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DOMAIN OPTIMIZATION PROBLEMS FOR PARABOLIC CONTROL SYSTEMS

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In the paper, a class of shape optimization problems associated with the optimal location of distributed controls for parabolic systems is considered. The existence of solutions to optimization problems is established using a regularization technique. The first-order optimality conditions are obtained by an application of the material-derivative method. A relaxation method is proposed and the resulting parametric optimization problems are analysed.

1. Introduction

In the paper, we consider a class of shape optimization problems associated with the optimal location of controls for parabolic systems. The shape calculus is used in order to establish the first-order necessary optimality conditions for the optimization problem. Since the shape functional $I(\omega)$ defined in Section 3 actually depends on the characteristic function χ_ω, the relaxed optimization problems are introduced in Section 5 in a standard way.

The optimal location of controls is considered in (El Yacoubi, 1990; El Jay and Pritchard, 1988; Zolesio, 1984). The related shape optimization problems can be found in (Bendsoe and Sokołowski, 1995a; 1995b; Hoffmann and Sokołowski; 1991, 1994). The differential stability of solutions to the parametric optimization problems is considered in (Rao and Sokołowski, 1991; Sokołowski, 1981).

We refer the reader to (Piekarski, 1995) and the forthcoming paper by Piekarski and Sokołowski for results on the optimal location of controls for elastic plate models.

The standard notation is used throughout the paper.

2. Domain Optimization Problem

Let D be a bounded open subset of $\mathbb{R}^N$, $N = 1, 2$ or 3, with smooth boundary ∂D. We denote by $\Omega(D)$ the family of all open subsets of D such that $\omega \in \Omega(D)$ is (Lebesgue) mesurable and the volume of ω is prescribed, $|\omega| = \alpha$; α is given.

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Consider a diffusion system defined in D and described by the following equations:

$$\frac{\partial y}{\partial t} = Ay + u(x,t)\chi_\omega \quad \text{in } Q = (0,T) \times D \tag{1}$$

$$y_{|\Sigma} = 0 \quad \text{in } \Sigma = \partial D \times (0,T) \tag{2}$$

$$y(0) = 0 \quad \text{in } D \tag{3}$$

with ω fixed in $\Omega(D)$. Here χ_ω denotes the characteristic function of the subset $\omega \subset D$ and $(0,T)$ is the time interval. From now on, we assume that $A \in \mathcal{L}\left(H_0^1(D); H^{-1}(D)\right)$ is an elliptic operator of the second order.

Let us denote by $y(u,\omega)$ the solution to eqns. (1)–(3), where $u \in L^2(Q) = L^2(0,T;L^2(D))$, $u(x,t) = (u\chi_\omega)(x,t)$ a.e. in Q, and introduce the cost functional

$$J(u,\omega) = \frac{\delta}{2}\|u\|^2_{L^2(0,T;L^2(\omega))} + \|y(u,\omega;T) - y_d\|^2_{L^2(D)} \tag{4}$$

with y_d given in $L^2(D)$, $\delta > 0$. Set

$$J_\epsilon(u,\omega) = J(u,\omega) + \epsilon P_D(\omega) \tag{5}$$

where

$$P_D(\omega) = \sup \left\{ \int_\omega \operatorname{div} \phi \, dx \mid \phi \in \mathcal{D}^1(D, \mathbb{R}^N); \max_{x\in D} \|\phi(x)\|_{\mathbb{R}^N} = 1 \right\} \tag{6}$$

is the perimeter of ω in D and $\epsilon > 0$ is a constant.

Let us formulate the minimization problem as follows:

$$\min \left\{ J(u,\omega) \mid u \in U_{\text{ad}},\ \omega \in \mathcal{O}_{\text{ad}} \right\} \tag{7}$$

where the set of admissible controls $U_{\text{ad}} \subset L^2(0,T;L^2(\omega))$ is assumed to be convex and closed (an example of such a set is given in Section 4). For a given set ω we shall use the same symbol $u(x,t)$ to denote an admissible control $u \in L^2(0,T;L^2(\omega))$ and the function $u\chi_\omega \in L^2(0,T;L^2(D))$. The family of admissible domains $\mathcal{O}_{\text{ad}} \subset \Omega(D)$ satisfies the following assumption:

For any sequence $\omega_k \in \mathcal{O}_{\text{ad}}$, $k = 1,2,\ldots$, there exists a subset $\hat{\omega} \in \mathcal{O}_{\text{ad}}$ and a subsequence of the sequence $k = 1,2,\ldots$, still denoted by $k = 1,2,\ldots$, such that

$$\chi_{\omega_k} \rightharpoonup \chi_{\hat{\omega}} \tag{8}$$

weakly-(*) in $L^\infty(D)$ as $k \to \infty$.

