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AN INDIRECT MODEL REFERENCE ADAPTIVE CONTROL ALGORITHM BASED ON MULTIDETECTED-OUTPUT CONTROLLERS

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The use of sampled-data multidetected-output controllers for model reference adaptive control of linear systems with unknown parameters is investigated. Multidetected-output controllers contain a sampling mechanism in which the system output is detected many times over one period. Such a control allows us to assign an arbitrary discrete-time transfer function to the sampled closed-loop system and does not make any assumptions on the plant but controllability and observability. An indirect adaptive control scheme based on these sampled-data controllers is proposed, which estimates the controller parameters on-line. By using the proposed adaptive algorithm, the model reference adaptive control problem is reduced to the determination of a fictitious static state feedback controller, due to the merits of dynamic multidetected-output controllers. The known techniques usually resort to the direct computation of dynamic controllers. The controller determination reduces to the simple problem of solving a linear algebraic system of equations whereas in known techniques a matrix polynomial Diophantine equation is usually needed to be solved. Moreover, persistent excitation of the continuous-time plant is provided without making any special richness assumption on the reference signal.

1. Introduction

In the last 20 years, much research has been reported on the use of digital controllers in controlling continuous-time linear systems. In (Chammas and Leondes, 1978a; 1978b; 1978c; 1979a; 1979b), a certain type of periodically varying gain controllers is proposed. This type of controllers detects all the plant outputs once in the sampling period T_0 and changes all the feedback gains N times in T_0. In (Araki and Hagiwara, 1986), the multirate-input controllers (MRIC's) are introduced. MRIC's differ from the controllers proposed in (Chammas and Leondes, 1979a) in that they change the i-th plant input N_i times in T_0 with uniform sampling periods. In (Greshak and Vergese, 1982; Khargonekar *et al.*, 1985) another type of periodically varying controllers is proposed, which is different from the controllers reported in (Araki and Hagiwara, 1986; Chammas and Leondes, 1979a) in that the plant output

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is detected N times in T_0. The intersample-data controller is proposed in (Mita *et al.*, 1987), where instead of changing inputs frequently, a specific set of intersample output data is used for control. Multirate-output controllers (MROC's) are introduced in (Hagiwara and Araki, 1988; Hagiwara *et al.*, 1990). Such a type of controllers changes the plant inputs once in T_0 and detects the i-th plant output N_i times in T_0 with uniform sampling periods. Finally, the use of generalized sampled-data hold functions (GSHF) in controlling linear systems has been investigated in (Kabamba, 1987). The difference between the classes of digital controllers mentioned above and GSHF comes from the fact that in the latter case not only the controller gain but also the hold function involved is needed to be designed. A special class of GSHF are the multirate GSHF, first proposed in (Arvanitis, 1994), which incorporate some of the features of MRIC's. All the foregoing types of digital controllers have successfully been applied in solving many important control problems, such as pole assignment (Al-Rahmani and Franklin, 1989; Araki and Hagiwara, 1986; Chammas and Leondes, 1979a; Greshak and Vergese, 1982; Hagiwara and Araki, 1988; Hagiwara *et al.*, 1990; Kabamba, 1987), optimal control (Chammas and Leondes, 1979b; 1978c; Hagiwara and Araki, 1988; Mita *et al.*, 1987), exact model matching (Arvanitis and Paraskevopoulos, 1993; Kabamba, 1987; Paraskevopoulos and Arvanitis, 1994), decoupling (Kabamba, 1987), strong stabilization (Hagiwara *et al.*, 1990; Mita *et al.*, 1987), simultaneous stabilization (Kabamba and Yang, 1991), loop transfer recovery (Hagiwara *et al.*, 1990), noise rejection (Kabamba, 1987), robust controller synthesis (Kabamba, 1987; Khargonekar *et al.*, 1985), H^{∞}-control (Arvanitis and Paraskevopoulos, 1994; 1995b), adaptive stabilization (Ortega *et al.*, 1988), model reference adaptive control (Arvanitis, 1994; Arvanitis and Paraskevopoulos, 1995a; Ortega and Kreisselmeier, 1990), adaptive decoupling control (Arvanitis, 1995; 1996a; 1996b), etc. Some practical issues concerning the above-mentioned types of digital controllers are investigated in (Er and Anderson, 1991; Hagiwara and Araki, 1988; Hagiwara *et al.*, 1990).

Based on the previous research, it is well-recognized that digital control provides various advantages over conventional time-invariant feedback controls, such as the classical state feedback, dynamic compensation or state observers. The main of them are:

(a) Digital controllers designed along the lines reported in (Arvanitis, 1994; Araki and Hagiwara, 1986; Chammas and Leondes, 1978a; 1978b; 1978c; 1979a; 1979b; Greshak and Vergese, 1982; Hagiwara and Araki, 1988; Hagiwara *et al.*, 1990; Kabamba, 1987; Khargonekar *et al.*, 1985; Mita *et al.*, 1987) can be used in the cases where the state vector is not accessible for measurements and consequently for feedback, providing almost the same (and in some cases exactly the same) ability to adjust to the desired characteristics of the closed-loop system.

(b) As a consequence, they can be considered as an alternative to dynamic compensators and particularly to state observers. They provide more design freedom than the state observers while usually they do not introduce exogeneous dynamics (additional state variables) in the control loop, as it happens in the case of observers. Even in the case of MROC's, the exogeneous dynamics introduced is smaller than the dynamics introduced by an observer.

(c) They are always stable controllers whereas estimator-based controllers may become unstable when considered as dynamical systems. Even in the case of MROC's, it is possible to choose appropriately the transition matrices of the controllers themselves, thus providing that they are stable controllers.

(d) They have the ability to assure satisfactory robustness to the closed-loop system.

(e) They do not require many on-line computations. The computational complexity involved in the design procedure is almost the same as that involved in conventional time-invariant controllers.

(f) Finally, in digital control much more complex logic in control actions can be implemented by making use of the recent advances in computer technology.

As mentioned above, in several recent papers (Arvanitis, 1994; Arvanitis and Paraskevopoulos, 1995a; Ortega and Kreisselmeier, 1990), digital controllers and especially GSHF and MRIC's, have been used succesfully in the study of model reference adaptive control problem (MRAC). The MRAC is one of the most attractive adaptive schemes proposed in the literature concerning the area of adaptive control of dynamical systems with unknown parameters. Its basic idea, originaly proposed in (Whittaker *et al.*, 1958), is to force the system under control to behave like a given reference model. Many important contributions have been made and a great number of papers reported on the subject, wherein several techniques are used to treat the problem, see e.g. (Arvanitis, 1994; Arvanitis and Paraskevopoulos, 1995a; Goodwin and Chan, 1983; Goodwin and Sin, 1984; Goodwin *et al.*, 1980; 1986; Kreisselmeier and Anderson, 1986; Kreisselmeier and Narendra, 1982; Landau, 1974; 1979; Monopoli, 1974; Ortega and Kreisselmeier, 1990; Sastry, 1984; Sastry and Bodson, 1989) and the references therein.

In the present paper, the model reference adaptive control problem for linear time-invariant systems is treated using an approach based on the design of *multidetected-output controllers* (MDOC's). MDOC's can be considered as a special case of multirate-output controllers, originally proposed by Hagiwara and Araki (1988), in order to realize equivalently a stable state feedback controller in the case where the state vector cannot be available for feedback. To the best of our knowledge, there are no results in the literature concerning the use of this kind of sampled-data controllers in order to achieve model reference adaptive control for linear systems. The only available, partially relevant results, are reported in (Arvanitis, 1994; Arvanitis and Paraskevopoulos, 1995a; Ortega and Kreisselmeier, 1990), where the GSHF and MRIC approaches have been extended to the model reference control of linear time-invariant systems with unknown parameters. The technique presented in the paper in order to solve the discrete model reference adaptive control problem of continuous-time linear single-input, single-output time-invariant systems, is based on an indirect adaptive control scheme. The proposed technique makes use of a modified version of the fundamental result of (Hagiwara and Araki, 1988) concerning the equivalence between MDOC's and static state feedback, of some ideas reported in (Ortega and Kreisselmeier, 1990) regarding the introduction in the control loop of a signal providing persistent excitation of the continuous-time plant, and of the new approach to exact model matching reported in (Paraskevopoulos *et al.*, 1992).

The motivation to use MDOC's in order to control linear continuous-time systems is manifold. Foremostly, MDOC's as a special case of MROC's maintain all benefits of digital controllers, as mentioned above, over conventional feedback techniques. Moreover, control using MDOC's does not produce any serious drawbacks interwoven with other types of digital controllers in which the plant inputs change their values rapidly (see (Hagiwara and Araki, 1988) for details). In particular, as regards the model reference adaptive control problem treated in the present paper, it is also pointed out that the technique based on MDOC's has also the following advantages over the known techniques:

(a) It is not based on pole-zero cancellation and thus it is readily applicable to non-stable invertible plants and to reference models having arbitrary poles, zeros and a relative degree.
(b) It offers a solution to the problem of ensuring persistency of excitation of the continuous-time plant without any special requirement on the reference signal (except boundedness).
(c) It reduces the solution of the problem to the solution of a simple nonhomogeneous algebraic matrix equation, while known techniques resort to the computation of dynamic controllers through the solution of Diophantine equations.
(d) Finally, it reduces the original problem to that of the determination of a fictitious static state feedback controller. Thus, using the present technique, gain controllers are essentially needed to be computed rather than dynamic compensators, as in other techniques.

2. Preliminaries and Problem Formulation

Consider the continuous-time linear time-invariant single-input, single-output (SISO) system described in the state-space by the following equations:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \qquad y(t) = \boldsymbol{c}^T\boldsymbol{x}(t) \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the input, and $y(t) \in \mathbb{R}$ is the output of the system. As regards the system (1), we make the following two assumptions:

Assumption 1. The system (1) is controllable, observable and of known order n.

Assumption 2. There is a sampling period $T_0 \in \mathbb{R}^+$ such that the descretized system $(\exp(\boldsymbol{A}T_0),\ \int_0^{T_0} \exp[\boldsymbol{A}(T_0 - \lambda)\boldsymbol{b}\,\mathrm{d}\lambda,\ \boldsymbol{c}^T)$ is controllable and observable.

Except this prior information, the matrix triplet $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}^T)$ is arbitrary and unknown.

