_{t=0}\,d\Omega & \qquad\text{for } k=2,3,\dots,K \end{cases} \tag{6}$$

From now on, for every θ, we define ψ and $\hat\psi$ by

$$\begin{cases} \Delta\psi-\mathcal{W}\psi=\theta & \text{in } \Omega\times]0,T[\\ \Delta\psi=0 & \text{on } \partial\Omega\times]0,T[\\ \psi_k^*(x,y,t)=C_k^*(t) & \text{on } \partial\Omega\times]0,T[,\ k=2,3,\dots,K\\ \psi_1^*=0 & \text{on } \partial\Omega\times]0,T[\\ \int_\Omega\psi_k^*\,d\Omega=0 & \text{for } k=2,3,\dots,K\\ \psi_k=\hat\psi_k+C_k & \text{in } \Omega\times]0,T[,\ \hat\psi_k\in H_0^1(\Omega) \end{cases} \tag{7}$$

where $C_k = PC_k^*$. We introduce the norm on H^{-1} defined by the scalar product (Bernier, 1994):

$$((\theta, \theta')) = \langle \theta, H\hat{\psi}' \rangle = -\sum_{i=1}^{i=K} H_i \langle \theta_i, \hat{\psi}_i' \rangle_{-1,1} \tag{8}$$

where $(H\hat{\psi})_k = H_k \hat{\psi}_k$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{-1,1}$ denotes the duality product $H^{-1} \times H_0^1$.

Introducing $p_k = \dfrac{f_0^2 \rho_0}{g(\rho_{k+1} - \rho_k)}$, $p_K = 0$, we obtain

$$\|\theta\|_{-1}^2 = \sum_{i=1}^{i=K} H_i |\nabla \psi_i|^2 + p_i |\psi_{i+1} - \psi_i|^2$$

This norm is equivalent to the standard one. On L^2, we introduce the norm

$$\|\theta\|_2^2 = \sum_{i=1}^{i=K} H_i \, |\Delta \psi_i|^2 + p_i \, |\nabla \psi_{i+1} - \nabla \psi_i|^2 \tag{9}$$

The norms $\| \cdot \|_2$, $\left(\sum_{i=1}^{i=K} H_i \, |\Delta \psi_i|^2 \right)^{1/2}$ and $|\theta|^2 = \left(\sum_{i=1}^{i=K} H_i |\theta_i|^2 \right)^{1/2}$, where $| \cdot |$ denotes the usual scalar product on $L^2(\Omega)$, are equivalent (Bernier, 1995).

Theorem 2. *(Bernier, 1995). For $\theta(0, x, y) = \theta_0(x, y)$ in H^{-1}, the system (5) admits a unique solution $\theta(t, x, y)$ in $C([0, T], H^{-1}) \cap L^2(0, T, L^2) \cap L^2_{loc}(0, T, H^1)$.*

The third theorem links systems (1) and (5).

Theorem 3. *(Bernier, 1994). For $t > 0$, the semigroup $G(t)$ is uniformly differentiable on A. Its differential at $\bar{\theta}_0$ is the linear operator on $H^{-1}(\Omega)$ given by $\theta \to L(t_0, \bar{\theta}_0)\theta_0 = \theta(t_0)$, where $\theta(t_0)$ is the value at time $t = t_0$ of the solution $\theta(t)$ of the linearized system (5). Moreover, $\sup_{\bar{\theta}_0 \in A} \left| L(t_0, \bar{\theta}_0) \right|_{L(H^{-1})} < \infty$.*

In what follows, we consider $\bar{\theta}_0 \in H_0^1$ and denote by $B = B(t, \bar{\theta}_0)$ the unbounded operator on H^{-1} with $D(B) = H_0^1$, defined by

$$(B\theta)_k = -\mu \Delta^2 \psi_k + J(\bar{\psi}_k, \theta_k) + J(\psi_k, \bar{\theta}_k) + \beta \frac{\partial \psi_k}{\partial x}$$

Let us determine the adjoint operator B^* and the symmetric part S of B, defined by $S = (B + B^*)/2$. The adjoint B^* can be easily calculated as follows:

$$\begin{aligned} E &= ((B\theta^1, \theta^2))_{-1} = -\langle B\theta^1, H\hat{\psi}^2 \rangle \\ &= -\sum \Big(-\mu \langle \Delta^2 \psi_k^1, H_k \hat{\psi}_k^2 \rangle + \langle J(\bar{\psi}_k, \theta_k^1), H_k \hat{\psi}_k^2 \rangle \\ &\quad + \langle J(\psi_k^1, \bar{\theta}_k), H_k \hat{\psi}_k^2 \rangle + \beta \langle \frac{\partial \psi_k^1}{\partial x}, H_k \hat{\psi}_k^2 \rangle \Big) \end{aligned}$$

$$
\begin{aligned}
&= -\sum \Big(-\mu H_k \langle \Delta^2 \psi_k^2, \hat{\psi}_k^1 \rangle - H_k \langle J(\bar{\psi}_k, \psi_k^2), \Delta \psi_k^1 \rangle \\
&\quad - p_k \langle J(\bar{\psi}_k, \psi_k^2), \hat{\psi}_{k+1}^1 - \hat{\psi}_k^1 \rangle + p_{k-1} \langle J(\bar{\psi}_k, \psi_k^2), \hat{\psi}_k^1 - \hat{\psi}_{k-1}^1 \rangle \\
&\quad - H_k \langle J(\psi_k^2, \bar{\theta}_k), \hat{\psi}_k^1 \rangle - \beta H_k \langle \frac{\partial \psi_k^2}{\partial x}, \bar{\psi}_k^1 \rangle \Big) \\
&= -\sum \Big(-\mu H_k \langle \Delta^2 \psi_k^2, \hat{\psi}_k^1 \rangle - H_k \langle \Delta J(\bar{\psi}_k, \psi_k^2), \hat{\psi}_k^1 \rangle \\
&\quad - p_{k-1} \langle J(\bar{\psi}_{k-1}, \psi_{k-1}^2), \hat{\psi}_k^1 \rangle + p_k \langle J(\bar{\psi}_k, \psi_k^2), \bar{\psi}_k^1 \rangle \\
&\quad + p_{k-1} \langle J(\bar{\psi}_k, \psi_k^2), \hat{\psi}_k^1 \rangle - p_k \langle J(\bar{\psi}_{k+1}, \psi_{k+1}^2), \hat{\psi}_k^1 \rangle \\
&\quad - H_k \langle J(\psi_k^2, \bar{\theta}_k), \hat{\psi}_k^1 \rangle - \beta H_k \langle \frac{\partial \psi_k^2}{\partial x}, \hat{\psi}_k^1 \rangle \Big)
\end{aligned}
$$

and then

$$
\begin{aligned}
(B^*\theta^2)_k &= -\mu \Delta^2 \psi_k^2 - \Delta J(\bar{\psi}_k, \psi_k^2) - \frac{p_{k-1}}{H_k} J(\bar{\psi}_{k-1}, \psi_{k-1}^2) + \frac{p_k}{H_k} J(\bar{\psi}_k, \psi_k^2) \\
&\quad + \frac{p_{k-1}}{H_k} J(\bar{\psi}_k, \psi_k^2) - \frac{p_k}{H_k} J(\bar{\psi}_{k+1}, \psi_{k+1}^2) - J(\psi_k^2, \bar{\theta}_k) - \beta \frac{\partial \psi_k^2}{\partial x}
\end{aligned}
$$