The above condition constitutes a compactness assumption on the family $\mathcal{O}_{\text{ad}}$ of admissible domains. In particular, if there exists a constant M such that $P_D(\omega) \le M$ for any $\omega \in \mathcal{O}_{\text{ad}}$, then the compactness assumption is satisfied for $\mathcal{O}_{\text{ad}}$. We refer the reader to (Sokołowski and Zolesio, 1992) for details on regularization of shape optimization problems using the perimeter.

We need the compactness assumption on the family $\mathcal{O}_{\rm ad}$ of admissible domains to assure the existence of a solution to problem (7). For $\epsilon > 0$ we consider the regularized minimization problem

$$\min\left\{J_\epsilon(u,\omega)|\; u \in U_{\rm ad},\; \omega \in \Omega(D),\; P_D(\omega) < \infty\right\} \tag{9}$$

The optimization problem is considered as a two-level minimization one. First, for a given domain $\omega \subset D$, the following optimal control problem is solved and a unique optimal control $u^* = \chi_\omega u^*$ is determined such that

$$J(u^*,\omega) = \frac{\delta}{2}\|u^*\|^2_{L^2(0,T;L^2(\omega))} + \|y(u^*,\omega;T) - y_d\|^2_{L^2(D)} \leq J(u,\omega) \tag{10}$$

for all $u \in U_{\rm ad}$. The optimal control is denoted by $u^* = u(\omega)$ and the optimal value of the cost functional by $I(\omega) = J(u^*,\omega) = J(u(\omega),\omega)$.

Then, the minimization of the resulting shape functional is performed with respect to ω,

$$\min\left\{I(\omega) + \epsilon P_D(\omega)|\; \omega \in \Omega(D),\; P_D(\omega) < \infty\right\} \tag{11}$$

3. Optimality System

Let us consider the control problem (7) for a given set ω. We assume for simplicity that $\delta = 2$. The unique optimal control $u^* \in U_{\rm ad}$ is given by the following non-linear relation (Lions, 1968):

$$u^* = \mathcal{P}(-p\chi_\omega) \tag{12}$$

where $\mathcal{P}$ denotes the metric projection in $L^2(\omega \times (0,T))$ onto the set $U_{\rm ad} \subset L^2(\omega \times (0,T))$ of admissible controls. The adjoint state p is given by the unique solution of the following parabolic equation

$$-\frac{\partial p}{\partial t} = A^*p \qquad \text{in } Q \tag{13}$$

$$p_{|\Sigma} = 0 \qquad \text{on } \Sigma \tag{14}$$

$$p(T) = y(T) - y_d \qquad \text{in } D \tag{15}$$

Therefore the optimality system for the optimal control problem under consideration can be rewritten as the system of two equations

Find (y, p) such that

$$\frac{\partial y}{\partial t} = Ay + \mathcal{P}(-p\chi_\omega) \qquad \text{in } Q \tag{16}$$

$$y_{|\Sigma} = 0 \qquad \text{on } \Sigma \tag{17}$$

$$y(0) = 0 \qquad \text{in } D \tag{18}$$

$$-\frac{\partial p}{\partial t} = A^* p \qquad \text{in } Q \tag{19}$$

$$p_{|\Sigma} = 0 \qquad \text{on } \Sigma \tag{20}$$

$$p(T) = y(T) - y_d \qquad \text{in } D \tag{21}$$

The optimal value of the cost functional takes the form

$$I(\omega) = \|\mathcal{P}(-\chi_\omega p)\|^2_{L^2(0,T;L^2(\omega))} + \|y(T) - y_d\|^2_{L^2(D)}$$

Lemma 1. *The functional $I(\omega)$ is sequentially lower-semicontinuous with respect to the weak-$(*)$ convergence in $L^\infty(D)$ of characteristic functions.*

In view of the non-linear term $\mathcal{P}(-\chi_\omega p)$ in the optimal value of the cost functional, we cannot expect in general that a stronger result can be obtained, i.e. that the cost functional is sequentially lower-semicontinuous with respect to the weak-$(*)$ convergence in $L^\infty(D)$. In Lemma 1 we assume that the weak-$(*)$ limit of a sequence of characteristic functions is a characteristic function.

