Appl. Math. and Comp. Sci., 1996, vol.6, No.2, 353–374

DOMAIN OPTIMIZATION PROBLEM FOR STATIONARY HEAT EQUATION

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In the paper, the support of a Radon measure is selected in an optimal way. The solution of the second-order elliptic equation depends on the mesure via the mixed-type boundary conditions. The existence of a solution for a class of domain optimization problems is shown. A relaxed formulation of the optimization problems is proposed. The first-order necessary optimality conditions are derived.

1. Introduction

In this paper, we shall consider a problem related to the following question. Given a flat piece of material, e.g. a pane of glass in a window, we attach a heating wire to one surface of this material. This wire is modelled as a continuous curve connected to fixed points A and B. We want to investigate which curve would optimize the temperature distribution on the opposite surface at a given time?

To cope with the problem, we use the following mathematical model. Let Ω be a region in the two-dimensional Euclidean space. Define

$$\Sigma = \Omega \times (0,d), \quad d > 0$$

and

$$\Omega_0 = \Omega \times \{0\}, \quad \Omega_1 = \Omega \times \{d\}$$

as well as

$$\Gamma = \partial\Omega \times (0,d)$$

the "vertical" boundary of Σ. Let u_0 and u_1 be positive functions on Ω_0 and Ω_1, $A = (x_0, y_0)$ and $B = (x_1, y_1)$ two distinct points in Ω_0, and $\gamma : [0,1] \to \Omega_0$ a

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continuous curve of finite length in Ω_0. Let U be the solution to the stationary heat equation

$$-\Delta U = 0$$

on Σ, with boundary values

$$\partial_n U|_\Gamma = 0$$

$$\partial_n U|_{\Omega_1} = U - u_1$$

and

$$\partial_n U|_{\Omega_0} = U - u_0 - f_\gamma$$

where f_γ is a positive function concentrated along the curve γ. One could think of f_γ to be an approximation of a delta-function at γ. The problem can now be stated as follows: Given a target function U^* on Ω_1, find γ such that

$$\| U|_{\Omega_1} - U^* \|_X^2$$

becomes minimal in a suitable Banach space X.

The crux of the matter is to find a suitable admissible set for the curves γ, as well as a convenient metric on this set of curves. First of all, it would be tempting to replace any curve by its parametrization in order to have a Banach structure on the set of curves. However, it is obvious that this point of view is not convenient, since a parametrization is not "intrinsic" enough to measure distances of two curves, as the following example illustrates. Let

$$\gamma_1 : \begin{cases} x(t) = t^4 \\ y(t) = 0 \end{cases}$$

and

$$\gamma_2 : \begin{cases} x(t) = t \\ y(t) = 0 \end{cases}$$

Both of these parametrizations give the same curve, but

$$\int_0^1 |\gamma_1(t) - \gamma_2(t)|^2 \, dt > 0$$

i.e. this integral does not define a metric on the set of curves.

A more classical idea is to work with the Hausdorf metric. For two curves parametrized respectively by $\gamma_1(t)$, $t \in [0,1]$, and $\gamma_2(t)$, $t \in [0,1]$, this distance is defined by

$$d(\gamma_1, \gamma_2) = \max\left\{ \max_{t\in[0,1]} \left(\min_{s\in[0,1]} |\gamma_1(t) - \gamma_2(s)| \right), \max_{s\in[0,1]} \left(\min_{t\in[0,1]} |\gamma_1(t) - \gamma_2(s)| \right) \right\}$$

This metric has good compactness properties. For example, for any sequence of compact sets K_n which are included in some large ball, there is a subsequence which converges in the Hausdorf metric to a compact set K. Unfortunately, for a sequence of curves K_n the K does not have to be a curve, as the following example illustrates. The curves $K_n = \{y = \sin(nx), x \in [0, \pi]\}$ converge to the set $K = [0, \pi] \times [-1, 1]$. It therefore seems natural that one has to impose some additional constraints on the set of curves considered. These constraints could be on the length or on the Hausdorff measure of the curves. The following sections will elaborate on these ideas.

It is well-known (e.g. see (Ziemer,1989)) that if γ_n is a sequence of continuous curves whose Hausdorff measure is uniformly bounded by a number M, and if γ_n converges to γ in the Hausdorff metric, then γ is also a continuous curve. However, it is not generally true that the Dirac-measures δ_{γ_n} converge weakly* to δ_γ (see Section 3). But this is exactly the kind of convergence necessary to prove the continuity of the solution to the problem above with respect to curves.

We are faced with a classical situation in shape optimization: the Hausdorff distance has very good compactness properties, but is not strong enough to ensure that the cost functional is lower semi-continuous.

Finally, we want to point out that the results of this paper also hold if the Laplacian is replaced by more general uniform elliptic operators.

2. Existence of a Classical Solution

We assume that Ω is a simply-connected domain in $\mathbb{R}^2$ and let $\Sigma = \Omega \times (0, d)$. We write $\Omega_0 = \Omega \times \{0\}$, $\Omega_1 = \Omega \times \{d\}$ and $\Gamma = \partial\Omega \times (0, d)$. Therefore

$$\partial\Sigma = \Omega_0 \cup \Omega_1 \cup \Gamma$$

Given a curve $\gamma \subset \Omega_0$ parametrized by $s \in [0, 1]$, we assume that $A = \gamma(0)$ and $B = \gamma(1)$ are fixed points in Ω_0. For the stationary heat equation, γ is the heat source. Let us consider the following elliptic equation:

$$(\mathcal{P}_1(\gamma)) \qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \\ -\dfrac{\partial u}{\partial n} = u - u_1 & \text{on } \Omega_1 \\ -\dfrac{\partial u}{\partial n} = u - u_0 - \delta_\gamma & \text{on } \Omega_0 \end{cases}$$

where u_0 and u_1 are given L^2-functions, and δ_γ is a Dirac mesure supported on the curve γ. The variational formulation of the stationary heat equation is given as follows:

Find $u \in H^1(\Sigma)$ such that for all functions $v \in H^1(\Sigma)$

$$(\mathcal{P}_2(\gamma)) \qquad a(u, v) = L(v)$$

where

$$a(u,v) = \int_\Sigma \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\Omega_1} uv \,\mathrm{d}\sigma + \int_{\Omega_0} uv \,\mathrm{d}\sigma \tag{1}$$

$$L(v) = \int_{\Omega_1} u_1 v \,\mathrm{d}\sigma + \int_{\Omega_0} u_0 v \,\mathrm{d}\sigma + \int_\gamma v \,\mathrm{d}\gamma \tag{2}$$

In order to have a well-defined problem, it is sufficient to show that the linear form

$$\delta_\gamma \ : \ v \mapsto \int_\gamma v \,\mathrm{d}\gamma$$

is continuous on the space $H^1(\Sigma)$. We are going to define the set of admissible curves γ in such a way that the linear form is continuous. To this end, we denote by Q the cube $Q = (0,1) \times (0,1)$, and by $I \subset Q$ the interval $I = \left[-\frac{1}{2}, \frac{1}{2}\right] \times \{0\}$.