Furthermore, for the operator $S = (B + B^*)/2$ we have

$$
\begin{aligned}
(S\theta)_k &= -\mu \Delta^2 \psi_k + \frac{1}{2} \Big(J(\bar{\psi}_k, \theta_k) - \Delta J(\bar{\psi}_k, \psi_k) \Big) \\
&\quad - \frac{p_{k-1}}{2H_k} \Big(J(\bar{\psi}_{k-1}, \psi_{k-1}) - J(\bar{\psi}_k, \psi_k) \Big) + \frac{p_k}{2H_k} \Big(J(\bar{\psi}_k, \psi_k) - J(\bar{\psi}_{k+1}, \psi_{k+1}) \Big)
\end{aligned}
$$

or

$$
\begin{aligned}
(S\theta)_k &= -\mu \Delta^2 \psi_k + \frac{1}{2} \Big(J(\bar{\psi}_k, \Delta \psi_k) - \Delta J(\bar{\psi}_k, \psi_k) \Big) \\
&\quad + \frac{p_{k-1}}{2H_k} J(\bar{\psi}_k - \bar{\psi}_{k-1}, \psi_{k-1}) - \frac{p_k}{2H_k} J(\bar{\psi}_{k+1} - \bar{\psi}_k, \psi_{k+1})
\end{aligned}
$$

3.2. Eigenvalues of the Operator S and Predictability

Let us consider m solutions of the system (5), $\theta_1(t), \theta_2(t), \dots, \theta_m(t)$, corresponding to m initial conditions $\theta_1^0, \theta_2^0, \dots, \theta_m^0$, respectively. We denote by $|\theta_1^0 \wedge \theta_2^0 \wedge \dots \wedge \theta_m^0|$ the m-volume of the parallelepiped with edges $\theta_1^0, \theta_2^0, \dots, \theta_m^0$, and by $|\theta_1(t) \wedge \theta_2(t) \wedge \dots \wedge \theta_m(t)|$, the m-volume of the parallelepiped with edges $\theta_1(t), \theta_2(t), \dots, \theta_m(t)$. Define

$$
\omega_m(t, \xi_0) = \sup_{\substack{\theta_i^0 \in H^{-1}(\Omega) \\ \|\theta_i^0\| \le 1}} \frac{|\theta_1(t) \wedge \theta_2(t) \wedge \dots \wedge \theta_m(t)|}{|\theta_1^0 \wedge \theta_2^0 \wedge \dots \wedge \theta_m^0|}
$$

The numbers $\omega_m(t,\xi_0)$ determine the largest distortion of an infinitesimal m-dimensional volume generated by $G(t)$ around the point $\xi_0 \in H^{-1}(\Omega)$.

One can prove (Temam, 1988) that

$$|\theta_1(t)\ \wedge\ \dots\ \wedge\ \theta_m(t)| = |\theta_1^0\ \wedge\ \dots\ \wedge\ \theta_m^0| \exp\left(\int_0^t \mathrm{Tr}(-B(s,\xi_0)\circ Q_m(s))\,\mathrm{d}s\right)$$

where $Q_m(s) = Q_m(s,\theta_1^0,\theta_2^0,\dots,\theta_m^0)$ is the projector from $H^{-1}(\Omega)$ onto the space spanned by $\theta_1(t),\theta_2(t),..,\theta_m(t)$, and Tr denotes the trace operator. It follows that

$$\frac{1}{t}\ln\omega_m(t,\xi_0) = \sup_{\substack{\theta_i^0\,\in\, H^{-1}(\Omega)\\ \|\theta_i^0\|\,\leq\, 1}} \left(\frac{1}{t}\int_0^t \mathrm{Tr}(-B(s,\xi_0)\circ Q_m(s))\,\mathrm{d}s\right) \tag{10}$$

But by definition, we have

$$\mathrm{Tr}\Big(-B(s,\xi_0)\circ Q_m(s)\Big) = \sum_{i=1}^{i=m}((-B\phi_i(s),\phi_i(s)\ ))$$

where $\phi_i(s)$, $i \in \mathbb{N}$, is an orthonormal basis of $H^{-1}(\Omega)$, $\phi_i(s) \in L^2(\Omega)$, with $\phi_1(s),\dots,\phi_m(s)$ spanning $Q_m(s)H^{-1}(\Omega) = \mathrm{Span}\,[\theta_1(t),\theta_2(t),..,\theta_m(t)]$. Noticing that

$$((-B\phi_i(s),\phi_i(s)\ )) = ((-S\phi_i(s),\phi_i(s)\ ))$$

we conclude that

$$\mathrm{Tr}\Big(-B(s,\xi_0)\circ Q_m(s)\Big) = \mathrm{Tr}\Big(-S(s,\xi_0)\circ Q_m(s)\Big)$$

and, since S is self-adjoint,

$$\mathrm{Tr}\Big(-S(s,\xi_0)\circ Q_m(s)\Big) = \sum_{i=1}^{i=m}((-S\phi_i(s),\phi_i(s)\ )) \leq \sum_{i=1}^{i=m}\lambda_i(-S(s,\xi_0))$$

From (10) we obtain

$$\lim_{t\to 0}\ \frac{1}{t}\ \log\omega_m(t,\xi_0)\ \leq\ \lim_{t\to 0}\ \left(\frac{1}{t}\int_0^t \sum_{i=1}^{i=m}\lambda_i(-S(s,\xi_0))\,\mathrm{d}s\right) \tag{11}$$

Theorem 4. *The following equality holds:*

$$\lim_{t\to 0}\frac{1}{t}\log\omega_m(t,\bar\theta_0) = \sum_{i=1}^{i=m}\lambda_i(-S(0,\bar\theta_0))$$

In the proof we use the following lemma proved in (Bernier, 1995):

Lemma 1. *The eigenvalue* $\lambda_i(S(t,\bar\theta_0))$ *converges to* $\lambda_i(S(t_0,\bar\theta_0))$ *continuously when* t *tends to* t_0.

Proof of Theorem 4. Since the operator $\mathrm{Tr}(-S(s,\xi_0)\circ Q_m(s))$ is continuous in s, we have

$$\mathrm{Tr}\Big(-S(0,\xi_0)\circ Q_m(0)\Big) = \lim_{t\to 0}\frac{1}{t}\int_0^t \mathrm{Tr}\Big(-S(s,\xi_0)\circ Q_m(s)\Big)\,\mathrm{d}s$$

Thus by (10), it follows that

$$\mathrm{Tr}\Big(-S(0,\xi_0)\circ Q_m(0)\Big) \le \lim_{t\to 0}\frac{1}{t}\log\omega_m(t,\xi_0)$$

Taking the supremum of the left-hand side, we get

$$\sup_{\substack{\theta_i^0\in H^{-1}(\Omega)\\ \|\theta_i^0\|\le 1}} \mathrm{Tr}\Big(-S(0,\xi_0)\circ Q_m(0)\Big) \le \lim_{t\to 0}\frac{1}{t}\log\omega_m(t,\xi_0) \tag{12}$$

We now consider an orthonormal basis $\phi_1,\dots,\phi_m$ consisting of the first m eigenvectors of $S(0,\xi_0)$ and we denote by Q_o the projector in $H^{-1}(\Omega)$ onto $\psi_o = \mathrm{Span}\,[\phi_1,\dots,\phi_m]$. We thus have

$$\mathrm{Tr}\Big(-S(0,\xi_0)\circ Q_o\Big) = \sum_{i=1,\dots,m}((-S\phi_i,\phi_i)) = \sum_{i=1,\dots,m}\lambda_i(-S(0,\xi_0))$$