We apply the following sampling mechanism to system (1): we connect a sampler and a zero-order hold with period T_0 which can be selected as suggested in (Er and Anderson, 1991), to the plant input such that

$$u(t) = u(kT_0) \quad \text{for} \quad t \in \Big[kT_0,\, (k+1)T_0\Big)$$

while the plant output $y(t)$ is detected at every $T^* = T_0/N$ such that

$$y(kT_0 + \mu T^*) = \boldsymbol{c}^T \boldsymbol{\xi}(kT_0 + \mu T^*), \qquad \mu = 0, 1, \ldots, N-1$$

where $\boldsymbol{\xi}(\cdot)$ is the discrete state vector obtained by sampling $\boldsymbol{x}(t)$ and $N \in \mathbb{Z}^+$ is the *output multiplicity of the sampling.*

Now, consider the dynamic output feedback control law of the form

$$u\Big[(k+1)T_0\Big] = l_u u(kT_0) - \boldsymbol{k}^T \hat{\boldsymbol{\gamma}}(kT_0) + g\omega\Big[(k+1)T_0\Big] \tag{2}$$

where the vector $\hat{\boldsymbol{\gamma}}(kT_0)$ is composed of the sampled data of the output in the interval $\big[kT_0,\ (k+1)T_0\big)$ and has the form

$$\hat{\boldsymbol{\gamma}}(kT_0) = \begin{bmatrix} y(kT_0) \\ y(kT_0 + T^*) \\ \vdots \\ y\Big[kT_0 + (N-1)T^*\Big] \end{bmatrix}$$

Controllers of the form (2), will be called *multidetected-output controllers (MDOC's).* They constitute a special case of the well-known *dynamic multirate-output controllers (DMROC)* (Hagiwara and Araki, 1988; Hagiwara *et al.*, 1990) in the case of SISO systems.

The following two lemmas will be useful in what follows.

Lemma 1. *With regard to the sampling mechanism given above, the following basic relation holds:*

$$\boldsymbol{H}\boldsymbol{\xi}\Big[(k+1)T_0\Big] = \hat{\boldsymbol{\gamma}}(kT_0) - \boldsymbol{d}u(kT_0) \quad \textit{for} \quad k = 0, 1, \ldots \tag{3}$$

where $\boldsymbol{H} \in \mathbb{R}^{N \times n}$ and $\boldsymbol{d} \in \mathbb{R}^N$ have the following forms:

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{c}^T \left(\hat{\boldsymbol{A}}^N\right)^{-1} \\ \boldsymbol{c}^T \left(\hat{\boldsymbol{A}}^{N-1}\right)^{-1} \\ \vdots \\ \boldsymbol{c}^T \hat{\boldsymbol{A}}^{-1} \end{bmatrix}, \qquad \boldsymbol{d} = \begin{bmatrix} \boldsymbol{c}^T \hat{\boldsymbol{b}}_N \\ \boldsymbol{c}^T \hat{\boldsymbol{b}}_{N-1} \\ \vdots \\ \boldsymbol{c}^T \hat{\boldsymbol{b}}_1 \end{bmatrix} \tag{4}$$

with

$$\hat{\boldsymbol{A}} = \exp(\boldsymbol{A}T^*), \quad \hat{\boldsymbol{b}}_j = \int_0^{-jT^*} \exp(\boldsymbol{A}\lambda)\boldsymbol{b}\,\mathrm{d}\lambda \quad \textit{for} \quad j = 1, 2, \ldots, N \tag{5}$$

Proof. The proof of Lemma 1 can be obtained as a special case of the results reported in (Hagiwara and Araki, 1988). ■

Lemma 2. *If we choose $N > n$, the matrix $\boldsymbol{H}$ has full column rank.*

Proof. The proof of Lemma 2 can also be obtained as a special case of the results of (Hagiwara and Araki, 1988). ∎

The model reference adaptive control problem treated in the present paper is as follows: we are given a discrete-time linear reference model M of the form

$$\mathcal{Z}\left\{y^*(kT_0)\right\} = M(z)\mathcal{Z}\left\{\omega(kT_0)\right\} \tag{6}$$

where $\mathcal{Z}\{\cdot\}$ denotes the usual $\mathcal{Z}$-transform, $M(z)$ is the discrete transfer function of the desired reference model, having the form

$$M(z) = \frac{b_m z^m + b_{m-1}z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}, \qquad m \le n \tag{7}$$

while $y^*(kT_0)$ is the output of the reference model and $\omega(kT_0)$ is an arbitrary uniformly bounded reference sequence. Find a dynamic controller of the form (2), which when applied to the system (1), achieves discrete-time asymptotic model i.e.

1. $\lim\limits_{k\to\infty}\left[y(kT_0) - y^*(kT_0)\right] = 0.$
2. All signals in the control loop are bounded.

To solve the above problem, we apply an indirect adaptive control scheme. In particular, we first solve the model matching problem, namely the exact matching of the system (1) to the model (7). This is done in Section 3 and the corresponding control strategy is given in Fig. 1. Next, using these results, the exact model matching problem is solved for the configuration of Fig. 2, where a persistent excitation signal is introduced in the control loop for future identification purposes. This is done in Section 4. It is remarked that the motivation to modify the control strategy as in Fig. 2 is that it facilitates the derivation of the indirect adaptive control scheme sought. The derivation of this scheme is presented in Section 5, where also the global stability of the proposed scheme is studied.

3. Solution of the Exact Model Matching Problem for Known Systems

In this Section, we present a solution to the exact model matching problem, via MDOC's, for the case of known systems. This is done by making use of some of the results reported in (Hagiwara and Araki, 1988; Paraskevopoulos *et al.*, 1992). More precisely, we first establish here the basic idea of equivalently realizing a desired state feedback via MDOC's, which is summarized in the following result.

Theorem 1. *Provided that the pair $(\boldsymbol{A}, \boldsymbol{c}^T)$ is observable and that the output multiplicity N is selected such that $N > n$, for almost every period T_0 we can make the control law of the form (2), equivalent to any state feedback control law of the form*

$$u(kT_0) = -\boldsymbol{f}^T\boldsymbol{\xi}(kT_0) + g\omega(kT_0) \tag{8}$$

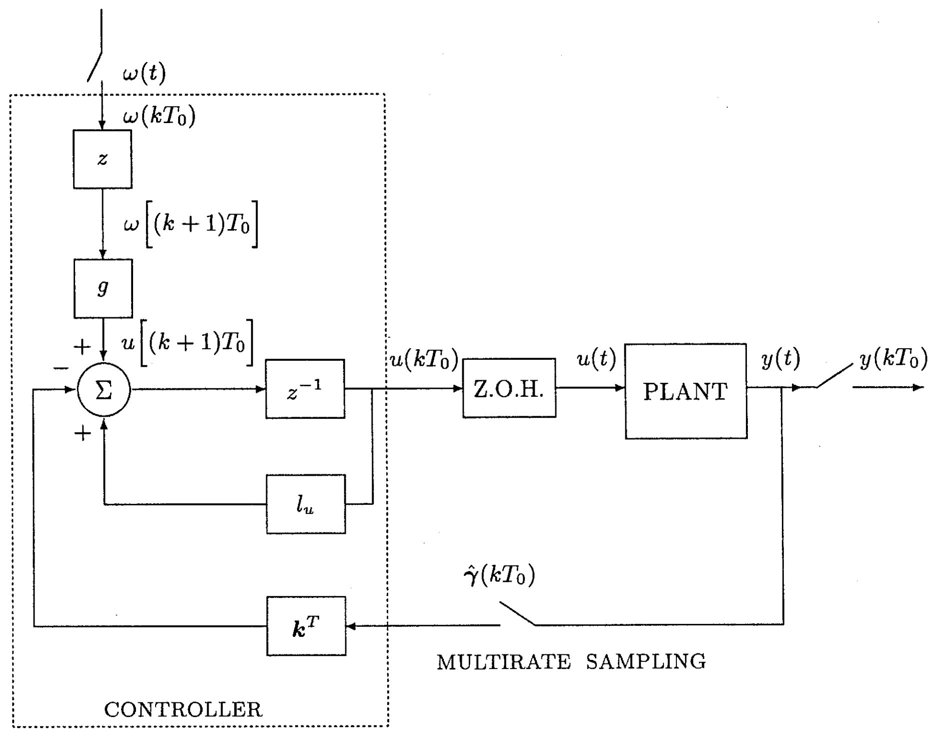

Fig. 1. Control strategy in the non-adaptive case.

by choosing properly the MDOC pair $(l_u, \boldsymbol{k}^T)$ *such that*

$$\boldsymbol{k}^T \boldsymbol{H} = \boldsymbol{f}^T \quad and \quad l_u = \boldsymbol{k}^T \boldsymbol{d} \tag{9}$$

Proof. The proof of Theorem 1 can be obtained as a special case of the results of (Hagiwara and Araki, 1988). ■

On the basis of Theorem 1, it is clear that in order to solve the exact model matching problem using MDOC's, one has essentialy to refer to an easier problem, i.e. to the design of a state feedback law of the form (8), which equivalently solves the exact model matching problem for the system with matrix triplet ($\boldsymbol{\Phi} \equiv e^{\boldsymbol{A}T_0}$, $\hat{\boldsymbol{b}} \equiv \int_0^{T_0} e^{\boldsymbol{A}\lambda}\boldsymbol{b}\,\mathrm{d}\lambda$, $\boldsymbol{c}^T$). Thus, in what follows, we will deal with this problem, recalling some basic results reported in (Paraskevopoulos *et al.*, 1992), regarding the state feedback exact model matching problem.

3.1. Solution of the Static State Feedback Exact Model Matching Problem

The system (1) can match the model (3) under the control law (8) iff

$$\boldsymbol{c}^T\left(z\boldsymbol{I} - \boldsymbol{\Phi} + \hat{\boldsymbol{b}}\boldsymbol{f}^T\right)^{-1}\hat{\boldsymbol{b}}g = M(z)$$