Definition 1. A given curve γ is called admissible if there exists a one-to-one mapping $F \ : \ Q \mapsto \mathcal{O}$, where $\mathcal{O}$ denotes an open neighbourhood of γ in Ω_0 such that

$$F(Q) = \mathcal{O}, \qquad F(I) = \gamma \tag{3}$$

$$\|F\|_{W^{1,\infty}(Q)} \leq L_1, \qquad \|F^{-1}\|_{W^{1,\infty}(\mathcal{O})} \leq L_2 \tag{4}$$

Prescribing uniform bounds $L = L_1 = L_2 > 0$ and assuming that the following compactness condition is satisfied:

($\mathcal{H}$) Given a sequence F_n which satifies uniformly the bounds (4), there exists a subsequence, still denoted by F_n, such that

$$|F_n'(\cdot,0)| \to |F'(\cdot,0)| \qquad \text{weakly in } L^2\left(-\frac{1}{2},\frac{1}{2}\right) \tag{5}$$

we define an admissible family

$$\mathcal{F}_L = \{\gamma \text{ is admissible} \,|\, (\mathcal{H}) \text{ is satisfied}, \|F\|_{W^{1,\infty}(Q)} \leq L \text{ and } \|F^{-1}\|_{W^{1,\infty}(\mathcal{O})} \leq L\}$$

where $L > 0$ is a given constant.

Remark 1. Without the assumption ($\mathcal{H}$) on the family $\mathcal{F}_L$ we cannot expect that, for any sequence $\{\gamma_n\} \subset \mathcal{F}_L$, there exists a subsequence, still denoted by $\{\gamma_n\}$, such that

$$\delta_{\gamma_n} \to \delta_\gamma \qquad \text{weakly in the space } (H^1(\Sigma))'$$

A counterexample can be constructed using $F_n(x,y) = \{x, y + \frac{1}{n}\sin(nx)\}$.

Our problem consists now in minimizing the cost functional

$$J(\gamma) = \|u_\gamma - u_d\|$$

where u_γ denotes a solution to the stationary heat equation for any $\gamma \in \mathcal{F}_L$ and the Dirac measure δ_γ in the boundary conditions, u_d is a given function, and $||\cdot||$ is a norm, or a seminorm on the space $H^1(\Sigma)$ which will be specified later.

Remark 2. We use the above definition of a set of admissible curves $\mathcal{F}_L$, since we want to apply an appropriate trace theorem on γ. Such a definition is better suited for our applications than the simple definition of curves parametrized over an interval.

Remark 3. We can replace Definition 1 by a more general notion of a Lipschitzian manifold, where the existence of a global parametrization is not required. We prefer to work with the global parametrization for the sake of simplicity. The same result can be obtained for a more general setting of the Lipschitzian manifold, provided that the uniform bounds are prescribed with the same Lipschitz constant for any collection of charts. Using a partition of unity the problem can be localized in a standard way.

Remark 4. Some classes of admissible curves in the plane are introduced by Daniliuk (1975) in the framework of integral equations in non-smooth domains.

On the other hand, it seems to be possible to use some families of admissible curves defined by using constraints of capacity type, which probably assure the existence of a solution in a slightly wider class. But this approach is rather complicated and it is not evident that such families of admissible curves can be of any interest for numerical methods. We refer the reader to the monograph (Ziemer, 1989) for the definition and properties of capacity, and to (Bucur and Zolesio, 1996) for some results in the case of admissible domains with capacitary constraints for homogeneous Dirichlet problems. In the present paper we rather use the notion of a generalized solution to the problem defined in Section 3.

Proposition 1. *For any admissible curve $\gamma \in \mathcal{F}_L$ the linear form*

$$\langle \delta_\gamma, \varphi \rangle = \int_\gamma \varphi \, \mathrm{d}\gamma$$

is continuous with the norm in the dual space bounded:

$$||\delta_\gamma||_* \leq C_\gamma P(\gamma)$$

where $C_\gamma = C_\gamma(L, \Sigma)$ and $P(\gamma) = \int_\gamma \mathrm{d}\gamma$ is the length of γ.

Proof. For an element $\varphi \in H^1(\Sigma)$ the trace on Ω_0 is also denoted by φ and it satisfies $\varphi \in H^{\frac{1}{2}}(\Omega_0)$, we refer to (Adams, 1975; Lions and Magenes, 1968) for a proof. The first important question is whether or not it is possible to define a trace on γ for any element of the space $H^{\frac{1}{2}}(\Omega_0)$. The positive answer is obtained by applying the theorem of Besov–Uspienskii (Adams, 1975, Thm. 7.58): The injection of the space $H^{\frac{1}{2}}(\mathbb{R}^2)$ into $L^2(\mathbb{R})$ is continuous.

Let us show that if u is in $H^{\frac{1}{2}}(\Omega_0)$, the funtion $\hat{u} = u \circ F$ defined on Q belongs to the space $H^{\frac{1}{2}}(Q)$. Hence, it is sufficient to prove that the following integral is finite

$$I = \int_Q \int_Q \left(\frac{|\hat{u}(x) - \hat{u}(y)|}{|x-y|^{\frac{3}{2}}} \right)^2 \mathrm{d}x\mathrm{d}y$$

$$= \int_Q \int_Q \frac{|u(F(x)) - u(F(y))|^2}{|x-y|^3} \mathrm{d}x\mathrm{d}y \tag{6}$$

We set $x_1 = F(x), y_1 = F(y)$, $|DF^{-1}|$ the determinant of Jacobian DF^{-1}, hence

$$I = \int_O \int_O \frac{|u(x_1) - u(y_1)|^2}{|F^{-1}(x_1) - F^{-1}(y_1)|^3} |DF^{-1}| \,\mathrm{d}x_1\mathrm{d}y_1 \tag{7}$$

Since the mapping F is Lipschitz,

$$L_1|F^{-1}(x_1) - F^{-1}(y_1)| = L_1|x-y| \geq |F(x) - F(y)| = |x_1 - y_1| \tag{8}$$

and

$$|DF^{-1}| \leq C = 2L_2^2 \tag{9}$$

we have

$$I \leq 2L_2^2 L_1^3 \int_O \int_O \frac{|u(x_1) - u(y_1)|^2}{|x_1 - y_1|^3} \mathrm{d}x_1\mathrm{d}y_1 < \infty$$

since $u \in H^{\frac{1}{2}}(\Omega_0)$.

The trace operator maps $H^{\frac{1}{2}}(Q)$ into $L^2(Q_0)$ by the theorem of Besov and Uspienskii and is defined by means of the mapping F, as a trace operator for the space $H^{\frac{1}{2}}(O)$ into the space $L^2(\gamma)$. Furthermore,

$$\|\varphi\|_{L^2(\gamma)} \leq C\|\varphi\|_{H^{\frac{1}{2}}(O)} \leq C\|\varphi\|_{H^{\frac{1}{2}}(\Omega_0)}$$

where $C = C(L_1, L_2)$, L_1 and L_2 are Lipschitz constants of F and F^{-1}, respectively. In particular, $L = L_1 = L_2$ for $\gamma \in \mathcal{F}_L$. In view of the continuity of the trace operator $H^1(\Sigma) \mapsto H^{\frac{1}{2}}(\Omega_0)$ it follows that .

$$\|\varphi\|_{L^2(\gamma)} \leq C\|\varphi\|_{H^1(\Sigma)} \tag{10}$$

with a constant $C = C(L_1, L_2, \Sigma)$. Therefore

$$\left| \int_\gamma \varphi \,\mathrm{d}\gamma \right| \leq P(\gamma) \left(\int_\gamma \varphi^2 \,\mathrm{d}\gamma \right)^{\frac{1}{2}} \leq CP(\gamma)\|\varphi\|_{H^1(\Sigma)}$$

which completes the proof. ■

An admissible curve is defined in the parametric form

$$\begin{cases} x(t) = F_1(t, 0) \\ y(t) = F_2(t, 0) \end{cases} \quad t \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]$$

where $F = (F_1, F_2)$ is a bi-Lipschitz mapping. For $\gamma \in \mathcal{F}_L$ it follows that

$$P(\gamma) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{x'^2(t) + y'^2(t)}\,\mathrm{d}t = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial F_1}{\partial x}^2 (t,0) + \frac{\partial F_2}{\partial x}^2 (t,0) \right)^{\frac{1}{2}} \mathrm{d}t \leq \sqrt{2}L$$

and therefore the length of admissible curves in the set $\mathcal{F}_L$ is uniformly bounded, but the uniform boundedness of the length is a weaker condition for a curve than the condition to be a member of $\mathcal{F}_L$.