For the particular case of unconstrained control problems, we have a stronger result, i.e. the optimal value $I(\omega)$ of the cost functional is sequentially lower-semicontinuous with respect to the weak-$(*)$ convergence in $L^\infty(D)$. In such a case we introduce a relaxation of the shape optimization problem.

We consider the shape differentiability of the shape functional $I(\omega)$.

4. Shape Sensitivity Analysis

In the present paper, the method of sensitivity analysis of optimal control problems introduced in (Sokołowski, 1987; 1988) is used. The material-derivative method is used for the purposes of shape sensitivity analysis. We refer the reader to (Sokołowski and Zolesio, 1992) for a detailed description of the material-derivative method in shape optimization.

Using the material-derivative method, a one-parameter family of domains $\{\omega_s\} \subset D$, $s \in [0, \delta)$, is defined as follows.

The quantity $\omega_s = T_s(\omega)$ for $s \in [0,\beta)$ where $T_s = T_s(V) : \mathbb{R}^N \mapsto \mathbb{R}^N$, $N = 1,2,3$, is a smooth transformation. It is given by a sufficiently smooth vector field $V(\cdot,\cdot)$ with $V(s,x) = \frac{\partial T_s}{\partial s} \circ T_s^{-1}(x)$. We assume that $V(\cdot,\cdot) \in C^1([0,\tilde{\beta}); C^2(\mathbb{R}^N;\mathbb{R}^N))$ and write $\partial\omega_s = T_s(\partial\omega)$. For our applications, we assume that any admissible vector field V is compactly supported in D, and div $V = 0$ in D.

Let us consider the shape functional

$$I_\epsilon(\omega) = \min\left\{ J_\epsilon(u,\omega) |\ u \in U_{\rm ad}(\omega)\right\}$$

parametrized by a real variable $s \in [0,\beta)$

$$\mathcal{F}(s) = \min\left\{ J_\epsilon(u_s,\omega_s) |\ u_s \in U_{\rm ad}(\omega_s)\right\} = \min\left\{ \|u_s\|^2_{L^2(0,T;L^2(\omega_s))} \right.$$

$$\left. + \|y(u_s,\omega_s;T) - y_d\|^2_{L^2(D)} + \epsilon P_D(\omega_s) |\ u_s \in U_{\rm ad}\right\}$$

Here the set of admissible controls $U_{\rm ad}(\omega_s)$ depends on the parameter s, because it depends on the set ω_s. We show that the function $s \mapsto \mathcal{F}(s)$ is differentiable at 0^+. To this end, using the transformation T_s in a standard way, we can rewrite the minimization problem in a fixed domain, i.e. with a fixed set ω, and we obtain

$$\mathcal{F}(s) = \min\left\{ J_{\epsilon,s}(u,\omega) |\ u \in U_{\rm ad}(\omega)\right\} = \min\left\{ \|\gamma(s)^{\frac{1}{2}} u\|^2_{L^2(0,T;L^2(\omega))} \right.$$

$$\left. + \|\gamma(s)^{\frac{1}{2}} \left(y_s(u,\omega;T) - y_d\right)\|^2_{L^2(D)} + \epsilon P^s_D(\omega) |\ u \in U_{\rm ad}(\omega)\right\} \tag{22}$$

where $\gamma(s) = \det DT_s$, DT_s denotes the Jacobian of the transformation $T_s : \mathbb{R}^N \mapsto \mathbb{R}^N$. We write $^*DT_s$ for the transpose of DT_s and DT_s^{-1} for the inverse.