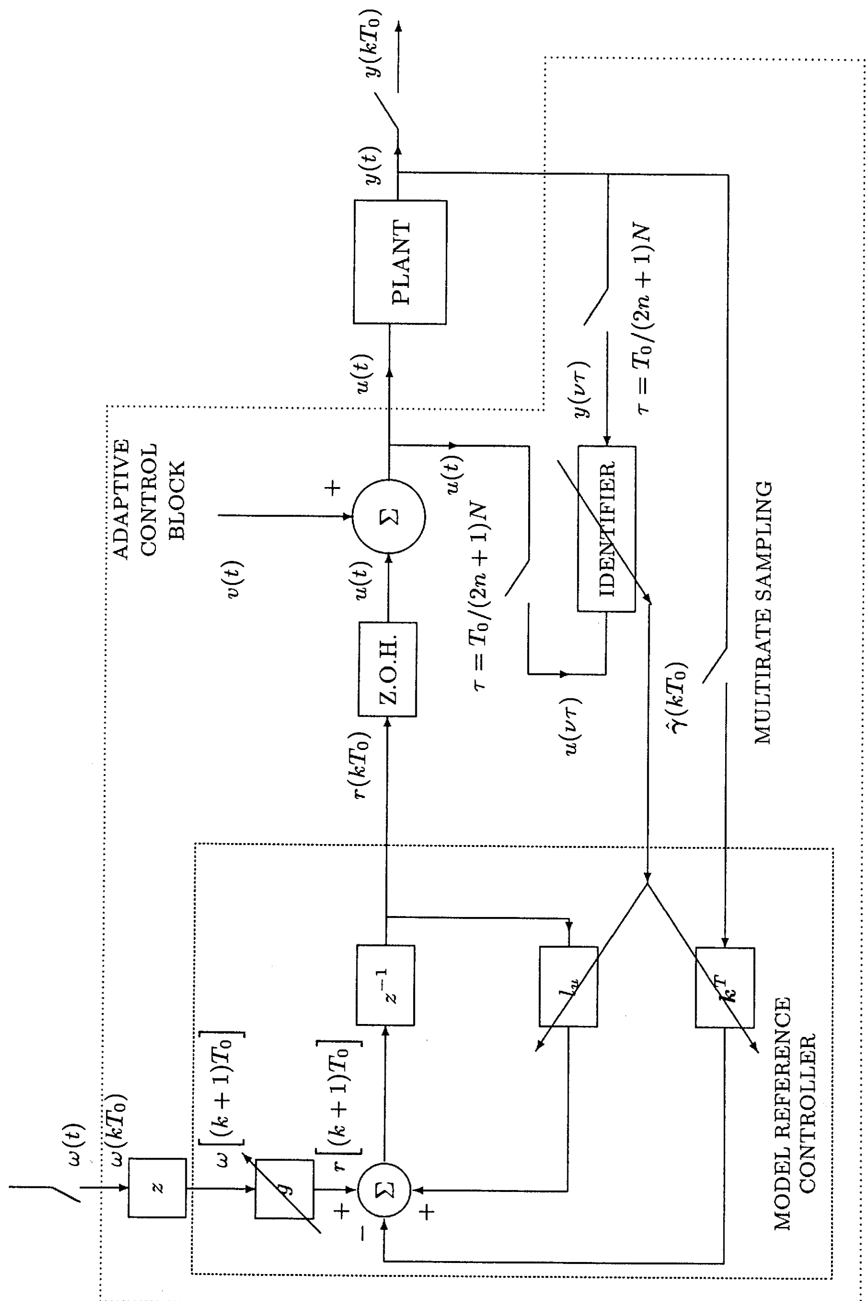

Fig. 2. The structure of the adaptive control system.

or equivalently iff

$$\boldsymbol{c}^T\left(z\boldsymbol{I}-\boldsymbol{\Phi}\right)^{-1}\hat{\boldsymbol{b}} = M(z)\left\{\gamma + \boldsymbol{\psi}^T\left(z\boldsymbol{I}-\boldsymbol{\Phi}\right)^{-1}\hat{\boldsymbol{b}}\right\} \tag{10}$$

where

$$\gamma = g^{-1} \quad \text{and} \quad \boldsymbol{\psi}^T = \gamma \boldsymbol{f}^T \tag{11}$$

Let $\hat{M}(z) = M^{-1}(z)$ and $z^{n-m}\hat{M}(z)$ be the $(n-m)$-delay inverse of $M(z)$ and define

$$L^*(z) = z^{n-m}\hat{M}(z)\boldsymbol{c}^T\left(z\boldsymbol{I}-\boldsymbol{\Phi}\right)^{-1}\hat{\boldsymbol{b}} \tag{12}$$

Then the relation (10) yields

$$L^*(z) = \gamma + \boldsymbol{\psi}^T\left(z\boldsymbol{I}-\boldsymbol{\Phi}\right)^{-1}\hat{\boldsymbol{b}} \tag{13}$$

From the definition of the σ-delay inverse of a rational function $M(z)$ it can be seen that the maximal positive power of z in the rational function $L^*(z)$ is $n-m-1$. Then expanding the rational functions $L^*(z)$ and $(z\boldsymbol{I}-\boldsymbol{\Phi})^{-1}$ in the formal power series of z in the relation (13) yields

$$\begin{aligned} &L_{-n+m+1}z^{n-m-1} + \cdots + L_{-1}z + L_0 + L_1 z^{-1} + L_2 z^{-2} + \cdots \\ &\quad = \gamma + \boldsymbol{\psi}^T\hat{\boldsymbol{b}}z^{-1} + \boldsymbol{\psi}^T\boldsymbol{\Phi}\hat{\boldsymbol{b}}z^{-2} + \cdots \end{aligned} \tag{14}$$

Equating the coefficients of the same powers of z in (14), we obtain the following system of algebraic equations:

$$L_{-j} = 0, \qquad j = 1, 2, \ldots, n-m-1 \tag{15}$$

$$\gamma = L_0 \tag{16}$$

$$\boldsymbol{\psi}^T\boldsymbol{\Phi}^{k-1}\hat{\boldsymbol{b}} = L_k, \qquad k \geq 1 \tag{17}$$

The relation (17) can be truncated to its first $2n$ equations (Ho and Kalman, 1966). Hence the relations (15)–(17) can be rewritten as

$$L_{-j} = 0, \qquad j = 1, 2, \ldots, n-m-1 \tag{18}$$

$$\gamma = L_0 \tag{19}$$

$$\boldsymbol{\psi}^T\boldsymbol{\Pi} = \boldsymbol{\lambda}^T \tag{20}$$

where

$$\boldsymbol{\Pi} = \left[\hat{\boldsymbol{b}} \;\vdots\; \boldsymbol{\Phi}\hat{\boldsymbol{b}} \;\vdots\; \cdots \;\vdots\; \boldsymbol{\Phi}^{2n-1}\hat{\boldsymbol{b}}\right] \tag{21}$$

and

$$\boldsymbol{\lambda}^T = \left[L_1 \ \vdots \ L_2 \ \vdots \ \cdots \ \vdots \ L_{2n-1} \right] \tag{22}$$

Equation (20) has a solution iff

$$\operatorname{rank} \begin{bmatrix} \boldsymbol{\Pi} \\ \boldsymbol{\lambda}^T \end{bmatrix} = \operatorname{rank} \boldsymbol{\Pi} \tag{23}$$

On the basis of the Cayley-Hamilton theorem, eqn. (20), yields

$$\boldsymbol{\psi}^T \boldsymbol{S} = \hat{\boldsymbol{\lambda}}^T \tag{24}$$

where $\boldsymbol{S}$ is the controllability matrix of the pair $(\boldsymbol{\Phi}, \hat{\boldsymbol{b}})$ and

$$\hat{\boldsymbol{\lambda}}^T = \left[L_1 \ \vdots \ L_2 \ \vdots \ \cdots \ \vdots \ L_n \right]$$

Since the pair $(\boldsymbol{\Phi}, \hat{\boldsymbol{b}})$ is assumed to be controllable, eqn. (24) admits the solution

$$\boldsymbol{\psi}^T = \hat{\boldsymbol{\lambda}}^T \boldsymbol{S}^{-1} \tag{25}$$

Furthermore, from (19) it is obvious that

$$g = L_0^{-1} \tag{26}$$

Finally, combining (11) and (26), we obtain

$$\boldsymbol{f}^T = L_0^{-1} \hat{\boldsymbol{\lambda}}^T \boldsymbol{S}^{-1} \tag{27}$$

3.2. Computation of the Admissible MDOC's

So far, we have established that the MDOC pair $(l_u, \boldsymbol{k}^T)$ is related to the vector $\boldsymbol{f}^T$ via the relations

$$\boldsymbol{k}^T \boldsymbol{H} = \boldsymbol{f}^T, \qquad l_u = \boldsymbol{k}^T \boldsymbol{d}$$

Since g is derived in (26), it only remains to determine the controller pair $(l_u, \boldsymbol{k}^T)$. To this end, let $\boldsymbol{E}$ be the $N \times N$ nonsingular matrix having the form

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{E}_1 \\ \hdashline \boldsymbol{E}_2 \end{bmatrix} \tag{28}$$

where

$$E_1 = \begin{bmatrix} e_{N-n+1} \\ e_{N-n+2} \\ \vdots \\ e_N \end{bmatrix}, \quad E_2 = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{N-n} \end{bmatrix} \quad \text{and} \quad e_j = \Big[0 \ \dots \ 0 \ \underset{\uparrow\, j\text{-th position}}{1} \ 0 \ \dots \ 0\Big] \tag{29}$$

Also, let

$$\hat{\boldsymbol{H}} \triangleq \boldsymbol{E}\boldsymbol{H} \equiv \begin{bmatrix} \hat{\boldsymbol{H}}_1 \\ \hdashline \hat{\boldsymbol{H}}_2 \end{bmatrix} \tag{30}$$

where the matrices $\hat{\boldsymbol{H}}_1 \in \mathbb{R}^{n\times n}$ and $\hat{\boldsymbol{H}}_2 \in \mathbb{R}^{(N-n)\times n}$ are defined as

$$\hat{\boldsymbol{H}}_1 = \begin{bmatrix} \boldsymbol{c}^T\left(\hat{\boldsymbol{A}}^n\right)^{-1} \\ \boldsymbol{c}^T\left(\hat{\boldsymbol{A}}^{n-1}\right)^{-1} \\ \vdots \\ \boldsymbol{c}^T\hat{\boldsymbol{A}}^{-1} \end{bmatrix}, \qquad \hat{\boldsymbol{H}}_2 = \begin{bmatrix} \boldsymbol{c}^T\left(\hat{\boldsymbol{A}}^N\right)^{-1} \\ \boldsymbol{c}^T\left(\hat{\boldsymbol{A}}^{N-1}\right)^{-1} \\ \vdots \\ \boldsymbol{c}^T\left(\hat{\boldsymbol{A}}^{n+1}\right)^{-1} \end{bmatrix} \tag{31}$$

Using the above definitions, we may determine $\boldsymbol{k}^T$ by mere inspection, having the following form:

$$\boldsymbol{k}^T = \left[\, L_0^{-1}\hat{\boldsymbol{\lambda}}^T\boldsymbol{S}^{-1}\hat{\boldsymbol{H}}_1^{-1} \ \vdots \ \boldsymbol{O} \,\right]\boldsymbol{E} \tag{32}$$

Furthermore, l_u takes on the form

$$l_u = \left[\, L_0^{-1}\hat{\boldsymbol{\lambda}}^T\boldsymbol{S}^{-1}\hat{\boldsymbol{H}}_1^{-1} \ \vdots \ \boldsymbol{O} \,\right]\boldsymbol{E}\boldsymbol{d} \tag{33}$$

3.3. Computation of Stable Dynamic and Static MDOC's

Relations (32) and (33) provide the possibility to determine the dynamic MDOC parameters $\boldsymbol{k}^T$ and l_u in the case where l_u (which corresponds to the transition matrix of the MDOC) is not prespecified to have any desired value. In the case of this additional requirement (in which case we set $l_u \equiv l_{u,sp}$), in order to compute the vector $\boldsymbol{k}^T$ of the MDOC, it is necessary to make the following additional assumption concerning the system (1):

Assumption 3. For the system (1), the following relationship holds:

$$\operatorname{rank}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}^T & 0 \end{pmatrix} = n+1 \tag{34}$$

Note that the relation (34) is equivalent to the following two statements:

(i) The system (1) is nondegenerate (Davison and Wang, 1974),

(ii) The system (1) has no invariant zeros at the origin (McFarlane and Karcanias, 1976).