The class $\mathcal{F}_L$ is sufficiently small to obtain an existence result for the problem under consideration.

Proposition 2. *Given a sequence of curves γ_n in $\mathcal{F}_L$, there exists a curve $\gamma \in \mathcal{F}_L$ and a subsequence γ_{n_k} such that*

$$\delta_{\gamma_{n_k}} \to \delta_\gamma \qquad \text{weakly in the space } (H^1(\Sigma))'$$

i.e.

$$\langle \delta_{\gamma_{n_k}}, \varphi \rangle \to \langle \delta_\gamma, \varphi \rangle \qquad \text{for all } \varphi \in H^1(\Sigma)$$

Proof. Given $\gamma_n = F_n(Q_0) \in \mathcal{F}_L$, we have

$$\|F_n\|_{W^{1,\infty}} \leq L \qquad \text{and} \qquad \|F_n^{-1}\|_{W^{1,\infty}} \leq L$$

By the Ascoli theorem there exists a function F which is continuous over Q such that for a subsequence F_{n_k}

$$F_{n_k}(x) \to F(x) \qquad \text{uniformly over } \overline{Q}$$

The functions F_{n_k} are uniformly Lipschitz continuous with a constant L, and the same remains valid for F, thus $F \in W^{1,\infty}(Q)$ with $\|F\|_{W^{1,\infty}} \leq L$. We set $\gamma = F(Q_0)$.

Furthermore, the inequality $\|F_n^{-1}\|_{W^{1,\infty}} \leq L$ implies that

$$|F_n(x) - F_n(y)| \geq \frac{1}{L}|x-y| \qquad \forall x, y \in Q \tag{11}$$

Hence taking the limit leads to

$$|F(x) - F(y)| \geq \frac{1}{L}|x-y| \qquad \forall x, y \in Q \tag{12}$$

which shows that F is one-to-one. We write $\mathcal{O} = F(Q)$, thus there exists the inverse mapping $F^{-1} : \mathcal{O} \mapsto Q$, F^{-1} being Lipschitz continuous with constant L in view of the latter inequality. Therefore $\gamma \in \mathcal{F}_L$.

For the sake of simplicity we denote by γ_n the subsequence γ_{n_k}. We are going to show that δ_{γ_n} converges to δ_γ. To this end, we assume that there is a function φ continuous on $\overline{\Sigma}$. Hence

$$\langle \delta_{\gamma_n}, \varphi \rangle = \int_{\gamma_n} \varphi \,\mathrm{d}\gamma_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi\Big(F_n(t,0)\Big) |F_n'(t,0)|\,\mathrm{d}t$$

The sequence F_n satisfies (4) uniformly, and using the assumption $(\mathcal{H})$ we have that

$$|F_n'(\cdot,0)| \to |F'(\cdot,0)| \qquad \text{weakly in } L^2\left(-\frac{1}{2},\frac{1}{2}\right)$$

Since φ is continuous, it is uniformly continuous on $\overline{\Omega}_0$:

$$\varphi\Big(F_n(\cdot,0)\Big) \to \varphi\Big(F(\cdot,0)\Big) \text{ in } L^\infty\left(-\frac{1}{2},\frac{1}{2}\right)$$

Thus

$$\langle\delta_{\gamma_n},\varphi\rangle \to \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi\Big(F(t,0)\Big)|F'(t,0)|\,\mathrm{d}t = \langle\delta_\gamma,\varphi\rangle$$

The same result can be obtained for an arbitrary $\varphi \in H^1(\Sigma)$ since $C(\overline{\Sigma}) \cap H^1(\Sigma)$ is dense in $H^1(\Sigma)$. ■

Let us consider a sequence of admissible curves γ_n and an admissible curve γ such that δ_{γ_n} converges to δ_γ weakly in $\big(H^1(\Sigma)\big)'$. We denote by u_n and u solutions to $(\mathcal{P}_1)$ or $(\mathcal{P}_2)$ for the boundary data δ_{γ_n} and δ_γ, respectively. We are interested in the convergence $u_n \to u$.

Proposition 3. *Let* $\{\gamma_n\}, \gamma \in \mathcal{F}_L$ *be given, such that* $\delta_{\gamma_n} \to \delta_\gamma$ *weakly in* $\big(H^1(\Sigma)\big)'$. *Then*

$$u_n \to u \qquad \text{in } H^1(\Sigma) \text{ weakly and in } L^2(\Sigma) \text{ strongly}$$

Proof. The element u_n is the unique solution to the following variational problem:

$$a(u_n,v) = \mathcal{L}_n(v) \qquad \forall v \in H^1(\Sigma) \tag{13}$$

where

$$\mathcal{L}_n(v) = \int_{\Omega_1} u_1 v\,\mathrm{d}\sigma + \int_{\Omega_0} u_0 v\,\mathrm{d}\sigma + \int_{\gamma_n} v\,\mathrm{d}\gamma_n$$

From Proposition 1 it follows that $\|\mathcal{L}_n\|_* \le C$, where the constant C is independent of $n = 1,2,\ldots$. Since the bilinear form $a(\cdot,\cdot)$ is coercive by the inequality of Friedrichs-Poincaré, we obtain directly from the variational formulation that

$$\alpha\|u_n\|^2_{H^1(\Sigma)} \le a(u_n,u_n) = \mathcal{L}_n(u_n) \le C\|u_n\|_{H^1(\Sigma)}$$

and therefore the sequence u_n, $n = 1,2,\ldots$, is bounded in $H^1(\Sigma)$. There exists a subsequence of the sequence u_n, still denoted by u_n, such that

$$u_n \to u^* \qquad \text{weakly in } H^1(\Sigma) \text{ and strongly in } L^2(\Sigma)$$

(the strong convergence follows from the Rellich theorem). We show that $u^* = u$.