But we also have

$$\mathrm{Tr}\Big(-S(0,\xi_0)\circ Q_o\Big) \le \sup_{\substack{\theta_i^0\in H^{-1}(\Omega)\\ \|\theta_i^0\|\le 1}} \mathrm{Tr}\Big(-S(0,\xi_0)\circ Q_m(0)\Big)$$

and consequently

$$\sum_{i=1,\dots,m}\lambda_i\Big(-S(0,\xi_0)\Big) \le \sup_{\substack{\theta_i^0\in H^{-1}(\Omega)\\ \|\theta_i^0\|\le 1}} \mathrm{Tr}\Big(-S(0,\xi_0)\circ Q_m(0)\Big)$$

This result, together with (12), gives

$$\sum_{i=1,\dots,m}\lambda_i\Big(-S(0,\xi_0)\Big) \le \lim_{t\to 0}\frac{1}{t}\log\omega_m(t,\xi_0)$$

To obtain the converse inequality, we recall that (cf. (11))

$$\lim_{t\to 0}\frac{1}{t}\log\omega_m(t,\xi_0) \le \lim_{t\to 0}\left(\frac{1}{t}\int_0^t\sum_{i=1}^{i=m}\lambda_i\Big(-S(s,\xi_0)\Big)\,\mathrm{d}s\right)$$

Now, by using the continuity of the eigenvalues of S, the result follows. ∎

3.3. Operator S

Theorem 5. *The operator S is a self-adjoint closed one on H^{-1} with compact resolvent. Its eigenvalues λ_i are real and bounded from below:*

$$\begin{aligned} &\lambda_i \geq \mu\nu_i - \|\bar{\psi}\|_{H^2}\sqrt{3M\nu_i} && \text{if} \quad \|\bar{\psi}\|_{H^2} \leq 2\mu\sqrt{\frac{\nu_i}{3M}} \\ &\lambda_i \geq -\|\bar{\psi}\|^2_{H^2}\frac{3M}{4\mu} && \text{otherwise} \end{aligned} \tag{13}$$

where

$$M = \max_k \left(1, p_k\left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right) \tag{14}$$

and ν_i are the critical points of the functional

$$G(\theta) = \sum_k H_k|\Delta\psi_k|^2 \tag{15}$$

subject to the constraint $K = \left\{\theta \in L^2, \Delta\psi = 0 \text{ on } \partial\Omega, \|\theta\|_{-1} = 1\right\}$.
We use the lemma proved in (Bernier, 1995):

Lemma 2. *For all $\varepsilon > 0$, the following inequality holds:*

$$\begin{aligned} \left|\sum_{k=1}^{k=K} H_k\langle J(\bar{\psi}_k, \theta_k), \hat{\psi}_k\rangle\right| \leq \mu\varepsilon \sum_{k=1}^{k=K} H_k|\Delta\psi_k|^2 \\ + \frac{3\|\bar{\psi}\|^2_{H^2}}{4\mu\varepsilon} \max_k \left(1, \left(\frac{1}{H_k} + \frac{p_k}{H_{k+1}}\right)\right) \|\theta\|^2_{-1} \end{aligned} \tag{16}$$

Proof of Theorem 5. The operator S is self-adjoint by construction. Let us prove that S is closed. We examine the scalar product $((S\theta, \theta))$:

$$\begin{aligned} ((S\theta,\theta)) &= -\langle S\theta, H\hat{\psi}\rangle \\ &= -\sum \Big(-\mu\langle\Delta^2\psi_k, H_k\hat{\psi}_k\rangle \\ &\quad + \frac{1}{2}\Big(\langle J(\bar{\psi}_k,\theta_k), H_k\hat{\psi}_k\rangle - \langle\Delta J(\bar{\psi}_k,\psi_k), H_k\hat{\psi}_k\rangle\Big) \\ &\quad - \frac{1}{2}\Big(p_{k-1}\langle J(\bar{\psi}_{k-1},\psi_{k-1}),\hat{\psi}_k\rangle - p_{k-1}\langle J(\bar{\psi}_k,\psi_k),\hat{\psi}_k\rangle\Big) \\ &\quad + \frac{1}{2}\Big(p_k\langle J(\bar{\psi}_k,\psi_k),\hat{\psi}_k\rangle - p_k\langle J(\bar{\psi}_{k+1},\psi_{k+1}),\hat{\psi}_k\rangle\Big)\Big) \\ &= -\sum\Big(-\mu H_k|\Delta\psi_k|^2 \\ &\quad + \frac{1}{2}\Big(-\langle J(\bar{\psi}_k,\psi_k), H_k\hat{\theta}_k\rangle - \langle J(\bar{\psi}_k,\psi_k), H_k\Delta\hat{\psi}_k\rangle\Big) \end{aligned}$$

$$-\frac{1}{2}\left(p_k\langle J(\bar{\psi}_k,\psi_k),\hat{\psi}_{k+1}\rangle - p_k\langle J(\bar{\psi}_k,\psi_k),\hat{\psi}_k\rangle\right)$$

$$+\frac{1}{2}\left(p_{k-1}\langle J(\bar{\psi}_k,\psi_k),\hat{\psi}_k\rangle - p_{k-1}\langle J(\bar{\psi}_k,\psi_k),\hat{\psi}_{k-1}\rangle\right)\Bigg)$$

$$= \sum \mu H_k|\Delta\psi_k|^2 + \sum\langle J(\bar{\psi}_k,\psi_k), H_k\hat{\theta}_k\rangle$$

From (16) with $\varepsilon = \frac{1}{2}$, we have

$$\left|\sum_{k=1}^{k=N}\langle J(\bar{\psi}_k,\psi_k), H_k\hat{\theta}_k\rangle\right| \le \frac{\mu}{2}\sum_{k=1}^{k=N} H_k|\Delta\psi_k|^2 + \frac{3\|\bar{\psi}\|_{H^2}^2}{2\mu}\max_k\left(1, p_k\left(\frac{1}{H_k}+\frac{1}{H_{k+1}}\right)\right)\|\theta\|_{-1}^2$$

Hence

$$((S\theta,\theta)) \ge \frac{\mu}{2}\sum_{k=1}^{k=N} H_k|\Delta\psi_k|^2 - \frac{3\|\bar{\psi}\|_{H^2}^2}{2\mu}\max_k\left(1, p_k\left(\frac{1}{H_k}+\frac{1}{H_{k+1}}\right)\right)\|\theta\|_{-1}^2$$

Let us consider the operator $T = S + hId$ where Id is the identity operator on H^{-1} and h is a non-negative real. We have

$$((T\theta,\theta)) = ((S\theta,\theta)) + h\|\theta\|_{-1}^2 > \frac{\mu}{2}\sum_{k=1}^{k=N} H_k|\Delta\psi_k|^2 \ge 0 \qquad (17)$$

for any real h such that $h > \frac{3\|\bar{\psi}\|_{H^2}^2}{2\mu}\max_k\left(1, p_k\left(\frac{1}{H_k}+\frac{1}{H_{k+1}}\right)\right)$. On the other hand, from Hölder's inequality we see that $((T\theta,\theta)) \le \|T\theta\|_{-1}\|\theta\|_{-1}$. From (17), we deduce

$$\frac{\mu}{2}\sum_{k=1}^{k=N} H_k|\Delta\psi_k|^2 \le \|T\theta\|_{-1}\|\theta\|_{-1} \qquad (18)$$

The Poincaré inequality implies that

$$\frac{\mu}{2}\sqrt{\mu_1}\sum_{k=1}^{k=N} H_k|\nabla\psi_k|\,|\Delta\psi_k| \le \|T\theta\|_{-1}\|\theta\|_{-1}$$

and by Hölder's inequality and the equivalence of the norms

$$\sum_{k=1}^{k=N} H_k|\nabla\psi_k|^2 \ge c\|\theta\|_{-1}^2, \qquad \sum_{k=1}^{k=N} H_k|\Delta\psi_k|^2 \ge c\|\theta\|_2^2$$

we have $c\|\theta\|_{-1}\|\theta\|_2 \le \|T\theta\|_{-1}\|\theta\|_{-1}$, and then $c\|\theta\|_2 \le \|T\theta\|_{-1}$. Since $\|\theta\|_{-1} \le \|\theta\|_2$, we deduce that the operator T^{-1} is bounded. Thus the operator T is closed and so is S. From the compactness of the embedding of L^2 into H^{-1}, we conclude that T^{-1} is a compact operator. Then S is an operator with compact resolvent.