We now establish the following result:

Theorem 2. *Let* $(\boldsymbol{A}, \boldsymbol{c}^T)$ *be an observable pair and assume that (34) holds. Then, for almost every sampling period* T_0*, the matrix* $[\, \boldsymbol{H} \;\vdots\; \boldsymbol{d}\,]$ *has full column rank if* N *is selected such that* $N \geq n+1$*, where* $n+1$ *is the observability index of the observable matrix pair*

$$\left(\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \hline \boldsymbol{0} & 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{c}^T & 0 \end{bmatrix} \right) \tag{35}$$

Proof. The proof of Theorem 2 is given in Appendix A. ■

On the basis of Theorem 2, the MDOC vector $\boldsymbol{k}^T$, can be computed as follows:

Case $N=n+1$: In this case, the matrix $[\, \boldsymbol{H} \;\vdots\; \boldsymbol{d}\,]$ is a square nonsingular matrix. Hence $\boldsymbol{k}^T$ can be computed according to the following relationship:

$$\boldsymbol{k}^T = \left[L_0^{-1} \hat{\boldsymbol{\lambda}}^T \boldsymbol{S}^{-1} \;\vdots\; l_{u,sp} \right] \left[\boldsymbol{H} \;\vdots\; \boldsymbol{d} \right]^{-1} \tag{36}$$

Case $N>n+1$: In this case, $\boldsymbol{k}^T$ can be obtained as follows: Denote by $\boldsymbol{S}^*$ the $N \times N$ nonsingular matrix of the following form:

$$\boldsymbol{S}^* = \begin{bmatrix} \boldsymbol{S}_1^* \\ \hline \boldsymbol{S}_2^* \end{bmatrix}, \text{ where } \boldsymbol{S}_1^* = \begin{bmatrix} \boldsymbol{e}_{N-n} \\ \boldsymbol{e}_{N-n+1} \\ \vdots \\ \boldsymbol{e}_N \end{bmatrix} \text{ and } \boldsymbol{S}_2^* = \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \vdots \\ \boldsymbol{e}_{N-n-1} \end{bmatrix} \tag{37}$$

Let also $\hat{\boldsymbol{H}}^*$ be the following matrix:

$$\hat{\boldsymbol{H}}^* \triangleq \boldsymbol{S}^* \left[\boldsymbol{H} \;\vdots\; \boldsymbol{d} \right] \equiv \begin{bmatrix} \hat{\boldsymbol{H}}_1^* \\ \hdashline \hat{\boldsymbol{H}}_2^* \end{bmatrix} \tag{38}$$

where the matrices $\hat{\boldsymbol{H}}_1^* \in \mathbb{R}^{(n+1)\times(n+1)}$ and $\hat{\boldsymbol{H}}_2^* \in \mathbb{R}^{(N-n-1)\times(n+1)}$ are defined on the basis of the following relationships:

$$\hat{\boldsymbol{H}}_1^* = \begin{bmatrix} \boldsymbol{c}^T\left(\hat{\boldsymbol{A}}^{n+1}\right)^{-1} & \vdots & \boldsymbol{c}^T\hat{\boldsymbol{b}}_{n+1} \\ \boldsymbol{c}^T\left(\hat{\boldsymbol{A}}^{n}\right)^{-1} & \vdots & \boldsymbol{c}^T\hat{\boldsymbol{b}}_{n} \\ \vdots & \vdots & \vdots \\ \boldsymbol{c}^T\hat{\boldsymbol{A}}^{-1} & \vdots & \boldsymbol{c}^T\hat{\boldsymbol{b}}_{1} \end{bmatrix}, \quad \hat{\boldsymbol{H}}_2^* = \begin{bmatrix} \boldsymbol{c}^T\left(\hat{\boldsymbol{A}}^{N}\right)^{-1} & \vdots & \boldsymbol{c}^T\hat{\boldsymbol{b}}_{N} \\ \boldsymbol{c}^T\left(\hat{\boldsymbol{A}}^{N-1}\right)^{-1} & \vdots & \boldsymbol{c}^T\hat{\boldsymbol{b}}_{N-1} \\ \vdots & \vdots & \vdots \\ \boldsymbol{c}^T\left(\hat{\boldsymbol{A}}^{n+2}\right)^{-1} & \vdots & \boldsymbol{c}^T\hat{\boldsymbol{b}}_{n+2} \end{bmatrix} \tag{39}$$

Using these definitions, it is plausible to determine $\boldsymbol{k}^T$ by mere inspection to have the form

$$\boldsymbol{k}^T = \left[\left[L_0^{-1}\hat{\boldsymbol{\lambda}}^T\boldsymbol{S}^{-1} \;\vdots\; l_{u,sp} \right] \left(\hat{\boldsymbol{H}}_1^*\right)^{-1} \;\vdots\; \boldsymbol{O} \right] \boldsymbol{S}^* \tag{40}$$

Remark 1. It is interesting at this point to observe that the above-mentioned analysis for the case where l_u is desired to take a prespecified value $l_{u,sp}$, provides us the following two important capabilities:

(a) The capability to choose l_u in such a way as to place its eigenvalues inside the unit circle, a fact that directly means that the corresponding dynamic MDOC, considered as a dynamical system, is a stable controller.

(b) The capability to choose l_u so that it will take the zero value, a fact which means that the corresponding MDOC is *static* and has the following configuration:

$$\boldsymbol{u}\left[(k+1)T_0\right] = -\boldsymbol{k}^T\hat{\boldsymbol{\gamma}}(kT_0) + g\omega\left[(k+1)T_0\right]$$

4. Solution to the Exact Model Matching Problem Appropriate for the Adaptive Case

In order to obtain a solution to the exact model matching problem which will be more appropriate for application in the case of systems with unknown parameters, we slightly modify the control stategy of Fig. 1, as it is shown in Fig. 2. In particular, we introduce in the control loop the persistent excitation signal $v(t)$, which is defined as

$$v(t) = \boldsymbol{q}^T(t)\boldsymbol{v}, \qquad \boldsymbol{q}^T(t) = \left[q_0(t), \ldots, q_{N-1}(t) \right]$$

Here, $\boldsymbol{q}(t)$ is the T^*-periodic vector function with elements having the form

$$q_i(t) = q_{i,\mu} \quad \text{for} \quad t \in \Big[\mu T^*, (\mu+1)T^*\Big), \quad i = 0, 1, \ldots, N-1, \ \mu = 0, 1, \ldots, N-1 \quad (41)$$

where

$$q_{i,\mu} = \begin{cases} 1 & \text{if} \quad \mu = i \\ 0 & \text{if} \quad \mu \neq i \end{cases} \quad (42)$$

It is pointed out that $\boldsymbol{v}$ is unknown. We remark that the additive term $v(t) = \boldsymbol{q}^T(t)\boldsymbol{v}$ at the input of the continuous-time system is used only for identification purposes and, as will be shown later, it is selected such that it will not influence the exact model matching problem.

We will now establish the following results:

Theorem 3. *If N is chosen such that $N > n$, the closed-loop system can be expressed in the form*

$$\left.\begin{aligned} \boldsymbol{\xi}\Big[(k+1)T_0\Big] &= \Big(\boldsymbol{\Phi} - \hat{\boldsymbol{b}}\boldsymbol{f}^T\Big)\boldsymbol{\xi}(kT_0) + \hat{\boldsymbol{b}}g\omega(kT_0) + \boldsymbol{B}^*\boldsymbol{v} \\ y(kT_0) &= \boldsymbol{c}^T\boldsymbol{\xi}(kT_0) \end{aligned}\right\} \quad k > 0 \quad (43)$$

where $\boldsymbol{B}^*$ is the $n \times N$ matrix of the form

$$\boldsymbol{B}^* = \Big[\, \hat{\boldsymbol{A}}^{N-1}\hat{\boldsymbol{b}}^* \ \vdots \ \hat{\boldsymbol{A}}^{N-2}\hat{\boldsymbol{b}}^* \ \vdots \ \cdots \ \vdots \ \hat{\boldsymbol{b}}^* \,\Big]$$

and

$$\hat{\boldsymbol{b}}^* = \int_0^{T^*} \exp\Big[\boldsymbol{A}(T^* - \lambda)\Big]\boldsymbol{b}\,\mathrm{d}\lambda$$

Proof. In order to show that the closed-loop system can be written in the form (43), we start by discretizing the system (1) with sampling period T_0 to yield

$$\boldsymbol{\xi}\Big[(k+1)T_0\Big] = \boldsymbol{\Phi}\boldsymbol{\xi}(kT_0) + \int_{kT_0}^{(k+1)T_0} \exp\Big\{\boldsymbol{A}\Big[(k+1)T_0 - \lambda\Big]\Big\}\boldsymbol{b}u(\lambda)\,\mathrm{d}\lambda \quad (44)$$

Observe now that $u(t) = r(t) + \boldsymbol{q}^T(t)\boldsymbol{v}$ and $r(t) = r(kT_0)$, $t \in \big[kT, (k+1)T_0\big)$. Hence the relation (44) yields

$$\boldsymbol{\xi}\Big[(k+1)T_0\Big] = \boldsymbol{\Phi}\boldsymbol{\xi}(kT_0) + \int_{kT_0}^{(k+1)T_0} \exp\Big\{\boldsymbol{A}\Big[(k+1)T_0 - \lambda\Big]\Big\}\boldsymbol{b}\,\mathrm{d}\lambda\, r(kT_0) + \boldsymbol{\Gamma}\boldsymbol{v} \quad (45)$$

where

$$\boldsymbol{\Gamma} = \int_{kT_0}^{(k+1)T_0} \exp\Big\{\boldsymbol{A}\Big[(k+1)T_0 - \lambda\Big]\Big\}\boldsymbol{b}\boldsymbol{q}^T(\lambda)\,\mathrm{d}\lambda$$

If N is chosen such that $N > n$, then according to the results of Section 3 one can take $r(kT_0) = -\boldsymbol{f}^T\boldsymbol{\xi}(kT_0) + g\omega(kT_0)$, with $\boldsymbol{f}^T = \boldsymbol{k}^T\boldsymbol{H}$ and $\boldsymbol{k}^T\boldsymbol{d} = l_u$. Hence the relation (45) equivalently yields

$$\boldsymbol{\xi}\Big[(k+1)T_0\Big] = \Big(\boldsymbol{\Phi} - \hat{\boldsymbol{b}}\boldsymbol{f}^T\Big)\boldsymbol{\xi}(kT_0) + \hat{\boldsymbol{b}}g\omega(kT_0) + \boldsymbol{\Gamma}\boldsymbol{v} \tag{46}$$