By the weak convergence of the sequence $\{u_n\}$ in $H^1(\Sigma)$, since the trace mapping is linear and continuous, we have the following convergence of the traces:

$$u_n \to u^* \qquad \text{in } L^2(\Omega_0) \text{ and in } L^2(\Omega_1)$$

Hence for any fixed test function $v \in H^1(\Sigma)$

$$a(u_n, v) \to a(u^*, v)$$

and with our assumptions

$$\int_{\gamma_n} v \, \mathrm{d}\gamma_n \to \int_{\gamma} v \, \mathrm{d}\gamma$$

whence

$$\mathcal{L}_n(v) \to \mathcal{L}(v)$$

We obtain

$$a(u^*, v) = \mathcal{L}(v)$$

and, since the solution to the problem $(\mathcal{P}_2(\gamma))$ is unique, it follows that $u^* = u$, which completes the proof. ■

Remark 5. In order to show that $u_n \to u$ strongly in $H^1(\Sigma)$, it is sufficient to have the following convergence:

$$\int_{\gamma_n} u_n \, \mathrm{d}\gamma_n \to \int_{\gamma} u \, \mathrm{d}\gamma \tag{14}$$

since, using the variational formulation of the problem $(\mathcal{P}_2)$, we obtain

$$\int_{\Sigma} |\nabla u_n|^2 \, \mathrm{d}x \to \int_{\Sigma} |\nabla u|^2 \, \mathrm{d}x \tag{15}$$

Using the above results, we are in a position to prove an existence result for the optimization problem under consideration. Assume that there is a given functional $J(\cdot)$ continuous with respect to $u = u(\gamma)$ in the norm topology of the space $L^2(\Sigma)$ or weakly lower semi-continuous on $H^1(\Sigma)$. Let us consider, as an example, the following cost functional:

$$J(\gamma) = \int_{\Sigma} \left(u(\gamma) - u_d\right)^2 \mathrm{d}x + \int_{\Sigma} |\nabla u(\gamma) - \nabla u_d|^2 \, \mathrm{d}x \tag{16}$$

Theorem 1. *There exists a solution to the minimization problem*

$$\inf_{\gamma \in \mathcal{F}_L} J(\gamma) \tag{17}$$

Proof. Let $\{\gamma_n\}$ denote a minimizing sequence. Then for its subsequence, still denoted by $\{\gamma_n\}$, we have

$$u(\gamma_n) \to u(\gamma) \qquad \text{weakly in } H^1(\Sigma) \tag{18}$$

Hence

$$\liminf J(\gamma_n) \geq J(\gamma)$$

which completes the proof of the theorem. ■

Let us present another formulation of the problem which provides a smooth solution. We denote by $D \subset \mathbb{R}^2 = \mathbb{C}$ the unit disk. We denote by $\mathfrak{O}$ the collection of holomorphic functions defined on D with values in Ω_0,

$$\Phi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \in \mathbb{C}, \qquad \text{the series converges in } D, \quad \Phi(z) \in \Omega_0 \quad .$$

The curve γ is defined by the following parametrization:

$$z(t) = \sum_{n=0}^{\infty} a_n t^n, \qquad t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

In fact, by the Stone-Weierstrass theorem, any curve located in Ω_0 can be approximated by the curve of this form.

Using the set $\mathfrak{O}$ we obtain the existence of the solution to our problem since if the sequence $\{\Phi_p\}_{p\in\mathbb{N}}$ is a minimizing sequence, by using the Montel theorem it follows, since Ω_0 is bounded and $\Phi_p(z) \in \Omega_0$, that there exists a subsequence which converges uniformly on any compact, along with all derivatives. In this case, the assumption $(\mathcal{H})$ is satisfied. In particular, Φ_p' converges to the limit Φ' on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, which implies that $\delta_{\gamma_p} \to \delta_\gamma$ weakly in $\left(H^1(\Sigma)\right)'$.

3. Generalized Solutions to the Domain Optimization Problem

We start with the classical definition of a solution $u \in W^{1,p}(\Sigma)$, $1 \leq p < \frac{3}{2}$, to the system $(\mathcal{P}_1)$ in the form

$$(\mathcal{P}_1(\mu)) \qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \\ -\dfrac{\partial u}{\partial n} = u - u_1 & \text{on } \Omega_1 \\ -\dfrac{\partial u}{\partial n} = u - u_0 - \mu & \text{on } \Omega_0 \end{cases}$$

where μ is a Radon measure supported on Ω_0.

We are going to prove the existence and uniqueness of the solution to $\mathcal{P}_1(\mu)$. First, we recall a Friedrich-type inequality related to our problem.

Lemma 1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded simply-connected domain with smooth boundary $\Gamma = \partial\Omega$, $\tilde{\gamma} \subset \Gamma$ a given set with $|\tilde{\gamma}| = \int_{\tilde{\gamma}} \mathrm{d}\Gamma(x) > 0$. Then there exists a constant $C = C(\Omega, \tilde{\gamma}, p)$ such that*

$$\|v\|_{L^p(\Omega)} \leq C \left[\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^p \mathrm{d}x + \int_{\tilde{\gamma}} |v|^p \, \mathrm{d}\Gamma(x)\right]^{\frac{1}{p}} \qquad \forall v \in W^{1,p}(\Omega)$$

Let $q > N$ be fixed. By the injection theorem of Sobolev, taking into account the variational formulation of $(\mathcal{P}'_\psi)$ and the definition of the norm by duality, we have

$$\|v\|_{L^\infty(\Omega)} \le C\|v\|_{W^{1,p}(\Omega)} \le C\left(\sum_{i=1}^{N} \|\xi_i\|_{L^q(\Omega)} + \|\xi_0\|_{L^q(\Gamma_0)}\right)$$

Furthermore,

$$\sum_{i=1}^{N} \int_\Omega \xi_i \frac{\partial u}{\partial x_i}\,\mathrm{d}x + \int_{\Gamma_0} \xi_0 u\,\mathrm{d}\Gamma(x) = -\int_\Omega u\Delta v\,\mathrm{d}x + \int_{\Gamma_0} \xi_0 u\,\mathrm{d}\Gamma(x)$$

$$= \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x + \int_{\Gamma_0} uv\,\mathrm{d}\Gamma(x) = \int_{\Gamma_0} \left(\frac{\partial u}{\partial n} + u\right) v\,\mathrm{d}\Gamma(x)$$

Hence, for all $\xi_0, \dots, \xi_N$,

$$\left|\sum_{i=1}^{N} \int_\Omega \xi_i \frac{\partial u}{\partial x_i}\,\mathrm{d}x + \int_{\Gamma_0} \xi_0 u\,\mathrm{d}\Gamma(x)\right| \le \left\|\frac{\partial u}{\partial n} + u\right\|_{L^1(\Gamma_0)} \|v\|_{L^\infty(\Gamma_0)}$$

$$\le C\left\|\frac{\partial u}{\partial n} + u\right\|_{L^1(\Gamma_0)} \left(\sum_{i=1}^{N} \|\xi_i\|_{L^q(\Omega)} + \|\xi_0\|_{L^q(\Gamma_0)}\right)$$

For p (the conjugate of q), $p < \frac{N}{N-1}$ since $q > N$, by the definition of the norm by duality, it follows that

$$\|u\|_{W^{1,p}(\Omega)} \le C\left\|\frac{\partial u}{\partial n} + u\right\|_{L^1(\Gamma_0)} = C\|\psi\|_{L^1(\Gamma_0)} = C\|\psi\|_{\mathcal{M}_b(\Gamma_0)}$$

Since the space $C(\Gamma_0)$ is dense in the space $\mathcal{M}_b(\Gamma_0)$, this completes the proof of the proposition. ∎

Furthermore, we have the following variational formulation

$$\int_\Omega \nabla u \cdot \nabla \xi\,\mathrm{d}x + \int_{\Gamma_0} u\xi\,\mathrm{d}\Gamma(x) = \int \xi\,\mathrm{d}\mu \qquad \forall \xi \in \mathcal{D}(\mathbb{R}^N) \tag{21}$$

For $N = 3$ the latter formulation remains valid, by density, for an arbitrary test function $\xi \in H^2(\Omega)$, since by the Sobolev imbedding theorem $H^2(\Omega) \subset C(\overline{\Omega})$ and the integral $\int \xi\,\mathrm{d}\mu$ is well-defined.

From Proposition 4 we obtain the following result.