Let

$$\mathcal{H}(u_s,\omega_s) = \|u_s\|^2_{L^2(0,T;L^2(\omega_s))} + \|y(u_s,\omega_s;T) - y_d\|^2_{L^2(D)}$$

$$\mathcal{G}(u,\omega,s) = \|\gamma(s)^{\frac{1}{2}} u\|^2_{L^2(0,T;L^2(\omega))} + \|\gamma(s)^{\frac{1}{2}} \left(y_s(u,\omega;T) - y_d\right)\|^2_{L^2(D)}$$

We have

$$\mathcal{H}(u_s,\omega_s) = \mathcal{G}(u^s,\omega,s), \quad u^s = u_s \circ T_s\ , \quad \omega_s = T_s(\omega), \quad s \in [0,\beta)$$

and

$$\mathcal{F}(s) = \min\left\{ \mathcal{H}(u_s,\omega_s) + \epsilon P_D(\omega_s) |\ u_s \in U_{\rm ad}(\omega_s)\right\}$$

$$= \min\left\{ \mathcal{G}(u,\omega,s) + \epsilon P^s_D(\omega) |\ u \in U_{\rm ad}(\omega)\right\}$$

where $P_D(\omega_s) = P_D\left(T_s(\omega)\right) = P^s_D(\omega)$.

In particular,

$$I(\omega_s) = \min\left\{\mathcal{H}(v_s,\omega_s) |\ v_s \in U_{\rm ad}(\omega_s)\right\} = \mathcal{H}(u_s,\omega_s)$$
$$= \min\left\{\mathcal{G}(v,\omega,s) |\ v \in U_{\rm ad}(\omega)\right\} = \mathcal{G}(u^s,\omega,s)$$

Therefore, for directional differentiability of the function $s \mapsto I(\omega_s)$ at $s = 0^+$, it is sufficient to show, for the problem under consideration, the directional differentiability of the function $s \mapsto \mathcal{G}(u,\omega,s)$ for fixed (u,ω).

Lemma 2. *Assume for simplicity that $Ay = \Delta y$. The function $\mathcal{G}(u,\omega,\cdot)$ is differentiable at 0^+. The derivative is given by the following formula:*

$$\partial s\mathcal{G}(u,\omega,0) = \int_0^T\int_\omega |u|^2 {\rm div}\, V(0)\,{\rm d}x\,{\rm d}t + \int_0^T\int_D |y(u,\omega;T) - y_d|^2 {\rm div}\, V(0)\,{\rm d}x\,{\rm d}t$$
$$+ \int_0^T\int_D \dot{y}(T)(y(u,\omega;T) - y_d)\,{\rm d}x\,{\rm d}t$$

where

$$\frac{\partial \dot{y}}{\partial t} = A\dot{y} - {\rm div}\, V(0)\frac{\partial y}{\partial t} + A'y + u(x,t)\chi_\omega {\rm div}\, V(0) \quad \textit{in } Q \qquad (23)$$
$$\dot{y}_{|\Sigma} = 0 \quad \textit{on } \Sigma \qquad (24)$$
$$\dot{y}(0) = 0 \quad \textit{in } D \qquad (25)$$

and, in the case of $Ay = \Delta y$,

$$A'z = {\rm div}\left(\left\{{\rm div}\, V(0)I - {}^*DV(0) - DV(0)\right\}\cdot\nabla z\right)$$

The proof of Lemma 2 is given below.

Remark 1. We show that we cannot directly differentiate the optimality system (16)–(21) in order to obtain the Eulerian semiderivative ${\rm d}I(\omega;V)$ of the optimal-value functional $I(\omega)$.

Assume that the metric projection $\mathcal{P}$ is differentiable at $-p^*\chi_\omega$, where p^* denotes the solution to (19)–(21), and denote by $\mathcal{P}'$ its differential. Then the Eulerian semiderivative ${\rm d}I(\omega;V)$ is formally given by the following system:

$${\rm d}I(\omega;V) = 2\int_0^T\int_\omega p^*(\omega)(x,t)p'(\omega;V)(x,t)\,{\rm d}x\,{\rm d}t$$
$$+ \int_0^T\int_{\partial\omega} |p^*(\omega)(x,t)|^2 V\cdot n\,{\rm d}\Gamma\,{\rm d}t$$
$$+2\int_D (y^*(\omega)(x,T) - y_d)y'(\omega;V)(x,t)\,{\rm d}x \qquad (26)$$

where the shape derivatives $y'(\omega;V), p'(\omega;V)$ satisfy formally the following parabolic equations:

$$\frac{\partial y'}{\partial t} = Ay' + \mathcal{P}'\left[-p'\chi_\omega - p\chi'_\omega\right] \qquad \text{in } Q \tag{27}$$

$$y'_{|\Sigma} = 0 \qquad \text{on } \Sigma \tag{28}$$

$$y'(0) = 0 \qquad \text{in } D \tag{29}$$

$$-\frac{\partial p'}{\partial t} = A^* p' \qquad \text{in } Q \tag{30}$$

$$p'_{|\Sigma} = 0 \qquad \text{on } \Sigma \tag{31}$$

$$p'(T) = y'(T) \qquad \text{in } D \tag{32}$$

We denote (Piekarski, 1995) by $\chi'_\omega \in H^{-1}(D)$ the distribution of the form

$$\langle \chi'_\omega, \varphi \rangle = \int_{\partial\omega} \varphi V(x,0)\cdot n(x)\,\mathrm{d}l(x)$$