It only remains to prove that $\boldsymbol{\Gamma} \equiv \boldsymbol{B}^*$. To this end, the $(i+1)$-th column of the matrix $\boldsymbol{\Gamma}$, denoted by $\boldsymbol{\Gamma}_{i+1}$, for $i = 0, 1, \ldots, N-1$, can be expressed as

$$\boldsymbol{\Gamma}_{i+1} = \int_0^{T_0} \exp\Big[\boldsymbol{A}(T-\lambda)\Big]\boldsymbol{b}q_i(\lambda)\,\mathrm{d}\lambda \quad \text{for } i = 0, 1, \ldots, N-1 \tag{47}$$

Substituting (41) and (42) into (47) yields

$$\boldsymbol{\Gamma}_{i+1} = \sum_{\mu=0}^{N-1} \int_{\mu T^*}^{(\mu+1)T^*} \exp\Big[\boldsymbol{A}(T-\lambda)\Big]\boldsymbol{b}q_{i,\mu}(\lambda)\,\mathrm{d}\lambda \quad \text{for } i = 0, 1, \ldots, N-1 \tag{48}$$

The relation (48) may further be written as

$$\boldsymbol{\Gamma}_{i+1} = \sum_{\mu=0}^{N-1} q_{i,\mu} \exp\Big\{\boldsymbol{A}(N-1-\mu)T^*\Big\} \int_0^{T^*} \exp\Big[\boldsymbol{A}(T-\lambda)\Big]\boldsymbol{b}\,\mathrm{d}\lambda$$

$$= \left\{\sum_{\xi=1}^{N} q_{i,N-\xi}\hat{\boldsymbol{A}}^{\xi-1}\right\}\hat{\boldsymbol{b}}^*$$

Making use of the relation (42), we get

$$\boldsymbol{\Gamma}_{i+1} = \hat{\boldsymbol{A}}^{N-i-1}\hat{\boldsymbol{b}}^*$$

Clearly, $\boldsymbol{\Gamma} \equiv \boldsymbol{B}^*$. This completes the proof. ∎

Theorem 4. *Let the* $n \times n$ *matrix* $\hat{\boldsymbol{S}}^*$ *be defined as follows:*

$$\hat{\boldsymbol{S}}^* = \Big[\, \hat{\boldsymbol{A}}^{n-1}\hat{\boldsymbol{b}}^* \;\vdots\; \hat{\boldsymbol{A}}^{n-2}\hat{\boldsymbol{b}}^* \;\vdots\; \cdots \;\vdots\; \hat{\boldsymbol{b}}^* \,\Big] \tag{49}$$

Then, for almost every sampling period T_0, *the matrix* $\hat{\boldsymbol{S}}^*$ *is nonsingular.*

Proof. The proof of Theorem 4 is given in Appendix B. ∎

Now, let $\boldsymbol{\Delta}$ be the $N \times N$ nonsingular permutation matrix with the property $\boldsymbol{\Delta}^{-1} \equiv \boldsymbol{\Delta}^T$ and having the form

$$\boldsymbol{\Delta} = \Big[\, \boldsymbol{\Delta}_1 \;\vdots\; \boldsymbol{\Delta}_2 \;\vdots\; \boldsymbol{\Delta}_3 \,\Big]$$

where

$$\boldsymbol{\Delta}_1 = \left[\varepsilon_{N-n+1} \ \vdots \ \varepsilon_{N-n+2} \ \vdots \ \cdots \ \vdots \ \varepsilon_N \right]$$

$$\boldsymbol{\Delta}_2 = \varepsilon_{N-n}$$

$$\boldsymbol{\Delta}_3 = \left[\varepsilon_{N-n-1} \ \vdots \ \varepsilon_{N-n-2} \ \vdots \ \cdots \ \vdots \ \varepsilon_1 \right], \qquad \varepsilon_j = \boldsymbol{e}_j^T$$

Furthermore, let

$$\tilde{\boldsymbol{B}}^* \triangleq \boldsymbol{B}^* \boldsymbol{\Delta} \equiv \left[\hat{\boldsymbol{S}}^* \ \vdots \ \hat{\boldsymbol{A}}^n \hat{\boldsymbol{b}}^* \ \vdots \ \hat{\boldsymbol{Q}}^* \right]$$

where

$$\hat{\boldsymbol{Q}}^* = \left[\hat{\boldsymbol{A}}^{N-1} \hat{\boldsymbol{b}}^* \ \vdots \ \hat{\boldsymbol{A}}^{N-2} \hat{\boldsymbol{b}}^* \ \vdots \ \cdots \ \vdots \ \hat{\boldsymbol{A}}^{n+1} \hat{\boldsymbol{b}}^* \right]$$

Since the MDOC gains $\boldsymbol{k}^T$ and l_u can be computed as in Section 3, it only remains to determine an appropriate vector $\boldsymbol{v}$ which guarantees that the exact model matching problem will not be influenced by the vector $\boldsymbol{v}$. In other words, $\boldsymbol{v} \in \ker \boldsymbol{B}^*$, or $\boldsymbol{B}^* \boldsymbol{v} = \boldsymbol{0}$. An obvious selection of such $\boldsymbol{v}$ obtained also by mere inspection is the following:

$$\boldsymbol{v} = \boldsymbol{\Delta} \begin{bmatrix} -\hat{\boldsymbol{S}}^{*-1} \hat{\boldsymbol{A}}^n \hat{\boldsymbol{b}} \\ \hdashline 1 \\ \hdashline \boldsymbol{0}_{(N-n-1)\times 1} \end{bmatrix} \tag{50}$$

The general form of $\boldsymbol{v}$ is

$$\boldsymbol{v} = \boldsymbol{B}_0^* \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{N-n} \end{bmatrix} \tag{51}$$

where $\boldsymbol{B}_0^*$ is the $N \times (N-n)$ matrix whose columns are linearly indepedent N-dimensional vectors which are orthogonal to the rows of $\boldsymbol{B}^*$ and where ρ_j, $j = 1, 2, \ldots, N-n$ are arbitrary real parameters.

Let us note that the vector $\boldsymbol{v}$, even though it does not affect the discrete model matching problem, provides persistent excitation useful for identification of the system, as will be shown in the next section.

The introduction of the reference signal $v(t)$ in the control loop greatly facilitates the estimation of the plant parameters in the case of unknown systems. For this reason, the control strategy of Fig. 2 is more appropriate than the control strategy of Fig. 1 for the achievement of the indirect adaptive control scheme presented in Section 5.

5. Control Strategy for the Adaptive Case

The control scheme presented in Section 4 has the corresponding scheme in the case where the system is unknown. For the case of unknown systems, the control strategy is largely based upon the computation of the MDOC gains $\boldsymbol{k}^T$, l_u, g and the vector $\boldsymbol{v}$ from suitable estimates of the parameters of the plant with updating taken every kT_0, $k = 0, 1, \dots$ and results in a globally stable closed-loop system whose output follows assymptotically the output of the desired model.

5.1. System Identification

The system (1), when descretized with sampling period $\tau = T^*/(2n+1)$, takes on the form

$$\boldsymbol{\xi}\Big[(\nu+1)\tau\Big] = \boldsymbol{\Phi}_\tau \boldsymbol{\xi}(\nu\tau) + \hat{\boldsymbol{b}}_\tau u(\nu\tau), \quad y(\nu\tau) = \boldsymbol{c}^T \boldsymbol{\xi}(\nu\tau), \quad \nu = 0, 1, \dots \tag{52}$$

where

$$\boldsymbol{\Phi}_\tau = \exp(\boldsymbol{A}\tau), \qquad \hat{\boldsymbol{b}}_\tau = \int_0^\tau \exp\Big[\boldsymbol{A}(\tau - \lambda)\Big] \boldsymbol{b}\, \mathrm{d}\lambda \tag{53}$$

Iterating equation (52) $2n+1$ times and observing that $u(\nu\tau)$ is constant for $\nu\tau \in \big[mT^*, (m+1)T^*\big)$, $m = 0, 1, \dots$, we obtain

$$\left.\begin{aligned} &\boldsymbol{\xi}\Big[(m+1)T^*\Big] = \boldsymbol{\Phi}_{T^*} \boldsymbol{\xi}(mT^*) + \hat{\boldsymbol{b}}_{T^*} u(mT^*) \\ &y(mT^*) = \boldsymbol{c}^T \boldsymbol{\xi}(mT^*) \end{aligned}\right\} \quad m = 0, 1, \dots$$

where

$$\boldsymbol{\Phi}_{T^*} \equiv \hat{\boldsymbol{A}} = \boldsymbol{\Phi}_\tau^{2n+1}, \qquad \hat{\boldsymbol{b}}_{T^*} \equiv \hat{\boldsymbol{b}}^* = \sum_{\rho=0}^{2n} \boldsymbol{\Phi}_\tau^\rho \hat{\boldsymbol{b}}_\tau \tag{54}$$

We also note that the matrix $\boldsymbol{\Phi}$ and vector $\hat{\boldsymbol{b}}$ can be written as

$$\boldsymbol{\Phi} \equiv \hat{\boldsymbol{A}}^N \equiv \boldsymbol{\Phi}_\tau^{(2n+1)N}, \qquad \hat{\boldsymbol{b}} \equiv \sum_{\rho=0}^{N-1} \hat{\boldsymbol{A}}^\rho \hat{\boldsymbol{b}}^* \equiv \sum_{\rho=0}^{(2n+1)N-1} \boldsymbol{\Phi}_\tau^\rho \hat{\boldsymbol{b}}_\tau \tag{55}$$

Furthermore, as can be shown, the vectors $\hat{\boldsymbol{b}}_j$, $j = 1, 2, \dots, N$ may be expressed as

$$\hat{\boldsymbol{b}}_j = -\boldsymbol{\Phi}_j \tilde{\boldsymbol{b}}_j \tag{56}$$

where

$$\mathbf{\Phi}_j = \exp(-\boldsymbol{A}jT^*), \qquad \tilde{\boldsymbol{b}}_j = \int_0^{jT^*} \exp(\ \boldsymbol{A}\lambda\ )\boldsymbol{b}\,\mathrm{d}\lambda \tag{57}$$

The matrices $\mathbf{\Phi}_j$ and $\tilde{\boldsymbol{b}}_j$ may also be written as

$$\mathbf{\Phi}_j = \left\{ \left[\ \mathbf{\Phi}_\tau\ \right]^{(2n+1)j} \right\}^{-1}, \qquad \tilde{\boldsymbol{b}}_j = \sum_{\rho=0}^{(2n+1)j-1} \mathbf{\Phi}_\tau^\rho \hat{\boldsymbol{b}}_\tau \tag{58}$$