Proposition 5. *Given a sequence $\{\mu_n\}$ of Radon measures supported on Γ_0, $\|\mu_n\|_{\mathcal{M}_b(\Gamma_0)} \le C$, there exists a subsequence, still denoted by $\{\mu_n\}$, and a Radon measure $\mu \in \mathcal{M}_b(\Gamma_0)$ such that*

$$\mu_n \to \mu \quad \text{weakly-}(*) \text{ in } \mathcal{M}_b(\Gamma_0)$$

$$u_n \to u \quad \text{weakly-}(*) \text{ in } W^{1,p}(\Omega)$$

$$u_n \to u \quad \text{in } L^p(\Omega), \quad 1 < p < \frac{N}{N-1}$$

where u_n denotes a solution to the problem $\mathcal{P}_1(\mu_n)$.

The proof of Proposition 5 is omitted here. It uses the Banach-Alaoglu theorem and the same argument as in the proof of Proposition 3.

We shall consider the admissible measures of the form

$$\mu = \psi\delta_\gamma$$

with some regularity properties imposed on the density $\psi \in L^\infty(\gamma)$ and on the curve $\gamma = \operatorname{supp}\mu$. The reason to consider such a class is that it is easy to construct a sequence γ_n such that the length of the curve γ_n is uniformly bounded and $\delta_{\gamma_n} \to \psi\delta_\gamma$ weakly-(*).

Example 1. Let us consider the family of curves, $n = 1, 2, \ldots,$

$$\gamma_n = \Big\{x_n(t), y_n(t)\Big\}, \qquad t \in [0,1]$$

where

$$x_n(t) = t, \quad y_n(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\ \frac{1}{n}\sin 3n\pi x, & t \in \left(\frac{1}{3}, \frac{2}{3}\right) \end{cases}$$

It can be shown that

$$\delta_{\gamma_n} \to \psi\delta_\gamma \quad \text{weakly-}(*)$$

where

$$\gamma = \begin{cases} x = x(t) = t, & t \in [0,1] \\ y = y(t) = 0, & t \in [0,1] \end{cases}$$

and

$$\psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\ L, & t \in \left(\frac{1}{3}, \frac{2}{3}\right) \end{cases}$$

with $L = \int_0^1 \sqrt{1 + \cos^2 \pi t}\, dt > 1$. ■

Let us recall that for a sequence of Radon measures $\{\mu_n\}$ such that

$$\mu_n \to \mu \quad \text{weakly-}(*)$$

we have in general the only information on the support of the limit

$$\operatorname{supp}\mu \subset \limsup_{n\to\infty} \operatorname{supp}\mu_n$$

where the $\limsup$ is taken in the sense of Kuratowski.

We choose $\mu = \psi\delta_\gamma$, since

$$\int \varphi\, \mathrm{d}\mu = \int_\gamma \varphi(\sigma)\psi(\sigma)\, \mathrm{d}\sigma$$

It follows that

$$\|\mu\|_{\mathcal{M}_b} \le L(\gamma)\|\psi\|_{L^\infty} , \quad \text{where } L(\gamma) = \int_\gamma d\sigma$$

i.e. $L(\gamma)$ denotes the length of the curve γ.

Now let $\alpha > \frac{1}{2}$ and $M > 0$ be given constants. We introduce the set of admissible Radon measures of the following form:

$$\mathfrak{M}_{\alpha,M} = \Big\{\mu = \psi\delta_\gamma | \, \gamma = \{x(\cdot), y(\cdot)\Big\} \in \Big[W^{1,2\alpha}(0,1)\Big]^2$$

$$\int_0^1 \Big(x'^2(t) + y'^2(t)\Big)^\alpha \, dt \le M, \int_0^1 \Big|\psi\Big(x(t), y(t)\Big)\Big| \sqrt{x'^2(t) + y'^2(t)} \, dt \le M\}$$

Theorem 2. *Given a sequence $\mu_n \in \mathfrak{M}_{\alpha,M}$, there exists a subsequence, still denoted by μ_n, a mesure $\mu \in \mathcal{M}_b(\Omega_0)$ and a curve γ such that*

$$\mu_n \to \mu \quad \textit{weakly-}(*) \textit{ in } \mathcal{M}_b(\Omega_0)$$

where $\gamma = \operatorname{supp} \mu = \{x(\cdot), y(\cdot)\} \in \left[W^{1,2\alpha}(0,1)\right]^2$. Furthermore, if the following conditions are satisfied:

$$\|\psi_n\|_{L^p(\gamma_n)} \le C$$

for some $p > 1$, with $C \le M^{\frac{p+1}{2p}}$ and

$$\sqrt{x'^2(t) + y'^2(t)} \ge \beta > 0 \quad \textit{for } t \in (0,1) \textit{ a.e.}$$

then there exists a function $\psi \in L^1(\gamma)$ such that

$$\mu = \psi\delta_\gamma$$

Proof. First, since $\{x_n\}$ and $\{y_n\}$ are bounded sequences in the Sobolev space $W^{1,2\alpha}(0,1)$ which is a reflexive Banach space compactly imbedded in the space of continuous functions by the Rellich theorem, it follows that there exist elements $x, y \in W^{1,2\alpha}(0,1)$ such that for subsequences, still denoted by $\{x_n\}$ and $\{y_n\}$,

$$x_n \to x \quad \text{uniformly in } [0,1] \text{ and weakly in } L^{2\alpha}(0,1) \tag{22}$$

$$y_n \to y \quad \text{uniformly in } [0,1] \text{ and weakly in } L^{2\alpha}(0,1) \tag{23}$$

By lower semicontinuity of the norm we obtain

$$\int_0^1 \Big(x'^2(t) + y'^2(t)\Big)^\alpha \, dt \le \liminf \int_0^1 \Big(x_n'^2(t) + y_n'^2(t)\Big)^\alpha \, dt \le M$$

Thus the curve $\gamma = \{x, y\}$ is the admissible support for the measure we are going to construct.

On the other hand, the sequence $\{\psi_n\}$ is bounded in $L^1(\gamma_n)$,

$$\int_0^1 \left|\psi_n\Big(x_n(t), y_n(t)\Big)\right| \sqrt{x_n'^2(t) + y_n'^2(t)}\, \mathrm{d}t \le M$$

i.e. the function $t \mapsto \psi_n(x_n(t), y_n(t))\sqrt{x_n'^2(t) + y_n'^2(t)}$ is bounded in $L^1(0,1)$. We denote by μ_n the measure defined in the following way:

$$\int v\, \mathrm{d}\mu_n = \int_0^1 v\Big(x_n(t), y_n(t)\Big)\psi_n\Big(x_n(t), y_n(t)\Big)\sqrt{x_n'^2(t) + y_n'^2(t)}\, \mathrm{d}t$$

for any $v \in C(\overline{\Omega}_0)$. Therefore, there exists a subsequence, still denoted by $\{\mu_n\}$, such that

$$\int v\, \mathrm{d}\mu_n \to \int v\, \mathrm{d}\mu \quad \text{for any } v \in C(\overline{\Omega}_0)$$

where the limit mesure satisfies

$$\operatorname{supp} \mu = \gamma$$

since

$$v\Big(x_n(t), y_n(t)\Big) \to v\Big(x(t), y(t)\Big) \quad \text{uniformly on } [0,1]$$

Let us show the second part of the theorem. To this end, we observe that by our assumptions the sequences $\{\psi_n(\cdot)\}$ and $\left\{\sqrt{x_n'^2(\cdot) + y_n'^2(\cdot)}\right\}$ are bounded in $L^p(\gamma_n)$ and $L^\alpha(0,1)$, respectively, where $p > 1$ and $\alpha > \frac{1}{2}$.