Unfortunately, the right-hand side of the equation for y' is not in general well-defined. On the other hand, we can only expect the so-called conical differentiability of the metric projection $\mathcal{P}$.

The related results on the conical differentiability of the metric projection can be found e.g. in (Sokołowski, 1985). A simple example of the set U_{ad} for which the assumption on the conical differentiability of the metric projection $\mathcal{P}$ is satisfied is the set

$$U_{\text{ad}} = \left\{ v \in L^2(0,T;L^2(\omega)) |\ 0 \le v(x,t) \le 1 \text{ a.e. in } \omega \right\}$$

In this case, the mapping $\mathcal{P}'$ evaluated at $u^* \in U_{\text{ad}}$ is given as the metric projection onto the cone

$$\mathcal{S} = \Big\{ v \in L^2(0,T;L^2(\omega)) |\ v(x,t) \ge 0 \quad \text{a.e. on} \quad \left\{(x,t) |\ u^*(x,t) = 0, x \in \omega\right\},$$
$$v(x,t) \le 0 \quad \text{a.e. on} \quad \left\{(x,t) |\ u^*(x,t) = 1, x \in \omega\right\}, \quad \int_0^T \int_\omega v(u^* + p)\,\mathrm{d}x\,\mathrm{d}t = 0 \Big\}$$

Proof of Lemma 2. We have

$$\mathcal{G}(u,\omega,s) = \int_0^T \int_\omega |u|^2 \gamma(s)\,\mathrm{d}x\,\mathrm{d}t + \int_D \left(y_s(u,\omega;T) - y_d\right)^2 \gamma(s)\,\mathrm{d}x$$

where y_s is a unique solution of the following parabolic equation:

$$\gamma(s)\frac{\partial y_s}{\partial t} = \operatorname{div}\left(\gamma(s) DT_s^{-1} \cdot {}^*DT_s^{-1} \cdot \nabla y_s\right) + \gamma(s) u \chi_\omega \qquad \text{in } Q$$

$$y_s = 0 \qquad \text{on } \Sigma$$

$$y_s(0) = 0 \qquad \text{in } D$$

Here we assume $Ay = \Delta y$. The result is obtained by an application of the formulae for material derivatives of solutions to parabolic equations given in (Sokołowski and Zolesio, 1992). ∎

Using the above results we obtain the necessary optimality conditions for the shape optimization problems under consideration. We assume that the volume of an admissible domain ω is prescribed, $|\omega| = \alpha$. We also assume that $\epsilon > 0$.

Theorem 1. *There exists an optimal domain ω which minimizes the shape functional $\mathcal{J}_\epsilon(\cdot)$,*

$$\mathcal{J}_\epsilon(\omega) = I(\omega) + \epsilon P_D(\omega) \tag{33}$$

subject to $|\omega| = \alpha$.

Furthermore, if the optimal domain ω is sufficiently regular, then for any vector field V with compact support in D and such that $\operatorname{div} V = 0$ it follows that

$$\mathrm{d}\mathcal{J}_\epsilon(\omega; V) = \mathrm{d}I(\omega; V) + \epsilon \int_{\partial\omega} \kappa(x) V(0,x) \cdot n(x)\, \mathrm{d}\Gamma(x) = 0 \tag{34}$$

where κ denotes the tangential divergence of the normal vector field n, $\kappa = -2\mathcal{H}$, $\mathcal{H}$ stands for the mean curvature of $\partial\omega$. The shape derivative $\mathrm{d}I(\omega; V)$ takes the following form:

$$\mathrm{d}I(\omega; V) = \int_0^T \int_\omega |u^*|^2 \operatorname{div} V(0)\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_D |y(u^*, \omega; T) - y_d|^2 \operatorname{div} V(0)\, \mathrm{d}x\, \mathrm{d}t$$
$$+ \int_0^T \int_D z(T)(y(u^*., \omega; T) - y_d)\, \mathrm{d}x\, \mathrm{d}t$$