Introducing (58) in (56) yields

$$\hat{\boldsymbol{b}}_j = -\left\{ \left[\ \mathbf{\Phi}_\tau\ \right]^{(2n+1)j} \right\}^{-1} \left\{ \sum_{\rho=0}^{(2n+1)j-1} \mathbf{\Phi}_\tau^\rho \hat{\boldsymbol{b}}_\tau \right\} \tag{59}$$

From the above analysis, it is clear that the matrices $\mathbf{\Phi}$, $\hat{\boldsymbol{b}}$, $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{b}}_j$ (which are the only matrices involved in computing the multidetected-output controller gains $\boldsymbol{k}^T$, l_u and g) can be computed on the basis of the pair $\left(\mathbf{\Phi}_\tau,\ \hat{\boldsymbol{b}}_\tau\right)$. Moreover, when fixing the coordinate system such that

$$\mathbf{\Phi}_\tau = \begin{bmatrix} 0 & 0 & \dots & 0 & -\alpha_n \\ 1 & 0 & \dots & 0 & -\alpha_{n-1} \\ 0 & 1 & \dots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\alpha_1 \end{bmatrix}, \quad \hat{\boldsymbol{b}} = \begin{bmatrix} \beta_n \\ \beta_{n-1} \\ \beta_{n-2} \\ \dots \\ \beta_1 \end{bmatrix}, \quad \boldsymbol{c}^T = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{60}$$

only α_i and β_i, $i = 1, 2, \dots, n$ are considered as the unknown parameters. Note that the relations (52) and (60) are equivalent to the following difference equation:

$$y(\nu\tau) + \sum_{\rho=1}^{n} \alpha_i y(\nu\tau - \rho\tau) = \sum_{\rho=1}^{n} \beta_i u(\nu\tau - \rho\tau), \quad \nu = 0, 1, \dots \tag{61}$$

The relation (61) can now be used for the identification of the parameters of the unknown system. To this end, the relation (61) can be written in the following linear regression form:

$$y(\nu\tau) = \boldsymbol{\phi}^T(\nu\tau)\boldsymbol{\theta}$$

where

$$\boldsymbol{\phi}^T(\nu\tau) = \left[\ -y(\nu\tau - \tau),\ \dots,\ -y(\nu\tau - n\tau),\ u(\nu\tau - \tau),\ \dots,\ u(\nu\tau - n\tau)\ \right]$$

and

$$\boldsymbol{\theta} = \left[\ \alpha_i,\ \dots,\ \alpha_n,\ \beta_1,\ \dots,\ \beta_n\ \right]$$

Define

$$\boldsymbol{Y}(kT_0) = \Big[\, y(kT_0),\ y(kT_0-\tau),\ \ldots,\ y\big((k-1)T_0\big) \,\Big]^T$$

$$\boldsymbol{Z}(kT_0) = \Big[\, \phi(kT_0),\ \phi(kT_0-\tau),\ \ldots,\ \phi\big((k-1)T_0\big) \,\Big]$$

and

$$\hat{\boldsymbol{\theta}}_k = \Big[\, \hat{\alpha}_1(kT_0),\ \ldots,\ \hat{\alpha}_n(kT_0),\ \hat{\beta}_1(kT_0),\ \ldots,\ \hat{\beta}_n(kT_0) \,\Big]$$

Clearly, we have the relation

$$\boldsymbol{Y}^T(kT_0) = \boldsymbol{Z}^T(kT_0)\boldsymbol{\theta}$$

We now choose the recursive algorithm for estimation of $\hat{\boldsymbol{\theta}}_k$ as

$$\hat{\boldsymbol{\theta}}_{k+1} = \hat{\boldsymbol{\theta}}_k - \Big[a\boldsymbol{I} + \boldsymbol{Z}(kT_0)\boldsymbol{Z}^T(kT_0)\Big]^{-1} \times \boldsymbol{Z}(kT_0)\Big[\boldsymbol{Z}^T(kT_0)\hat{\boldsymbol{\theta}}_k - \boldsymbol{Y}^T(kT_0)\Big], \quad a > 0 \tag{62}$$

5.2. Algorithm for Adaptive Controller Synthesis

On the basis of the estimated parameter vector $\hat{\boldsymbol{\theta}}$ obtained from (62), as well as on the basis of relations (54), (55), (59) and (60), one can take the estimates needed for the computation of the matrices $\hat{\boldsymbol{A}} \equiv \hat{\boldsymbol{A}}(\hat{\boldsymbol{\theta}})$, $\boldsymbol{\Phi} \equiv \boldsymbol{\Phi}(\hat{\boldsymbol{\theta}})$, $\hat{\boldsymbol{b}} \equiv \hat{\boldsymbol{b}}(\hat{\boldsymbol{\theta}})$ and $\hat{\boldsymbol{b}}_j = \hat{\boldsymbol{b}}_j(\hat{\boldsymbol{\theta}})$, which are involved in the algorithms presented in the previous sections. Moreover, since the matrices $\boldsymbol{\Pi}$, $\boldsymbol{H}$ and the vector $\boldsymbol{d}$ can be constructed on the basis of the matrices $\boldsymbol{\Phi}(\hat{\boldsymbol{\theta}})$, $\hat{\boldsymbol{b}}(\hat{\boldsymbol{\theta}})$, $\hat{\boldsymbol{A}}(\hat{\boldsymbol{\theta}})$ and $\hat{\boldsymbol{b}}_j(\hat{\boldsymbol{\theta}})$, provided that the matrix triplet $\big(\boldsymbol{\Phi}_{T^*}(\hat{\boldsymbol{\theta}}_k),\ \hat{\boldsymbol{b}}_{T^*}(\hat{\boldsymbol{\theta}}_k),\ \boldsymbol{c}^T\big)$ is minimal and that relations (15) and (23) hold for any possible value of $\hat{\boldsymbol{\theta}}_k$, we can obtain the following results sought:

$$\boldsymbol{k}^T \equiv \boldsymbol{k}^T(\hat{\boldsymbol{\theta}}_k), \quad l_u \equiv l_u(\hat{\boldsymbol{\theta}}_k) \text{ and } g \equiv g(\hat{\boldsymbol{\theta}}_k) \tag{63}$$

Consequently, the procedure for the synthesis of a model reference adaptive MDOC, for each case considered in Section 3, consists of the main steps given bellow:

1. Case of nonprespecified l_u and $N>n$:

Step 1. Choose the sampling period τ such that $\tau = T_0/(2n+1)N = T^*/(2n+1)$.

Step 2. Update the estimates using (62).

Step 3. Use (60) to compute the matrices $\boldsymbol{\Phi}_\tau$, $\hat{\boldsymbol{b}}_\tau$ and $\boldsymbol{c}^T$.

Step 4. Use (55) to compute the matrices $\boldsymbol{\Phi}$ and $\hat{\boldsymbol{b}}$ and (54) to compute the matrix $\hat{\boldsymbol{A}}$.

Step 5. Use (26) and (27) to implement the equivalent fictitious static state feedback controller parameters g and $\boldsymbol{f}^T$.

Step 6. Find the matrices $\boldsymbol{S}$ and $\hat{\boldsymbol{S}}^*$ using the relation (49).

Step 7. Form the matrix $\boldsymbol{H}$ and the vector $\boldsymbol{d}$, using the relation (4).

Step 8. Use (28) and (29) to compute the matrix $\boldsymbol{E}$ as well as (31) to compute the matrix $\hat{\boldsymbol{H}}_1$.

Step 9. Implement the dynamic MDOC sought using (32), (33) and (50) or (51).

2. Case of prespecified l_u and $N=n+1$:

In this case repeat Steps 1–7 of Case 1 and furthermore:

Step 8. Implement the MDOC sought using (36) and (50) or (51).

3. Case of prespecified l_u and $N>n+1$:

In this case repeat Steps 1–6 of Case 1 and furthermore:

Step 8. Use (59) to compute the vectors $\hat{\boldsymbol{b}}_j$.

Step 9. Use (37) to compute the matrix $\boldsymbol{S}^*$ as well as (39) to compute the matrix $\hat{\boldsymbol{H}}_1^*$.

Step 10. Implement the MDOC sought using (40) and (50) or (51).

5.3. Stability Analysis of the Adaptive Control Scheme

With regard to the stability of the proposed adaptive scheme, we next give the following fundamental theorem.

Theorem 5. *The regressor sequence $\phi(\nu\tau)$ is persistently exciting, i.e. there is a $\delta > 0$, such that*

$$\boldsymbol{Z}(kT_0)\boldsymbol{Z}^T(kT_0) = \sum_{\nu=0}^{(2n+1)N} \phi(kT_0 - \nu\tau)\phi^T(kT_0 - \nu\tau) \geq \delta \boldsymbol{I} \tag{64}$$

Proof. The proof of Theorem 5 is given in (Ortega and Kreisselmeier, 1990). ■

Since the regressor sequence is persistently exciting, the difference $\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}$, where $\boldsymbol{\theta}$ contains the true values of the parameters, converges to zero. This guarantees convergence of the controller parameter estimates to their true values, uniform boundedness of $\boldsymbol{\xi}(kT_0)$, $y(kT_0)$, $\forall k = 0, 1, \ldots$ and $y(t)$ and assymptotic discrete model. Moreover, the adaptive scheme ensures exponential convergence of the estimated parameters, since

$$\hat{\boldsymbol{\theta}}_{k+1} - \boldsymbol{\theta} = \left[1 + a^{-1}\boldsymbol{Z}(kT_0)\boldsymbol{Z}^T(kT_0) \right] \left(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta} \right) \tag{65}$$

The relation (65) together with (64) ensure that $\hat{\boldsymbol{\theta}}_k \to \boldsymbol{\theta}$ exponentially as $k \to \infty$ (see (Kreisselmeier, 1989) for details).

6. Simulation Example

In this section, the proposed model reference adaptive control algorithm is tested through an illustrative simulation example. The simulation results that follow are obtained with the use of the "mdmrac" procedure, which was coded for this purpose in MATLAB 4.2c for WINDOWS. In the example, we address the same continuous-time plant utilized in (Kinnaert and Blondel, 1992), i.e. the unstable second-order system of the form

$$H(s) = \frac{s-10}{s(s-1)}$$

which has a non-minimum-phase zero at $s = 10$. The plant is discretized by using a zero-order hold and a sampling interval of $T_0 = 0.02\,\mathrm{s}$, that yields the following discrete-time plant:

$$H(z) = \frac{0.0182z - 0.0222}{z^2 - 2.0202z + 1.0202}$$

having a pole at $z = 1$, an unstable pole at $z = 1.0202$ and an unstable zero at $z = 1.2221$. The reference model is selected to be the second-order system of the form

$$M(z) = \frac{0.0546z - 0.0667}{z^2 - 1.8462z + 0.8542}$$

The poles of the model are conjugate with values $z = 0.9231 \pm j0.046$. The reference signal $\omega(t)$ is a unit square wave with period of 12 s.

The output multiplicity of the sampling is selected as $N = 4$. Thus the sampling period has the value $\tau = 1\,\mathrm{ms}$.