We estimate the eigenvalues of S using some ideas of the theory of critical points (Kavian, 1993). We want to find the critical points of $E_0(\theta)$ on the constraint set K:

$$E_0(\theta) = \mu \sum_k H_k |\Delta\psi_k|^2 + J(\theta)$$

$$K = \Big\{\theta \in L^2,\ \Delta\psi = 0 \ \text{ on } \ \partial\Omega, \|\theta\|_{-1} = 1\Big\}$$

where

$$J(\theta) = -\sum_k H_k \langle J(\bar{\psi}_k, \psi_k), \Delta\psi_k\rangle + \sum_k p_k \langle J(\bar{\psi}_k, \psi_k), \psi_{k+1} - \psi_k\rangle$$
$$-\sum_k p_{k-1} \langle J(\bar{\psi}_k, \psi_k), \psi_k - \psi_{k-1}\rangle$$

Since $\|\theta\|_{-1}^2 = 1$, we obtain from (16)

$$E_0(\theta) \geq \mu(1-\varepsilon) \sum_k H_k |\Delta\psi_k|^2 - \frac{3M}{4\mu\varepsilon} \sum_k \|\bar{\psi}_k\|_{H^2}^2$$

where

$$M = \max_k \left(1, p_k \left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right)$$

We denote by ν_i the critical points of the functional

$$G(\theta) = \sum_k H_k |\Delta\psi_k|^2$$

on the constraint set K. By the inf-sup definition of the eigenvalues, we obtain

$$\lambda_i(S(t_0)) \geq \mu(1-\varepsilon)\nu_i - \frac{3M}{4\mu\varepsilon} \|\bar{\psi}\|_{H^2}^2$$

We now optimize this result with respect to $\varepsilon \in [0,1]$ and obtain precisely the assertion of the theorem. ■

Let us determine the critical points ν_i of the functional $G(\theta)$ on the constraint set

$$K = \Big\{\theta \in L^2,\ \Delta\psi = 0 \ \text{ on } \ \partial\Omega, \|\theta\|_{-1} = 1\Big\}$$

These points are the eigenvalues of the problem

$$H_k \Delta^2 \psi_k = \nu\Big(- H_k \Delta\psi_k - p_k(\psi_{k+1} - \psi_k) + p_{k-1}(\psi_k - \psi_{k-1})\Big)$$

i.e.

$$\Delta^2\psi = \nu(\Delta\psi - W\psi)$$

Let us restrict our attention to the eigenbasis of $\mathcal{W}$ (3). In this basis, our problem takes the form

$$\Delta^2\psi^* = \nu(\Delta\psi^* - \Lambda\psi^*)$$

with boundary conditions (6). As before, we introduce $\hat{\psi}^*$ such that

$$\psi^* = \hat{\psi}^* + C^* = \hat{\psi}^* - \frac{1}{|\Omega|}\int_\Omega \hat{\psi}^*$$

where $|\Omega| = \text{meas}(\Omega)$, $\hat{\psi}^* \in H_0^1$, and rewrite the problem in the following form:

$$\Delta^2\hat{\psi}^* = \nu\left(\Delta\hat{\psi}^* - \Lambda\hat{\psi}^* + \frac{\Lambda_k}{|\Omega|}\int_\Omega \hat{\psi}^*\right) \tag{19}$$

Thus, ν is a critical point of the functional $G^*(\theta^*) = \sum_k |\Delta\hat{\psi}_k^*|^2$ on the constraint set

$$K^* = \left\{\theta^* \in L^2,\ \Delta\hat{\psi}_k^* = 0 \text{ on } \partial\Omega, \sum_k |\nabla\hat{\psi}_k^*|^2 + \Lambda_k|\hat{\psi}_k^*|^2 - \frac{\Lambda_k}{|\Omega|}\left(\int_\Omega \hat{\psi}^*\right)^2 = 1\right\}$$

Recall that the Schwarz inequality gives

$$\Lambda_k|\hat{\psi}_k^*|^2 - \frac{\Lambda_k}{|\Omega|}\left(\int_\Omega \hat{\psi}^*\right)^2 \geq 0$$

Using the inf-sup definition, we obtain

$$\nu_i = \inf_{A\in L^2, \dim A = i}\ \sup_{\theta^*\in A\cap K^*} G^*(\theta^*)$$

or

$$\nu_i = \inf_{A\in L^2, \dim A = i}\ \sup_{\theta^*\in A, \Delta\hat{\psi}^*=0 \text{ on } \partial\Omega} \frac{\sum_k |\Delta\hat{\psi}_k^*|^2}{\sum_k |\nabla\hat{\psi}_k^*|^2 + \Lambda_k|\hat{\psi}_k^*|^2 - \frac{\Lambda_k}{|\Omega|}\left(\int_\Omega \bar{\psi}^*\right)^2}$$

which is equivalent to

$$\nu_i = \inf_{A\in L^2, \dim A = i}\ \sup_{\theta^*\in A, \Delta\hat{\psi}^*=0 \text{ on } \partial\Omega} \frac{G^*(\theta^*)}{\|\theta^*\|^2}$$

But since

$$\sum_k |\nabla\hat{\psi}_k^*|^2 \leq \sum_k |\nabla\hat{\psi}_k^*|^2 + \Lambda_k|\hat{\psi}_k^*|^2 - \frac{\Lambda_k}{|\Omega|}\left(\int_\Omega \hat{\psi}^*\right)^2 \leq \sum_k |\nabla\hat{\psi}_k^*|^2 + \Lambda_k|\hat{\psi}_k^*|^2$$

we obtain

$$\frac{\sum_k |\Delta\hat{\psi}_k^*|^2}{\sum_k |\bar{\nabla}\hat{\psi}_k^*|^2} \geq \frac{G^*(\theta^*)}{\|\theta^*\|^2} \geq \frac{\sum_k |\Delta\hat{\psi}_k^*|^2}{\sum_k |\nabla\hat{\psi}_k^*|^2 + \Lambda_k|\hat{\psi}_k^*|^2}$$

As before, we now take the supremum over $\theta \in A$ with $\Delta\psi_k^* = 0$ on $\partial\Omega$, and next the infimum over all subspaces A of $L^2(\Omega)$ of dimension i. We denote by μ_i the i-th eigenvalue of the Laplacian operator with homogeneous Dirichlet boundary condition. The left-hand side equals μ_i. Let us determine the right-hand side. We seek the critical points of the functional $G^*(\theta^*)$ on the constraint set

$$\bar{K} = \left\{\theta^* \in L^2,\ \Delta\hat{\psi}_k^* = 0 \ \text{ on } \ \partial\Omega, \sum_k |\nabla\hat{\psi}_k^*|^2 + \Lambda_k|\hat{\psi}_k^*|^2 = 1\right\}$$

These are the eigenvalues η_i of the problem

$$\Delta^2\hat{\psi}_k^* = \eta(-\Delta\hat{\psi}_k^* + \Lambda_k\hat{\psi}_k^*)$$

It is easy to see that

$$\{\eta_i,\, i = 1, \ldots, \infty\} = \left\{\frac{\mu_i^2}{\mu_i + \Lambda_k},\, k = 1, \ldots, N;\, i = 1, \ldots, \infty\right\}$$

Consequently,

$$\mu_i \geq \nu_i \geq \eta_i$$

Notice that for homogeneous boundary conditions on ψ, we have $\nu_i = \eta_i$.