where

$$\frac{\partial z}{\partial t} = Az - \operatorname{div} V(0) \frac{\partial y(u^*)}{\partial t} + A' y(u^*) + u^* \chi_\omega \operatorname{div} V(0) \qquad \textit{in } Q \tag{35}$$

$$z_{|\Sigma} = 0 \qquad \textit{on } \Sigma \tag{36}$$

$$z(0) = 0 \qquad \textit{in } D \tag{37}$$

Here we assume for simplicity that $Ay = \Delta y$.

5. Parametric Optimization—Relaxation of Shape Optimization Problems

A parametric optimization problem is introduced. It is constructed in the same way as the shape optimization problem in Section 2. The only difference is that the characteristic function χ_ω is replaced by an admissible function from a convex subset in $L^\infty(D)$, i.e. the non-convex set of admissible characteristic functions is replaced by its convex hull in $L^\infty(D)$.

Let us consider unconstrained control problems, and replace the characteristic function χ_ω by a function $\eta(x), x \in D$, within the admissible set

$$\mathcal{U}_{\text{ad}} = \left\{ \eta \in L^\infty(D) |\ 0 \leq \eta(x) \leq 1, \quad \int_D \eta(x)\, \mathrm{d}x = \alpha \right\}$$

Consider the problem of minimization with respect to η of the optimal value of the cost functional for the control problem, i.e.

$$\min_\eta h(\eta) = \min_\eta \ \min_u j(u, \eta) \tag{38}$$

where $u^* = u(\eta)$ denotes the unique optimal control u^* such that $u^* = \eta u^*$,

$$j(u,\eta) = \|\eta u\|^2_{L^2(Q)} + \|y(u,\eta;T) - y_d\|^2_{L^2(D)} \tag{39}$$

$$h(\eta) = \|\eta u^*\|^2_{L^2(Q)} + \|y(u^*,\eta;T) - y_d\|^2_{L^2(D)} \tag{40}$$

$$\frac{\partial y}{\partial t} = Ay + \eta(x) u(x,t) \quad \text{in } Q \tag{41}$$

$$y_{|\Sigma} = 0 \quad \text{on } \Sigma \tag{42}$$

$$y(0) = 0 \quad \text{in } D \tag{43}$$

In particular, we have $I(\omega) = h(\chi_\omega)$ for any characteristic function $\chi_\omega \in \mathcal{U}_{\text{ad}}$, and therefore the problem under consideration is a relaxation of the shape optimization problem defined in Section 2.

Using the optimality system for the unconstrained control problem, it follows that

$$h(\eta) = \|\eta p^*\|^2_{L^2(Q)} + \|y^*(T) - y_d\|^2_{L^2(D)} \tag{44}$$

$$\frac{\partial y^*}{\partial t} = Ay^* - p^*\eta \quad \text{in } Q \tag{45}$$

$$y^*_{|\Sigma} = 0 \quad \text{on } \Sigma \tag{46}$$

$$y^*(0) = 0 \quad \text{in } D \tag{47}$$

$$-\frac{\partial p^*}{\partial t} = A^* p^* \quad \text{in } Q \tag{48}$$

$$p_{|\Sigma} = 0 \quad \text{on } \Sigma \tag{49}$$

$$p^*(T) = y^*(T) - y_d \quad \text{in } D \tag{50}$$

Lemma 3. *The functional $h(\eta)$ is sequentially lower-semicontinuous with respect to the weak-$(*)$ convergence in $L^\infty(D)$.*

Proof. For a given sequence $\eta_k \in \mathcal{U}_{\rm ad}$ such that

$$\eta_k \rightarrow \eta \text{ weakly-}(*) \quad \text{in} \quad L^\infty(D)$$

we write

$$h(\eta_k) = \|\eta_k p_k\|^2_{L^2(Q)} + \|y_k(T) - y_d\|^2_{L^2(D)} \tag{51}$$

$$\frac{\partial y_k}{\partial t} = Ay_k - p_k\eta_k \qquad \text{in } Q \tag{52}$$