In the case where l_u is not prespecified, the parameters of the admissible MDOC, as computed by (32) and (33), are

$$\boldsymbol{k}^T = \begin{bmatrix} 0 & 0 & 234.9080 & -236.8954 \end{bmatrix}, \quad l = -1.2434, \quad g = 3$$

In the case of a static MDOC ($l_u = 0$), the parameters of the admissible MDOC, as computed by (40), are

$$\boldsymbol{k}^T = \begin{bmatrix} 0 & -4986 & 10182 & -5198 \end{bmatrix}, \quad g = 3$$

while in the case where l_u is chosen such that $l_u = 0.5$ (the case of a stable dynamic MDOC), the MDOC gains can be computed using (40), to yield

$$\boldsymbol{k}^T = \begin{bmatrix} 0 & -6991 & 14182 & -7193 \end{bmatrix}, \quad g = 3$$

In the case of an unknown plant, the simulation was performed using the proposed modified recursive least-squares algorithm. The nominal vector $\boldsymbol{\theta}$ was of the form

$$\boldsymbol{\theta} = \begin{bmatrix} -2.0010 & 1.0010 & 0.0010 & -0.0010 \end{bmatrix}^T$$

The identification algorithm was initialized with the following parameter vector:

$$\hat{\theta}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$$

and with $a = 0.2$. Simulation results are given in Figs. 3–20. Figures 3–8 represent simulation results for the case of non-prespecified l_u. In Figs. 9–14, simulation results are given for the case of static MDOC. Finally, Figs. 15–20 represent results of the simulation in the case of a stable dynamic MDOC, where $l_u = 0.5$. Note that similar results can be obtained in the case where $\hat{\theta}_0$ or a, take other values, e.g. $a = 0.5$ or 0.9 and

$$\hat{\theta}_0 = \begin{bmatrix} 4 & 4 & 4 & 3 \end{bmatrix}^T \text{ or } \hat{\theta}_0 = \begin{bmatrix} -2 & 4 & 2 & 1 \end{bmatrix}^T \text{ or } \hat{\theta}_0 = \begin{bmatrix} -2 & -2 & -2 & -2 \end{bmatrix}^T$$

From this example we can see that the proposed model reference adaptive control algorithm based on MDOCs has a good performance even if it is applied to non-minimum-phase plants. We remark at this point that the square wave $\omega(t)$ with period of 12 s, used as the reference signal, provides sufficient excitation to the plant. So in this case the excitation signal $v(t)$ is useless. In the case where $v(t)$ is added in the control loop, simulation results show that, in the case where v is evaluated by (50), the convergence of the identification algorithm is ameliorated. It is worth noticing that in this case the excitation signal $v(t)$ causes a static steady-state error of approximately 15%. However, this static error is eliminated by evaluating v through (51) and by appropriately selecting the arbitrary parameters ρ_j, $j = 1, 2$ (e.g. $\rho_1 = 0.5$ and $\rho_2 = -0.45$).

7. Conclusions

A new indirect adaptive scheme has been derived for model reference adaptive control of continuous-time linear time-invariant single-input, single-output systems using multidetected-output controllers. A simulation has been presented to demonstrate this adaptive controller. The approach proposed to solve the model reference adaptive control problem has, from both the theoretical and the practical point of view, advantages in comparison with the related known techniques. From the theoretical point of view, the model reference adaptive control problem is reduced to the determination of a fictitious static state feedback controller, due to the merits of multidetected-output controllers. The known techniques do not possess this flexibility and they resort to the direct computation of dynamic controllers. Moreover, the present technique does not rely on pole-zero cancellation and it is readily applicable to nonstably invertible plants and to reference models with arbitrary poles, zeros and a relative degree. Finally, in the present technique, persistency of excitations of the plant under control is provided, without making any special assumption on the reference signal, except boundedness. From the computational point of view, the controller determination reduces to the simple problem of solving a linear algebraic system of equations. In the known techniques, matrix polynomial Diophantine equations are usually needed to be solved.

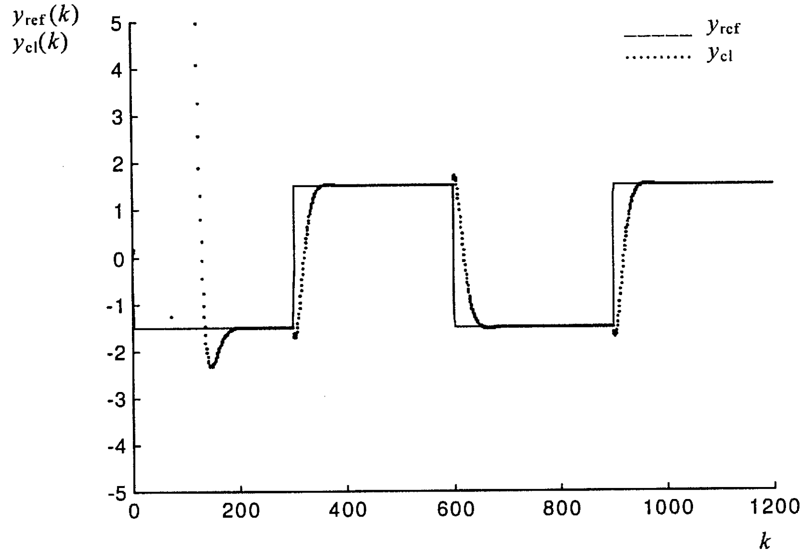

Fig. 3. Closed loop system output y_{cl} versus reference model output y_{ref} in the case of non-prespecified l_u.

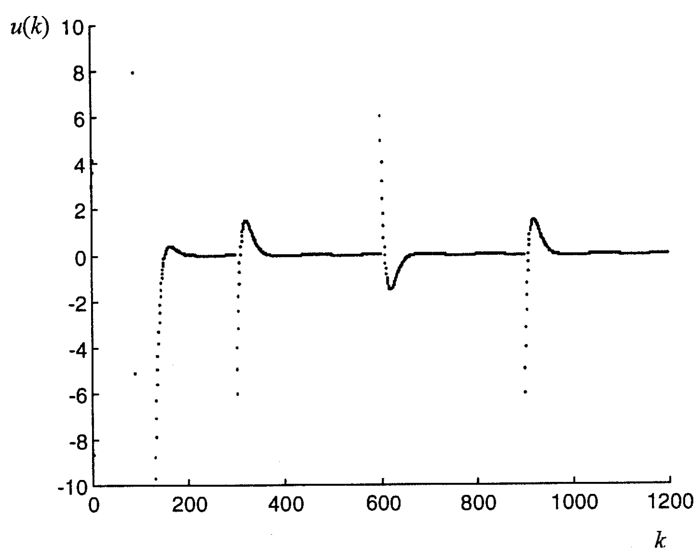

Fig. 4. Controller output $u(k)$ in the case of non-prespecified l_u.

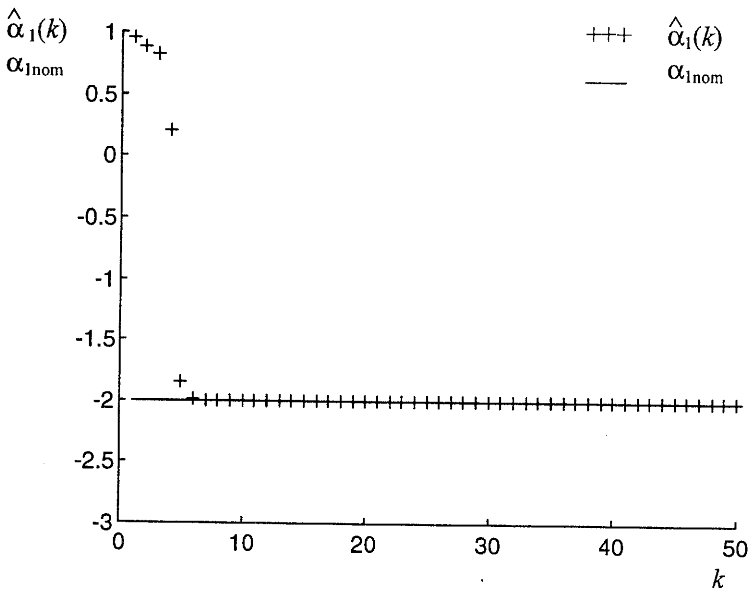

Fig. 5. Estimates of $\hat{\alpha}_1(k)$ versus $\alpha_{1\text{nom}}$ in the case of non-prespecified l_u.

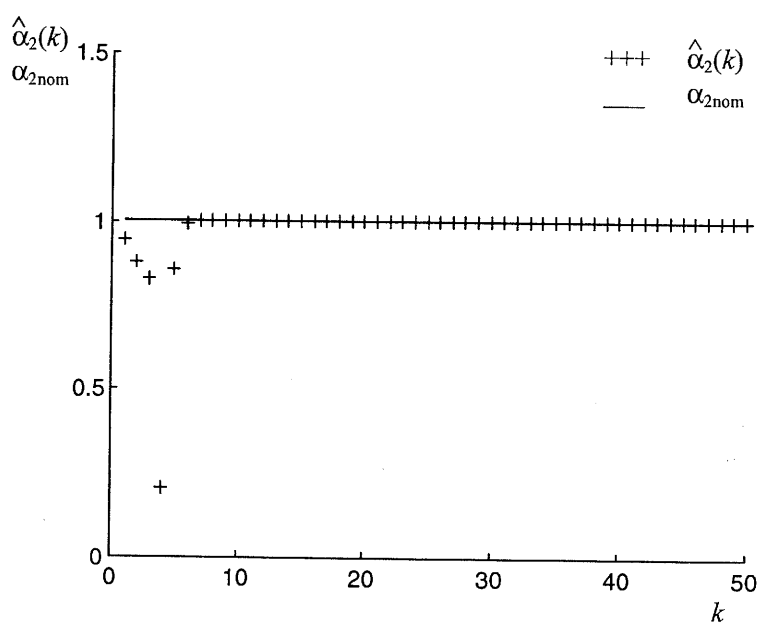

Fig. 6. Estimates of $\hat{\alpha}_2(k)$ versus $\alpha_{2\text{nom}}$ in the case of non-prespecified l_u.