Set $\beta = 2\alpha p/(p + 2\alpha - 1)$. Since $p > 1$ and $2\alpha > 1$, it follows that $\beta > 1$ and we verify that the sequence $\left\{\psi_n(x_n, y_n)\sqrt{x_n'^2 + y_n'^2}\right\}$ is bounded in $L^\beta(0,1)$. Let

$$m = \frac{p + 2\alpha - 1}{2\alpha}, \quad m^* = \frac{p + 2\alpha - 1}{p - 1}, \quad \frac{1}{m} + \frac{1}{m^*} = 1$$

From the Hölder inequality it follows that

$$\begin{aligned}
&\int_0^1 \left|\psi_n\Big(x_n(t), y_n(t)\Big)\right|^\beta \Big(x_n'^2(t) + y_n'^2(t)\Big)^{\frac{\beta}{2}} \mathrm{d}t \\
&\le \left(\int_0^1 \left|\psi_n\Big(x_n(t), y_n(t)\Big)\right|^{\beta m} \Big(x_n'^2(t) + y_n'^2(t)\Big)^{\frac{\alpha m}{p+2\alpha-1}} \mathrm{d}t\right)^{\frac{1}{m}} \\
&\quad \times \left(\int_0^1 \Big(x_n'^2(t) + y_n'^2(t)\Big)^{\frac{\alpha p - \alpha}{p+2\alpha-1}\cdot m^*} \mathrm{d}t\right)^{\frac{1}{m^*}} \\
&= \left(\int_0^1 \left|\psi_n(x_n, y_n)\right|^p \sqrt{x_n'^2 + y_n'^2}\, \mathrm{d}t\right)^{\frac{1}{m}} \left(\int_0^1 \Big(x_n'^2 + y_n'^2\Big)^\alpha \mathrm{d}t\right)^{\frac{1}{m^*}} \\
&= \left\|\psi_n\right\|_{L^p(\gamma_n)}^{\frac{p}{m}} \left\|x_n'^2 + y_n'^2\right\|_{L^\alpha(0,1)}^{\frac{\alpha}{m^*}} \le C^{\frac{p}{m}} M^{\frac{\alpha}{m^*}}
\end{aligned}$$

The last inequality follows from our assumptions. Hence

$$\left\|\psi_n(x_n, y_n)\sqrt{x_n'^{\,2} + y_n'^{\,2}}\right\|_{L^\beta(0,1)} \le C^{\frac{p}{m\beta}} M^{\frac{\alpha}{m^*\beta}} = CM^{\frac{p-1}{2p}} \le M$$

Therefore, there exists an element $\varphi \in L^\beta(0,1)$ such that

$$\psi_n(x_n, y_n)\sqrt{x_n'^{\,2} + y_n'^{\,2}} \to \varphi \quad \text{weakly in} \quad L^\beta(0,1)$$

$$\|\varphi\|_{L^\beta(0,1)} \le \liminf \left\|\psi_n(x_n, y_n)\sqrt{x_n'^{\,2} + y_n'^{\,2}}\right\|_{L^\beta(0,1)} \le M$$

and

$$\int v \, \mathrm{d}\mu_n = \int_0^1 v\Big(x_n(t), y_n(t)\Big)\psi_n\Big(x_n(t), y_n(t)\Big)\sqrt{x_n'^{\,2}(t) + y_n'^{\,2}(t)} \, \mathrm{d}t$$

$$\to \int v \, \mathrm{d}\mu = \int_0^1 v\Big(x(t), y(t)\Big)\varphi(t) \, \mathrm{d}t$$

so that we define

$$\psi\Big(x(t), y(t)\Big) = \frac{\varphi(t)}{\sqrt{x'^2(t) + y'^2(t)}}$$

with $\psi \in L^1(\gamma)$ and

$$\|\psi\|_{L^1(\gamma)} = \int_0^1 \left|\psi\Big(x(t), y(t)\Big)\right| \sqrt{x'^2(t) + y'^2(t)} \, \mathrm{d}t = \int_0^1 |\varphi(t)| \, \mathrm{d}t \le \|\varphi\|_{L^\beta(0,1)} \le M$$

Thus

$$\mu = \psi\delta_\gamma \in \mathcal{M}_{\alpha,M}$$

■

Conclusion. According to Theorem 2, there exists a solution to the minimization problem

$$\min_{\mu \in \mathcal{M}_{\alpha,M}} J(\mu)$$

for a class of cost functionals, e.g. $J(\mu) = \int_\Sigma (u - \overline{u})^2 \, \mathrm{d}x$, u being a solution to $\mathcal{P}_1(\mu)$.

4. Optimality Conditions

We start with auxiliary results on differentiability of the shape functional

$$\gamma \to \int_\gamma \mathcal{G}\,\mathrm{d}\gamma$$

We assume that the function $\mathcal{G} \in L^1(\gamma)$ may depend on the curve γ. We use the same approach as in the case of a thin shell, where we consider a curve γ on the manifold. Here Ω_0 is flat set.

Let a sufficiently smooth mapping $\mathcal{F}_s \;:\; \mathbb{R}^3 \mapsto \mathbb{R}^3$ be given, where $s \in [0,\delta)$ is a parameter, such that $F_s = \mathcal{F}_{s|Q}$ for any $s \in [0,\delta)$ satisfies the assumptions of Definition 1, i.e.

$$F_s(Q) = \mathcal{O}, \qquad F_s(I) = \gamma$$
$$\|F_s\|_{W^{1,\infty}(Q)} \le L_1, \qquad \|F_s^{-1}\|_{W^{1,\infty}(\mathcal{O})} \le L_2$$

Given a parametrization $\{x_s(t), y_s(t)\}$, $t \in [0,1]$, of the curve γ_s, we write

$$j(s) = \int_{\gamma_s} \mathcal{G}_s\,\mathrm{d}\gamma_s = \int_0^1 \mathcal{G}_s\Big(x_s(t), y_s(t)\Big)\sqrt{x_s'^2(t) + y_s'^2(t)}\,\mathrm{d}t$$

The derivative takes the form

$$j'(s) = \int_0^1 \left\{ \frac{\partial \mathcal{G}_s}{\partial s} + \nabla\mathcal{G}_s\left(x_s(t), y_s(t)\right)\cdot \xi_s(t)\right\} \sqrt{x_s'^2(t) + y_s'^2(t)}\,\mathrm{d}t$$
$$+ \int_0^1 \mathcal{G}_s\Big(x_s(t), y_s(t)\Big)\tau_s(t)\cdot\frac{\mathrm{d}\xi_s}{\mathrm{d}t}(t)\,\mathrm{d}t$$

where $\tau_s(t) = \frac{\left(x_s'(t), y_s'(t)\right)}{\sqrt{x_s'^2(t)+y_s'^2(t)}}$ is the unit vector tangent to γ and $\xi_s(t) = \frac{\mathrm{d}}{\mathrm{d}s}\left(x_s(t), y_s(t)\right)$.