We remark that the estimates take into account the inverse of the radius of deformation for each layer.

4. Calculation of Instantaneous Lyapunov Exponents

The operator of the linearized system (5) can be rewritten in the form

$$\begin{aligned}
&\frac{\partial\Theta_k^*}{\partial t} + P^{-1}J(\bar{\psi}, \Delta\psi - \mathcal{W}\psi) + P^{-1}J(\psi, \Delta\bar{\psi} + \beta y - \mathcal{W}\bar{\psi}) \\
&\qquad = \mu(\Delta\Theta_k^* + \Lambda_k\Theta_k^* + \Lambda_k^2\psi_k^*) \qquad (20) \\
&\Theta_k^* = \Delta\psi_k^* - \Lambda_k\psi_k^* \\
&\psi = P\psi^*
\end{aligned}$$

In order to find a weak solution to the problem (20) with boundary conditions (6), we use the variational formulation of this problem. We denote by $\langle\cdot,\cdot\rangle$ the scalar product on L_2:

$$\langle\psi, \varphi\rangle = \iint_\Omega \psi\varphi\, \mathrm{d}x\mathrm{d}y \qquad (21)$$

Let us multiply eqn. (20) by a test function $\varphi^{(0)}(x,y) \in H_0^1(\Omega)$ and integrate it over the domain Ω. Using notation (21) and the Green formula, we get

$$\left\langle \frac{\partial \Theta_k^*}{\partial t}, \varphi^{(0)} \right\rangle + \left\langle P^{-1}J(\bar{\psi}, \Delta\psi - \mathcal{W}\psi) + P^{-1}J(\psi, \Delta\bar{\psi} + \beta y - \mathcal{W}\bar{\psi}), \varphi^{(0)} \right\rangle$$

$$= \mu\Big(- \langle \nabla\Theta_k^*, \nabla\varphi^{(0)}\rangle + \langle \Lambda_k \Theta_k^* + \Lambda_k^2 \psi_k^*, \varphi^{(0)}\rangle \Big) \tag{22}$$

$$\langle \Theta_k^*, \varphi^{(0)}\rangle = -\langle \nabla\psi_k^*, \nabla\varphi^{(0)}\rangle - \langle \Lambda_k \psi_k^*, \varphi^{(0)}\rangle$$

$$\langle \psi_k, \varphi^{(0)}\rangle = \langle P\psi^*, \varphi^{(0)}\rangle$$

since $\varphi^{(0)} \in H_0^1(\Omega)$, i.e. $\varphi^{(0)}\mid_{\partial\Omega} = 0$.

In order to solve eqns. (20) and to satisfy the boundary conditions (6), we introduce auxiliary functions

$$\psi^{(0)}(x,y,t), \quad \psi^{(1)}(x,y), \quad \Theta^{(0)}(x,y,t), \quad \Theta^{(1)}(x,y)$$

so that

$$\begin{aligned} \psi_k &= {\psi_k}^{(0)}(x,y,t) + C_k(t){\psi_k}^{(1)}(x,y), && {\psi_k}^{(0)} = 0, {\psi_k}^{(1)} = 1 \text{ on } \partial\Omega \\ \Theta_k &= {\Theta_k}^{(0)}(x,y,t) - \Lambda_k C_k(t){\Theta_k}^{(1)}(x,y), && {\Theta_k}^{(0)} = 0, {\Theta_k}^{(1)} = 1 \text{ on } \partial\Omega \end{aligned} \tag{23}$$

The values of $C_k(t)$ are calculated according to the formulae

$$C_k(t) = \frac{\displaystyle\int_\Omega \psi_k\mid_{t=0} \mathrm{d}x\mathrm{d}y - \int_\Omega \psi_k^{(0)}(x,y,t)\,\mathrm{d}x\mathrm{d}y}{\displaystyle\int_\Omega \psi_k^{(1)}\,\mathrm{d}x\mathrm{d}y}, \qquad k = 2,3,\ldots,K, \quad C_1 = 0 \tag{24}$$

Hence, eqn. (22) can be written in the form

$$\left\langle \frac{\partial {\Theta_k}^{*,(0)}}{\partial t}, \varphi^{(0)} \right\rangle - \Lambda_k \langle {\Theta_k}^{*,(1)}, \varphi^{(0)}\rangle \frac{\partial C_k(t)}{\partial t}$$

$$+ \left\langle P^{-1}J(\bar{\psi}, \Delta\psi - \mathcal{W}\psi), \varphi^{(0)} \right\rangle + \left\langle P^{-1}J(\psi, \bar{\theta} + \beta y), \varphi^{(0)} \right\rangle$$

$$= \mu\Big(- \langle \nabla\Theta_k^*, \nabla\varphi^{(0)}\rangle + \langle \Lambda_k \Theta_k^* + \Lambda_k^2 \psi_k^*, \varphi^{(0)}\rangle \Big) \tag{25}$$

The representation (23) allows us to split the relationship between ψ^* and Θ^* into

$$\begin{cases} {\Theta_k}^{*,(0)} = \Delta{\psi_k}^{*,(0)} - \Lambda_k {\psi_k}^{*,(0)} \\ -\Lambda_k {\Theta_k}^{*,(1)} = \Delta{\psi_k}^{*,(1)} - \Lambda_k {\psi_k}^{*,(1)} \end{cases} \tag{26}$$

In the variational formulation, we obtain

$$\begin{aligned} \langle \Theta_k^{*,(0)}, \varphi^{(0)} \rangle &= -\langle \nabla \psi_k^{*,(0)}, \nabla \varphi^{(0)} \rangle - \langle \Lambda_k \psi_k^{*,(0)}, \varphi^{(0)} \rangle \\ -\Lambda_k \langle \Theta_k^{*,(1)}, \varphi^{(0)} \rangle &= -\langle \nabla \psi_k^{*,(1)}, \nabla \varphi^{(0)} \rangle - \langle \Lambda_k \psi^{*,(1)}, \varphi^{(0)} \rangle \end{aligned} \tag{27}$$

To develop a finite-dimensional approximation of the operator S, we shall use the finite-element method. The feasibility and utility of the FEM for modelling ocean dynamics was first recognized by (Fix, 1975). He observed that the use of the FEM has numerous advantages regarding modelling, such as the precision, natural conservation of model invariants, flexibility of discretization of complex domains, etc. These advantages remain even though irregular discretisation of the domain is performed.

Since the solution produced by the QG model of the North Atlantic typically includes a western boundary layer with intense velocity gradients, the possibility of refining the triangulation along the western boundary of the domain is clearly avantageous. This helps us to maintain the quality of explicit eddy resolution by the model while working with a lower number of grid nodes. The comparison of finite-element and finite-difference models performed in (Le Provost *et al.*, 1994) revealed that differences between simulations by FE and FD techniques can be considered as insignificant.