$$y_k = 0 \qquad \text{on } \Sigma \tag{53}$$

$$y_k(0) = 0 \qquad \text{in } D \tag{54}$$

$$-\frac{\partial p_k}{\partial t} = A^* p_k \qquad \text{in } Q \tag{55}$$

$$p_k = 0 \qquad \text{on } \Sigma \tag{56}$$

$$p_k(T) = y_k(T) - y_d \qquad \text{in } D \tag{57}$$

Under our assumptions,

$$\|p_k\|_{H^{2,1}(Q)} \le C, \qquad \|y_k\|_{H^{2,1}(Q)} \le C$$

and therefore there exist elements $y, p \in H^{2,1}(Q)$ such that, for subsequences, still denoted by y_k, p_k, $\eta_k p_k$, it follows that

$$y_k \rightarrow y \text{ weakly in } H^{2,1}(Q)$$

$$p_k \rightarrow p \text{ weakly in } H^{2,1}(Q)$$

Consequently, by the imbedding theorem,

$$y_k \rightarrow y \text{ strongly in } L^\infty\Big(0,T; L^2(D)\Big)$$

$$p_k \rightarrow p \text{ strongly in } L^2(Q)$$

Hence

$$\eta_k p_k \rightarrow \eta p \text{ weakly in } L^2(Q)$$

$$y_k(T) \rightarrow y(T) \text{ strongly in } L^2(D)$$

This completes the proof of the lemma. ∎

Theorem 2. *There exists a solution to the parametric optimization problem*

$$h(\eta^*) = \min\Big\{h(\eta) \mid \eta \in \mathcal{U}_{\rm ad}\Big\} \tag{58}$$

which satisfies the following necessary optimality conditions:

$$\mathrm{d}h(\eta^*; v - \eta^*) \geq 0 \quad \textit{for all} \;\; v \in \mathcal{U}_{\mathrm{ad}}$$

where

$$\mathrm{d}h(\eta; v) = \Big(\eta p^*, vp^* + \eta p'(v)\Big)_{L^2(Q)} + \Big(y^*(T) - y_d, y'(v;T)\Big)_{L^2(D)}$$

$$\frac{\partial y'(v)}{\partial t} = Ay'(v) - vp^* - \eta p'(v) \qquad \textit{in } Q \tag{59}$$

$$y'(v) = 0 \qquad \textit{on } \Sigma \tag{60}$$

$$y'(v)(0) = 0 \qquad \textit{in } D \tag{61}$$

$$-\frac{\partial p'(v)}{\partial t} = A^* p'(v) \qquad \textit{in } Q \tag{62}$$

$$p'(v) = 0 \qquad \textit{on } \Sigma \tag{63}$$

$$p'(v)(T) = y'(v)(T) \qquad \textit{in } D \tag{64}$$

It is an interesting question when the parametric optimization problem admits a solution in the form of the characteristic function χ_ω. From the necessary optimality conditions formulated in Theorem 2 it follows that the optimal solution $\eta^* \in \mathcal{U}_{\mathrm{ad}}$ is given in the form $\eta^* = \mathcal{P}_{\mathcal{U}}(\phi)$ of the metric projection onto the set of admissible parameters:

$$\int_D \Big[\eta^*(x) - \phi(x)\Big]\Big[v(x) - \eta^*(x)\Big]\psi(x)\,\mathrm{d}x \geq 0 \qquad \forall v \in \mathcal{U}_{\mathrm{ad}}$$

where

$$\psi(x) = \int_0^T |p^*(x,t)|^2\,\mathrm{d}t$$

$$\int_D \phi(x)v(x)\psi(x)\,\mathrm{d}x = \int_D \left(\int_0^T p^*(x,t)p'(v;x,t)\,\mathrm{d}t\right)\mathrm{d}x$$
$$+ \int_D (y^*(T,x) - y_d(x))y'(v;T,x)\,\mathrm{d}x$$

Here we assume that the function $\psi(x)$ is strictly positive. Therefore, the necessary condition to have

$$\eta^*(1 - \eta^*) = 0 \quad \text{a.e. in } D$$

is that

$$\phi(x) \notin (0,1) \qquad \text{a.e. in } D$$

However, the element $\phi(x)$ depends on the unknown optimal design $\eta^* \in L^\infty(D)$.

6. Conclusion

In the paper, a class of shape optimization problems is considered. In our opinion, an interesting question, is the investigation whether relaxed problems can be used for numerical solution to the optimal location of controls. In addition to that, let us note that in relaxed problems the control constraints are not directly present.

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