Under regularity assumptions, after integration by parts, the latter integral can be rewritten in the form

$$\int_0^1 \mathcal{G}_s\Big(x_s(t), y_s(t)\Big)\tau_s(t)\cdot\frac{\mathrm{d}\xi_s}{\mathrm{d}t}(t)\,\mathrm{d}t$$
$$= -\int_0^1 \Big\{\nabla\mathcal{G}_s\Big(x_s(t), y_s(t)\Big)\cdot\Big(x_s(t), y_s(t)\Big)\tau_s(t)\cdot\xi_s(t)$$
$$+ \mathcal{G}_s\Big(x_s(t), y_s(t)\Big)\frac{\mathrm{d}\tau_s}{\mathrm{d}t}(t)\cdot\xi_s(t)\Big\}\,\mathrm{d}t$$
$$+ \mathcal{G}_s\Big(x_s(1), y_s(1)\Big)\tau_s(1)\cdot\xi_s(1) - \mathcal{G}_s\Big(x_s(0), y_s(0)\Big)\tau_s(0)\cdot\xi_s(0)$$

On the other hand, we can use the material-derivative method to obtain the same derivative $j'(s)$. Namely, we introduce the vector field

$$V(s,x,y,z) = \left(\frac{\partial \mathcal{F}_s}{\partial s} \circ \mathcal{F}_s^{-1}\right)(x,y,z)$$

and assume that its support is included in a small neighbourhood $\mathcal{O}(\gamma)$ of the curve γ in $\mathbb{R}^3$. Furthermore, we assume that for $(x,y,z) \in \mathcal{O}(\gamma)$ and a sufficiently small $z \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, the field is of the form

$$V(s,x,y,z) = \begin{pmatrix} V_1(s,x,y) \\ V_1(s,x,y) \\ 0 \end{pmatrix} = V(s,x,y,0)$$

The shape functional we consider takes the form

$$J(\gamma) = \int_\gamma \mathcal{G}\,\mathrm{d}\gamma$$

With the vector field V we associate the mapping

$$T_s(V)\ :\ \mathbb{R}^3 \mapsto \mathbb{R}^3$$

In particular, under our assumptions on the support of the field V, $\operatorname{supp} V \subset \mathcal{O}(\gamma)$, it follows that $T_s(V) \equiv I$ on $\mathbb{R}^3 \setminus \mathcal{O}(\gamma)$, where I denotes the identity mapping.

Let us define the Eulerian semiderivative

$$\mathrm{d}J(\gamma; V) = \lim_{s \downarrow 0} \frac{1}{s}\Big(J(T_s(\gamma)) - J(\gamma)\Big)$$

For $\gamma_s = T_s(\gamma)$, $s \in [0, \delta)$ it follows that

$$j'(0^+) = \mathrm{d}J(\gamma; V)$$

and therefore, by application of the structure theorem for the shape gradient, we obtain

$$\begin{aligned} \mathrm{d}J(\gamma; V) = &\int_0^1 \left\{ \frac{\partial \mathcal{G}_s}{\partial s}\Big|_{s=0} + \nabla \mathcal{G}\Big(x(t), y(t)\Big) \cdot \xi(t) \right\} \sqrt{x'^2(t) + y'^2(t)}\,\mathrm{d}t \\ &+ \int_0^1 \mathcal{G}\Big(x(t), y(t)\Big) \frac{\mathrm{d}\tau}{\mathrm{d}t}(t) \cdot \xi(t)\,\mathrm{d}t \\ &+ \mathcal{G}\Big(x(1), y(1)\Big)\tau(1^-) \cdot \xi(1) - \mathcal{G}\Big(x(0), y(0)\Big)\tau(0^+) \cdot \xi(0) \end{aligned}$$

since $V(s, x(t), y(t), 0) = (\xi_s(t), 0)$ for $t \in [0,1]$, and the vector $\tau(t) \in \Omega_0$, $t \in (0,1)$, is tangent to γ. If $\nu(t) \in \Omega_0$, $t \in (0,1)$, denotes the normal vector field on γ, the equivalent form of the first integral reads

$$\int_0^1 \left\{ \frac{\partial \mathcal{G}_s}{\partial s}\bigg|_{s=0} + \nabla\mathcal{G}\Big(x(t),y(t)\Big)\cdot\xi(t)\right\}\sqrt{x'^2(t)+y'^2(t)}\,\mathrm{d}t$$

$$= \int_0^1 \left\{ \frac{\partial \mathcal{G}_s}{\partial s}\bigg|_{s=0} + \left[\nabla\mathcal{G}\Big(x(t),y(t)\Big)\cdot\nu(t)\right]\xi(t)\cdot\nu(t)\right\}\sqrt{x'^2(t)+y'^2(t)}\,\mathrm{d}t$$

since, by the structure theorem, the integral part of $\mathrm{d}J(\gamma;V)$ depends only on the normal component $V(0,x(t),y(t),0)\cdot n = \xi(t)\cdot\nu(t),\ t\in(0,1)$, of the field $V(0,x(t),y(t),0)$. We write

$$\int_\gamma \dot{\mathcal{G}}\,\mathrm{d}\gamma = \int_0^1 \left\{ \frac{\partial \mathcal{G}_s}{\partial s}\bigg|_{s=0} + \nabla\mathcal{G}\Big(x(t),y(t)\Big)\cdot\xi(t)\right\}\sqrt{x'^2(t)+y'^2(t)}\,\mathrm{d}t$$

$$\int_\gamma \mathcal{G}\tau'\cdot V\,\mathrm{d}\gamma = \int_0^1 \mathcal{G}\Big(x(t),y(t)\Big)\frac{\mathrm{d}\tau}{\mathrm{d}t}(t)\cdot\xi(t)\,\mathrm{d}t$$

$$\Big(x(1),y(1)\Big) = (x_1,y_1), \qquad \Big(x(0),y(0)\Big) = (x_0,y_0)$$

Proposition 6. *The shape functional $J(\gamma)=\int_\gamma \mathcal{G}\,\mathrm{d}\gamma$ is shape-differentiable, and the Eulerian semiderivative takes the following form:*

$$\mathrm{d}J(\gamma;V) = \int_\gamma \dot{\mathcal{G}}\,\mathrm{d}\gamma + \int_\gamma \mathcal{G}\tau'\cdot V\,\mathrm{d}\gamma + \mathcal{G}(x_1,y_1)\tau(x_1^-,y_1^-)\cdot V(0,x_1,y_1,0)$$
$$- \mathcal{G}(x_0,y_0)\tau(x_0^+,y_0^+)\cdot V(0,x_0,y_0,0)$$

where $\dot{\mathcal{G}}$ denotes the material derivative of $\mathcal{G}$ in the direction of the vector field V.

Remark 6. In particular, for $\mathcal{G}=1$ and $J(\gamma)=|\gamma|=\int_\gamma \mathrm{d}\gamma$

$$\mathrm{d}J(\gamma;V) = \int_\gamma \tau\cdot DV\cdot\tau\,\mathrm{d}\gamma$$

We have the property

$$|\gamma_s| = |T_s(\gamma)| = \int_{\gamma_s}\mathrm{d}\gamma_s = \int_\gamma \mathrm{d}\gamma, \quad \gamma_s = T_s(\gamma)$$

provided that the vector field V satisfies the equation

$$\int_\gamma \tau'\cdot V\,\mathrm{d}\gamma + \tau(x_1^-,y_1^-)\cdot V(0,x_1,y_1,0) - \tau(x_0^+,y_0^+)\cdot V(0,x_0,y_0,0) = 0$$

■

Now, we are in a position to obtain the shape differentiability of solutions to the problem $\mathcal{P}_1(\gamma)$.