Although the number of operations per time step and grid node is much higher for a FE model, by exploiting the possibility of the reduction of the number of grid points one can considerably diminish the computational cost of a model run. The possibility of reaching good precision while working with a lower number of grid points is especially valuable for Lyapunov exponent calculations where a very high number of operations per point is needed.

In this paper, we use $P2$ finite elements, i.e. polynomials of second-degree $p_i(x,y) = a_i x^2 + b_i xy + c_i y^2 + d_i x + e_i y + f_i$. We cover our domain Ω by a set of non-intersecting triangles and we define the set of integration points as the union of vertices and mi-edges of the triangles. We construct finite elements $p_i(x,y)$ so that they are equal to one at the i-th integration point and zero at all other points. We enumerate the integration points beginning with internal points of the domain, and pushing all the boundary points at the end of the set. In other words, we require

$$\begin{aligned} &(x_i, y_i) \in \Omega \backslash \partial\Omega \quad \text{for } i = 1, \ldots, N^{(0)} \\ &(x_i, y_i) \in \partial\Omega \qquad \text{for } i = N^{(0)} + 1, \ldots, N \end{aligned}$$

Since $\psi_k^{(0)}, \Theta_k^{(0)} \in H_0^1(\Omega)$, they can be expressed as linear combinations

$$\begin{aligned} \psi_k^{(0)}(x,y,t) &= \sum_{i=1}^{N^{(0)}} \psi_{i,k}^{(0)}(t) p_i^{(0)}(x,y) \\ \Theta_k^{(0)}(x,y,t) &= \sum_{i=1}^{N^{(0)}} \Theta_{i,k}^{(0)}(t) p_i^{(0)}(x,y) \end{aligned} \tag{28}$$

The functions $\psi_k^{(1)}, \Theta_k^{(1)} \in H^1$ and therefore they can be represented as

$$\psi_k^{(1)}(x,y) = \sum_{i=1}^{N} \psi_{i,k}^{(1)} p_i(x,y)$$
$$\Theta_k^{(1)}(x,y) = \sum_{i=1}^{N} \Theta_{i,k}^{(1)} p_i(x,y)$$

Using these expressions, we can write down the discretized version of the system (25), (27):

$$\mathcal{M}^{(0)} \frac{\partial \Theta_k{}^{*,(0)}}{\partial t} - \Lambda_k \frac{\partial C_k(t)}{\partial t} \mathcal{M}^{(1/2)} \Theta_k{}^{*,(1)}$$
$$= - \sum_m P^{-1} \Big(\sum_i \bar{\psi}_i \langle J(p_i, p_m), p_j^{(0)} \rangle (\Delta\psi - \mathcal{W}\psi)_m \Big)$$
$$- \sum_i P^{-1} \Big(\sum_m (\Delta\bar{\psi} + \beta y - \mathcal{W}\bar{\psi})_m \langle J(p_i, p_m), p_j^{(0)} \rangle \psi_i \Big)$$
$$+ \mu \Big(-\mathcal{C}^{(1/2)} \Theta_k^* + \Lambda_k \mathcal{M}^{(1/2)} \Theta_k^* + \Lambda_k^2 \mathcal{M}^{(1/2)} \psi_k^* \Big) \qquad (29)$$

$$\mathcal{M}^{(0)} \Theta_k^{*,(0)} = -\mathcal{C}^{(0)} \psi_k^{*,(0)} - \Lambda_k \mathcal{M}^{(0)} \psi_k^{*,(0)} \qquad (30)$$

$$-\Lambda_k \mathcal{M}^{(1/2)} \Theta_k^{*,(1)} = -\mathcal{C}^{(1/2)} \psi_k^{*,(1)} - \Lambda_k \mathcal{M}^{(1/2)} \psi_k^{*,(1)} \qquad (31)$$

$$\psi = P\psi^* \qquad (32)$$

where $\mathcal{M}$ and $\mathcal{C}$ are respectively the matrices of mass and rigidity:

$$\mathcal{M}_{i,j}^{(0)} = \langle p_i^{(0)}, p_j^{(0)} \rangle, \quad \mathcal{C}_{i,j}^{(0)} = \langle \nabla p_i^{(0)}, \nabla p_j^{(0)} \rangle \quad \begin{cases} i = 1, \dots, N^{(0)} \\ j = 1, \dots, N^{(0)} \end{cases}$$
$$\mathcal{M}_{i,j}^{(1/2)} = \langle p_i, p_j^{(0)} \rangle, \quad \mathcal{C}_{i,j}^{(1/2)} = \langle \nabla p_i, \nabla p_j^{(0)} \rangle \quad \begin{cases} i = 1, \dots, N \\ j = 1, \dots, N^{(0)} \end{cases} \qquad (33)$$

We choose $\Theta_k{}^{*,(1)}$ so as to have $\mathcal{M}^{(1/2)} \Theta_k{}^{*,(1)} = 0$. Then by (30) and (31)

$$-(\mathcal{C}^{(1/2)} + \Lambda_k \mathcal{M}^{(1/2)}) \psi_k^* = \mathcal{M}^{(1/2)} \Theta_k^* \qquad (34)$$
$$-\Lambda_k \psi_{k_i}^* = \Theta_{k_i}^*, \qquad i = N^{(0)} + 1, \dots, N$$

or simply

$$\psi_k^* = \mathcal{H} \Theta_k^*. \qquad (35)$$

Equation (29) can be rewritten as

$$\mathcal{M}^{(0)}\frac{\partial \Theta_k^{*(0)}}{\partial t} = \sum_{k1}\left(J^{(1)}_{(k,k1)}(\bar{\psi}) - J^{(2)}_{(k,k1)}(\bar{\psi})\right)\Theta^*_{k1} + \mathcal{A}_k\Theta^*_k = \mathcal{G}\Theta^*_{k1} \quad (36)$$

where

$$J^{(1)}_{(k,k2),(j,i)}(\bar{\psi}) = \sum_{k1} P^{-1}_{k,k1}\left(\sum_m \bar{\psi}_{m,k1}\langle J(p_i,p_m),p_j^{(0)}\rangle\right)P_{k1,k2}$$

$$J^{(2)}_{(k,k2),(j,l)}(\bar{\psi}) = \sum_{k1} P^{-1}_{k,k1}\sum_i\left(\sum_m (\bar{\theta})_{m,k1}\langle J(p_i,p_m),p_j^{(0)}\rangle\right)P_{k1,k2}\mathcal{H}_{k2,(i,l)}$$

$$\mathcal{A}_k = \mu\left(-\mathcal{C}^{(1/2)} + \Lambda_k\mathcal{M}^{(1/2)} + \Lambda_k^2\mathcal{M}^{(1/2)}\mathcal{H}_k\right)$$

Taking into account the evolution of the boundary constants obtained from (24), (30) and (36), we get

$$\begin{aligned}\frac{\mathrm{d}C_k}{\mathrm{d}t} &= -\frac{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega {\psi_k}^{(0)}\,\mathrm{d}x\mathrm{d}y}{\int_\Omega {\psi_k}^{(1)}\,\mathrm{d}x\mathrm{d}y} = \frac{\int_\Omega (\mathcal{C}^{(0)} + \Lambda_k\mathcal{M}^{(0)})^{-1}\mathcal{M}^{(0)}\frac{\partial \Theta_k^{*(0)}}{\partial t}\,\mathrm{d}x\mathrm{d}y}{\int_\Omega {\psi_k}^{(1)}\,\mathrm{d}x\mathrm{d}y} \\ &= \frac{1}{\int_\Omega {\psi_k}^{(1)}\,\mathrm{d}x\mathrm{d}y}(\mathcal{C}^{(0)} + \Lambda_k\mathcal{M}^{(0)})^{-1}\mathcal{G}\Theta^*\end{aligned} \quad (37)$$

Hence

$$\begin{aligned}\frac{\partial\Theta^*}{\partial t} &= \frac{\partial\Theta_k^{*,(0)}}{\partial t} - \Lambda_k\Theta_k^{*,(1)}\frac{\mathrm{d}C_k}{\mathrm{d}t} \\ &= \left(\mathcal{G} - \frac{\Lambda_k\Theta_k^{*,(1)}}{\int_\Omega {\psi_k}^{(1)}\,\mathrm{d}x\mathrm{d}y}\left(\mathcal{C}^{(0)} + \Lambda_k\mathcal{M}^{(0)}\right)^{-1}\mathcal{G}\right)\Theta^* = A\Theta^*\end{aligned}$$

We calculate the adjoint operator using the finite-dimensional approximation of the scalar product on L_2:

$$\langle\theta,\phi\rangle = \int_\Omega \theta\phi\,\mathrm{d}x\mathrm{d}y = \sum_i^N\sum_j^N \theta_i\phi_j\int_\Omega p_ip_j\,\mathrm{d}x\mathrm{d}y = \sum_i^N \theta_i(\mathcal{M}\phi)_i = (\mathcal{M}\theta,\phi) \quad (38)$$

Consequently,

$$\langle A\theta,\phi\rangle = (\mathcal{M}A\theta,\phi) = (\theta,A^t\mathcal{M}\phi) = (\mathcal{M}\theta,\mathcal{M}^{-1}A^t\mathcal{M}\phi) = \langle\theta,A^*\phi\rangle \quad (39)$$

hence

$$A^* = \mathcal{M}^{-1}A^t\mathcal{M} \quad (40)$$

and

$$S = \frac{A+A^*}{2} = \frac{A+\mathcal{M}^{-1}A^t\mathcal{M}}{2} \quad (41)$$

5. Comparison of Eigenvalues of S with Their Estimates

The package MODULEF (Bernadou, 1988) was used to perform triangulation of a domain. This package produces quasi-regular triangulation of the domain based on prescribed grid nodes on its boundary. We need to refine the triangulation near the western boundary and especially in the middle of the domain where velocity gradients are extremely sharp. Due to a high computational cost we take a very simple triangulation of the domain. This triangulation is composed of 26 triangles. The integration point set, being a union of vertices and mi-edges of triangles, counts 67 nodes. Thus the resolution of the grid varies between 1/10 of the side length (about 400 km) near the western boundary and 1/4 of the side length (about 1000 km) near the eastern one.

In this experiment, we take a square basin of characteristic length $L = 4000\,\text{km}$. We suppose that the depth of the ocean is composed of three layers of different densities. The depth thickness for each layer is respectively equal to $H_1 = 300\,\text{m}$, $H_2 = 700\,\text{m}$ and $H_3 = 4000\,\text{m}$. The difference in the water density between layers is taken so that the Rossby deformation radii of the baroclinic modes $\mathcal{R}_k = 1/\sqrt{\Lambda_k}$ are about $\mathcal{R}_2 = 34$ and $\mathcal{R}_3 = 18$ km.

The mean wind stress field applied to the upper layer of the ocean is approximated by a steady zonal wind composed of two gyre antisymmetric patterns:

$$F_1^{wind} = F_1^{wind}(y) = -\tau_0 \sin \frac{2\pi y}{L} \tag{42}$$

To calculate theoretical estimates of instantaneous Lyapunov exponents (Theorem 5) we first have to find the critical points of the functional $G(U)$ in (15). As was noted, we can obtain them by solving the eigenvalue problem (19). The variational formulation of this problem takes the form

$$\begin{aligned} &\langle \omega_k, \phi \rangle = \langle \Delta \hat{\psi}_k^*, \phi \rangle \\ &\langle \Delta \omega_k, \phi \rangle = \nu \Big\langle (\Delta \hat{\psi}_k^* - \Lambda_k \hat{\psi}_k^* + \frac{\Lambda_k}{|\Omega|} \int_\Omega \hat{\psi}_k^*), \phi \Big\rangle, \qquad \forall \phi \in H_1^0(\Omega) \end{aligned} \tag{43}$$

The approximation of this problem by the finite-element method can be written as follows:

$$\begin{aligned} &\mathcal{M}^{(0)} \omega_k = -\mathcal{C}^{(0)} \hat{\psi}_k^* \\ &\mathcal{C}^{(0)} \omega_k = -\nu (\mathcal{C}^{(0)} + \Lambda_k \mathcal{M}^{(0)} + \Lambda_k \mathcal{K}) \hat{\psi}_k^* \end{aligned} \tag{44}$$

where $\mathcal{M}$ and $\mathcal{C}$ are respectively the matrices of mass and rigidity (33), and $\mathcal{K}_{i,j} = \int_\Omega p_i \,\mathrm{d}\Omega \int_\Omega p_j \,\mathrm{d}\Omega$. Hence the eigenvalue problem to be solved is of the form

$$(\mathcal{C}^{(0)} + \Lambda_k \mathcal{M}^{(0)} - \Lambda_k \mathcal{K})^{-1} \mathcal{C}^{(0)} (\mathcal{M}^{(0)})^{-1} \mathcal{C}^{(0)} \hat{\psi}_k^* = \nu \hat{\psi}_k^* \tag{45}$$

The eigenvalues ν_i are the critical points of G which we shall use.

The norm $\|\bar{\psi}\|_{H^2}$ in the estimates (13) is given by

$$\|\bar{\psi}\|^2_{H^2} = \sum_{k=1}^{K} \frac{H_k}{H}\left(c_o 0 \int_\Omega \bar{\psi}_k^2 \, d\Omega + c_1 L^2 \int_\Omega |\nabla \bar{\psi}_k^2| \, d\Omega + c_2 L^4 \int_\Omega \Delta \bar{\psi}_k^2 + (\frac{\partial^2 \bar{\psi}_k}{\partial xy})^2 \, d\Omega\right)$$

where $H = H_1 + H_2 + H_3$ is the total depth of the ocean. Since the coefficients c_0, c_1, c_2 are not known exactly, we choose an overestimate of this norm

$$\|\bar{\psi}\|^2_{H^2} \leq c\frac{L^4}{H} \sum_{k=1}^{K} H_k \int_\Omega \Delta \bar{\psi}_k^2 \, d\Omega \tag{46}$$

where the constant c is to be determined. We note that this norm is proportional to the enstrophy of the solution.

For the problem described above, we calculate the eigenvalues of the operator S and their theoretical estimates. We perform a number of computational experiments with our model corresponding to different forcing magnitudes τ_0 and lateral friction coefficients μ.

The value of τ_0 was choosen from the range $1.6 \times 10^{-13} \cdots 1.6 \times 10^{-12}\,\mathrm{s}^{-2}$, i.e. the characteristic values of velocity of the flow given by the linear Sverdrup balance $U = \tau_0/\beta$ are comprised between $U = 0.8\,\mathrm{cm/s}$ and $U = 8\,\mathrm{cm/s}$. The lateral friction coefficient μ was choosen to avoid numerical instability which occurred due to the concentration of variability of the model at grid scales. The parameters of each experiment, i.e. the characteristic velocity values, which indicate the forcing magnitudes, and lateral friction coefficients are shown in Table 1.

Table 1. Experiment parameters.