We write $\Sigma_s = T_s(\Sigma)$ and let $u_s \in W^{1,p}(\Sigma_s)$ stand for the unique solution to the following integral identity:

$$\int_{\Sigma_s} \nabla u_s \cdot \nabla\varphi \, \mathrm{d}\Sigma_s + \int_{\Omega_0^s} u_s\varphi \, \mathrm{d}\sigma_s + \int_{\Omega_1^s} u_s\varphi \, \mathrm{d}\sigma_s = \int_{\Omega_0^s} u_0\varphi \, \mathrm{d}\sigma_s + \int_{\Omega_1^s} u_1\varphi \, \mathrm{d}\sigma_s + \int_{\gamma_s} \varphi \, \mathrm{d}\gamma_s$$

for all $\varphi \in W^{1,q}(\Sigma_s)$, where $\Omega_i^s = T_s(\Omega_i)$, $i = 0, 1$, $\gamma_s = T_s(\gamma)$.

The integral identity is transferred to a fixed domain Σ, so we set $u^s = u_s \circ T_s \in W^{1,p}(\Sigma)$, $\varphi = v \circ T_s^{-1}$, and by the standard change of variables it follows that u^s is the unique solution of the following inegral identity:

$$\int_\Sigma \Big\langle A(s) \cdot \nabla u_s, \nabla v \Big\rangle_{\mathbb{R}^3} \mathrm{d}\Sigma + \int_{\Omega_0} u^s v\omega(s) \, \mathrm{d}\sigma + \int_{\Omega_1} u^s v\omega(s) \, \mathrm{d}\sigma$$

$$= \int_{\Omega_0} u_0^s v\omega(s) \, \mathrm{d}\sigma + \int_{\Omega_1} u_1^s v\omega(s) \, \mathrm{d}\sigma + \int_\gamma v\rho(s) \, \mathrm{d}\gamma$$

for all $v \in W^{1,q}(\Sigma)$, where the matrix $A(s)$ and the boundary terms $\omega(s), \rho(s)$ are given, sufficiently smooth functions of space variables, and $s \in [0, \delta)$,

$$A(s) = \det(DT_s)DT_s^{-1} \cdot {}^*DT_s^{-1}$$

$$\omega(s) = \|\det(DT_s)^* DT_s^{-1} \cdot n\|_{\mathbb{R}^3}$$

$$\rho(s) = \left(\frac{x_s'^2(t) + y_s'^2(t)}{x'^2(t) + y'^2(t)} \right)^{\frac{1}{2}}, \quad \Big(x(t), y(t)\Big) \in \gamma, \ \gamma_s = T_s(\gamma), \ t \in (0, 1)$$

By application of the implicit-function theorem for solutions to the last integral identity we obtain the existence of the weak material derivative in $W^{1,p}(\Sigma)$, $1 < p < \frac{3}{2}$,

$$\dot{u} = \lim_{s \downarrow 0} \frac{1}{s}(u^s - u)$$

The material derivative $\dot{u} \in W^{1,p}(\Sigma)$ satisfies the following integral identity:

$$\int_\Sigma \nabla\dot{u} \cdot \nabla v \, \mathrm{d}\Sigma + \int_\Sigma \Big\langle A'(0) \cdot \nabla u, \nabla v \Big\rangle_{\mathbb{R}^3} \mathrm{d}\Sigma + \int_{\Omega_0} \dot{u} v \, \mathrm{d}\sigma + \int_{\Omega_0} uv\omega'(0) \, \mathrm{d}\sigma$$

$$+ \int_{\Omega_1} \dot{u} v \, \mathrm{d}\sigma + \int_{\Omega_1} uv\omega'(0) \, \mathrm{d}\sigma$$

$$= \int_{\Omega_0} \Big(\dot{u}_0 + u_0\omega'(0)\Big) v \, \mathrm{d}\sigma + \int_{\Omega_1} \Big(\dot{u}_1 + u_1\omega'(0)\Big) v \, \mathrm{d}\sigma + \int_\gamma v\rho'(0) \, \mathrm{d}\gamma$$

where

$$A'(0) = \mathrm{div}V(0)I - DV(0) - {}^*DV(0)$$

$$\omega'(0) = \mathrm{div}V(0) - \langle DV(0) \cdot n, n\rangle_{\mathbb{R}^3}$$

$$\rho'(0) = \tau \cdot DV(0) \cdot \tau$$

Finally, the weak shape derivative $u' = \dot{u} - \nabla u \cdot V$ in $L^p(\Sigma)$, $1 < p < \frac{3}{2}$, satisfies the following integral identity:

$$-\int_\Sigma u' \Delta v \,\mathrm{d}x = -\int_\gamma v\tau' \cdot V \,\mathrm{d}\gamma + v(x_1, y_1)\tau(x_1^-, y_1^-) \cdot V(0, x_1, y_1, 0)$$
$$- v(x_0, y_0)\tau(x_0^+, y_0^+) \cdot V(0, x_0, y_0, 0)$$

for all test functions $v \in W^{2,q}(\Sigma)$, $\frac{\partial v}{\partial n} = 0$ on Γ, $\frac{\partial v}{\partial n} + v = 0$ on $\Omega_0 \cup \Omega_1$.

Theorem 3. *A solution to the minimization problem*

$$\inf_{\gamma \in \mathcal{F}_L} J(\gamma)$$

satisfies the first-order necessary optimality conditions

$$\mathrm{d}J(\gamma; V) = 0$$

for all admissible vector fields V, where

$$\mathrm{d}J(\gamma; V) = 2\int_\Sigma \Big(u(\gamma) - u_d\Big)\dot{u}\,\mathrm{d}\Sigma + 2\int_\Sigma \Big\langle \nabla u(\gamma) - \nabla u_d, \nabla \dot{u}\Big\rangle_{\mathbb{R}^3} \mathrm{d}\Sigma$$
$$- 2\int_\Sigma \Big\langle {}^*DV \cdot \nabla\Big(u(\gamma) - u_d\Big), \nabla\Big(u(\gamma) - u_d\Big)\Big\rangle_{\mathbb{R}^3} \mathrm{d}\Sigma$$
$$+ \int_\Sigma \left(\Big|\nabla\Big(u(\gamma) - u_d\Big)\Big|^2 + \Big|u(\gamma) - u_d\Big|^2\right) \mathrm{div}V \,\mathrm{d}\Sigma$$

Remark 7. For any vector field V such that

$$V(0) \cdot \nu = 0 \text{ on } \gamma, \qquad V(0, A) = V(0, B) = 0$$

it follows that $\mathrm{d}J(\gamma; V) = 0$. Therefore we obtain the following Green formula for such fields:

$$0 = 2\int_\Sigma \Big(u(\gamma) - u_d\Big)\nabla u \cdot V(0)\,\mathrm{d}\Sigma$$
$$+ 2\int_\Sigma \Big\langle \nabla u(\gamma) - \nabla u_d, \nabla(\nabla u \cdot V(0))\Big\rangle_{\mathbb{R}^3} \mathrm{d}\Sigma$$
$$- 2\int_\Sigma \Big\langle {}^*DV \cdot \nabla\Big(u(\gamma) - u_d\Big), \nabla\Big(u(\gamma) - u_d\Big)\Big\rangle_{\mathbb{R}^3} \mathrm{d}\Sigma$$
$$+ \int_\Sigma \left(\Big|\nabla\Big(u(\gamma) - u_d\Big)\Big|^2 + \Big|u(\gamma) - u_d\Big|^2\right) \mathrm{div}V \,\mathrm{d}\Sigma$$

Remark 8. In the particular case of the cost functional $I(\gamma) = \int_{\Omega_1} \Big(u(\gamma) - u_d\Big)^2 \mathrm{d}\Omega$, it follows that

$$\mathrm{d}I(\gamma; V) = 2\int_{\Omega_1} \Big(u(\gamma) - u_d\Big)u'(\gamma; V)\,\mathrm{d}\Omega$$

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