Number	$U(\frac{\mathrm{cm}}{\mathrm{s}})$	$\mu(\frac{\mathrm{m}^2}{\mathrm{s}})$	$\overline{\lambda_{theor}}(\mathrm{day}^{-1})$	$\overline{\lambda_{num}}(\mathrm{day}^{-1})$	$\alpha(\lambda_{theor}, \lambda_{num})$
1	0.8	200	154.5	0.88	0.71 ± 0.03
2	0.8	400	54.5	0.80	0.75 ± 0.03
3	0.8	600	24.9	0.71	0.63 ± 0.04
4	2.0	500	24.5	1.29	0.79 ± 0.02
5	2.0	800	10.2	1.12	0.68 ± 0.03
6	4.0	600	11.2	1.79	0.75 ± 0.03
7	4.0	800	7.4	1.73	0.80 ± 0.02
8	4.0	1000	4.95	1.61	0.73 ± 0.03
9	6.0	1000	4.14	2.06	0.63 ± 0.04
10	6.0	1500	2.01	1.81	0.66 ± 0.04
11	8.0	1500	1.81	2.22	0.54 ± 0.04
12	8.0	2000	1.02	1.97	0.64 ± 0.04

The QG model in this experiment was integrated over 20 years from the zero state. We suppose that after this period the spin-up phase is terminated and the solution of the model attains its attractor. After the spin-up phase, the model was integrated over 100 years. We used this part of the trajectory as $\bar{\psi}(x,y,t)$ to create the operator S in (41) and to calculate instantaneous Lyapunov exponents of the model as well as their estimates (13). From this trajectory we took 1000 samples $\bar{\psi}_l(x,y) = \bar{\psi}(x,y,t_l)$ spaced by one month.

Altough Theorem 5 gives us only overestimates of the instantaneous Lyapunov exponents, we shall compare their temporal variability with calculated instantaneous exponents. This will provide us with a characteristic of the quality of these estimates. The comparison of temporal variability helps us to avoid difficulties in the choice of the exact value of the parameter c in the expression (46). Since this constant does not influence the variability of the norm, we can choose it arbitrarily. Here we use $c = 2.5 \times 10^{-13}$.

As the characteristic of the behaviour of temporal variability of these two values, we use the correlation coefficient

$$\alpha(\lambda_{theor}, \lambda_{num}) = \frac{\int_0^T (\lambda_{theor} - \overline{\lambda_{theor}})(\lambda_{num} - \overline{\lambda_{num}})\, dt}{\sqrt{\int_0^T (\lambda_{theor} - \overline{\lambda_{theor}})^2\, dt}\sqrt{\int_0^T (\lambda_{num} - \overline{\lambda_{num}})^2\, dt}} \tag{47}$$

where $\overline{\lambda} = \frac{1}{T}\int_0^T \lambda\, dt$. In Table 1 we present the time mean values of calculated exponents and their estimates, as well as their correlation coefficients α with 95% confidence interval.

As can be noted, the correlation coefficients are always positive, and, for experiments with relatively low forcing, they are even rather high. This indicates a good matching of the temporal variability of instantaneous exponents and their estimates.

The plot of normalized values of exponents and estimates is shown in Fig. 1. As you can see, estimates and exponents exhibit similar behaviour. The positions of local maxima and minima of curves are close to each other.

To illustrate the variability of exponents and estimates we plot the scatter diagram of their values Fig. 2. One can see that points on the scatter diagram form a cloud around the line representing the proportionality of instantaneous exponents and their estimates

$$\lambda_{theor} = A\lambda_{num} + B$$

the coefficients A and B of this line are about $A = 6.4 \pm 0.3$ and $B = -3.7$. Since A is positive, we can state that, generally, a greater H^2 norm of $\bar{\psi}$ corresponds to a less stable situation. The major part of points on this diagram are concentrated in the vicinity of this line. The mean distance between the line and points is about 0.18.

In this paper, we have used the enstrophy of the solution as its H^2-norm to obtain the best correlation of the temporal variability of instantaneous exponents and their estimates. Consequently, to estimate the QG model predictability at time t, we can calculate the enstrophy of the solution at this time as the first approximation of predictability estimates.

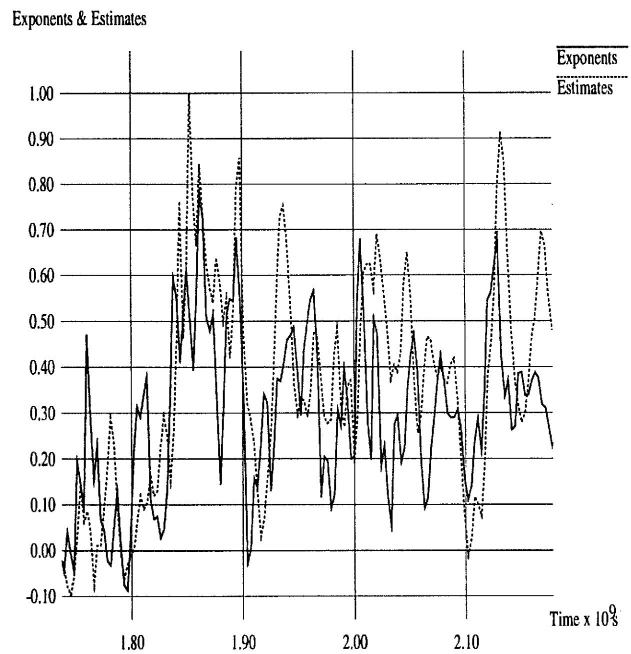

Fig. 1. Temporal variability of exponents and their estimates.

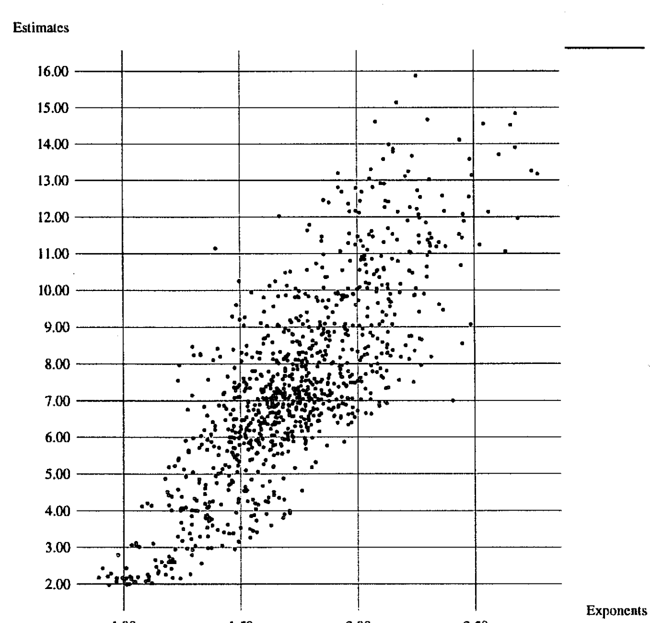

Fig. 2. Scatter diagram estimates v. exponents.

6. Conclusions

In this paper, we propose *a priori* estimates of instantaneous Lyapunov exponents of a quasi-geostrophic multi-layer ocean model. The *a priori* estimates we obtained give us not only a bound from below for each eigenvalue, but also a bound from above for the number of negative eigenvalues. These estimates remain valid for a very small coefficient of viscosity.

The temporal variability of these estimates demonstrates a rather high correlation with variability of calculated exponents. Consequently, we can use them as an easy calculable approximation of predictability estimates of the multi-layer quasi-geostrophic ocean model.

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