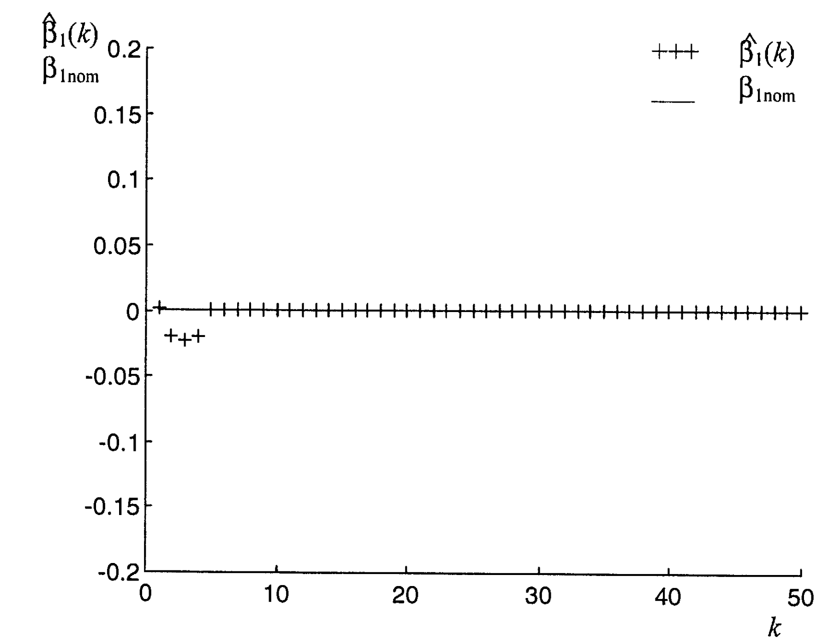

Fig. 7. Estimates of $\hat{\beta}_1(k)$ versus $\beta_{1\mathrm{nom}}$ in the case of non-prespecified l_u.

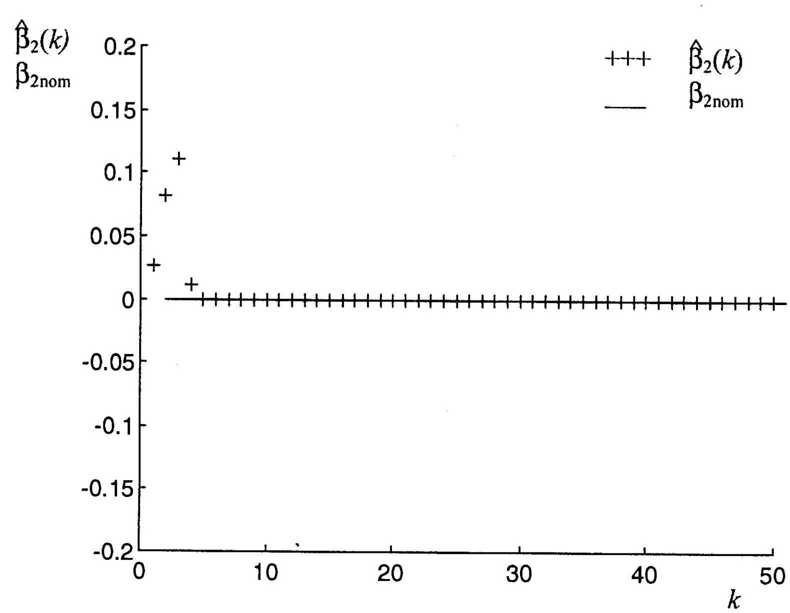

Fig. 8. Estimates of $\hat{\beta}_2(k)$ versus $\beta_{2\mathrm{nom}}$ in the case of non-prespecified l_u.

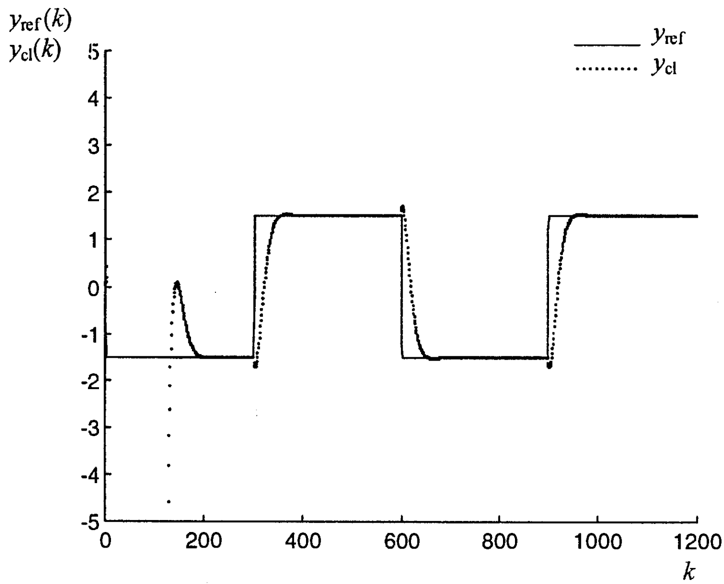

Fig. 9. Closed loop system output y_{cl} versus reference model output y_{ref} in the case of $l_u = 0$ (static MDOC).

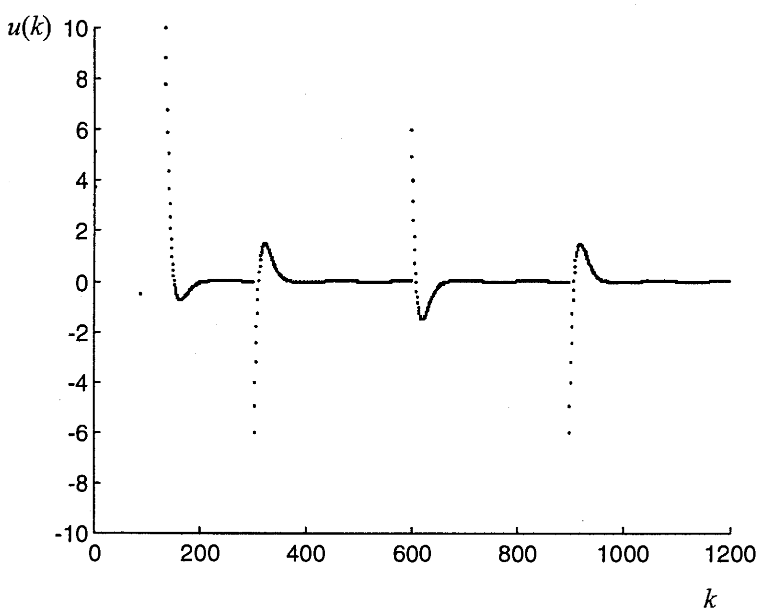

Fig. 10. Controller output $u(k)$ in the case of $l_u = 0$ (static MDOC).

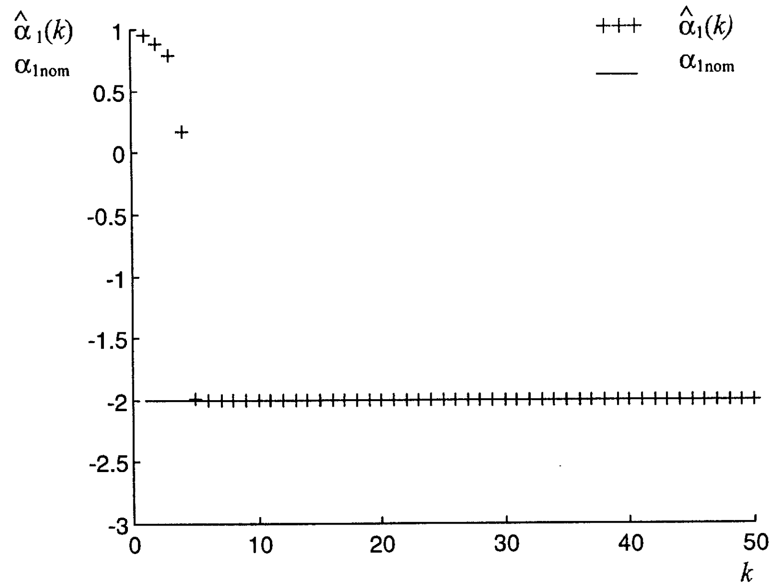

Fig. 11. Estimates of $\hat{\alpha}_1(k)$ versus α_{1nom} in the case of $l_u = 0$ (static MDOC).

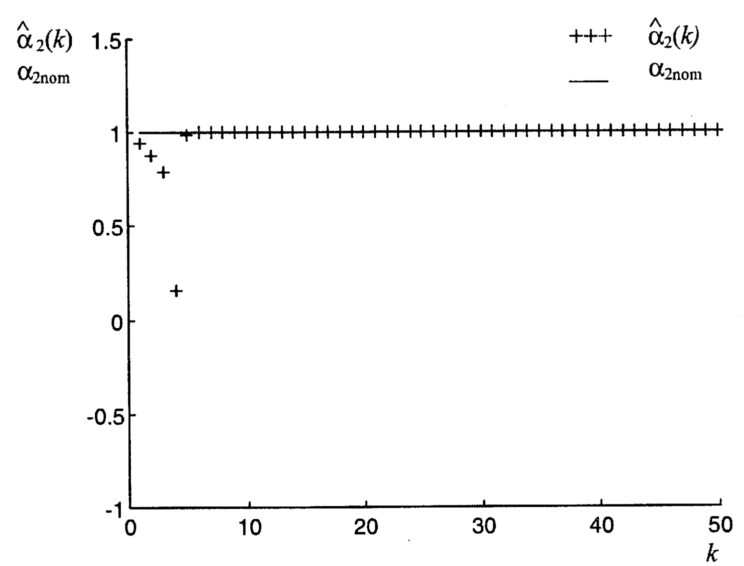

Fig. 12. Estimates of $\hat{\alpha}_2(k)$ versus α_{2nom} in the case of $l_u = 0$ (static MDOC).

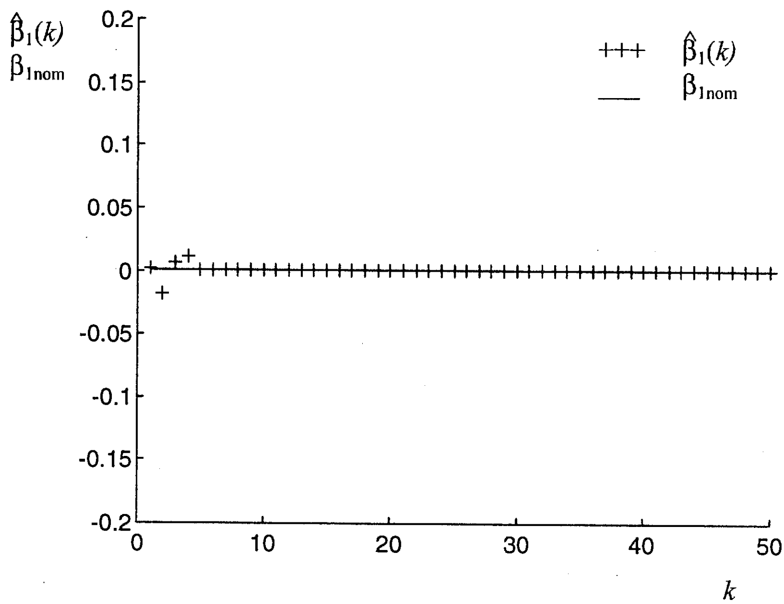

Fig. 13. Estimates of $\hat{\beta}_1(k)$ versus $\beta_{1\text{nom}}$ in the case of $l_u = 0$ (static MDOC).

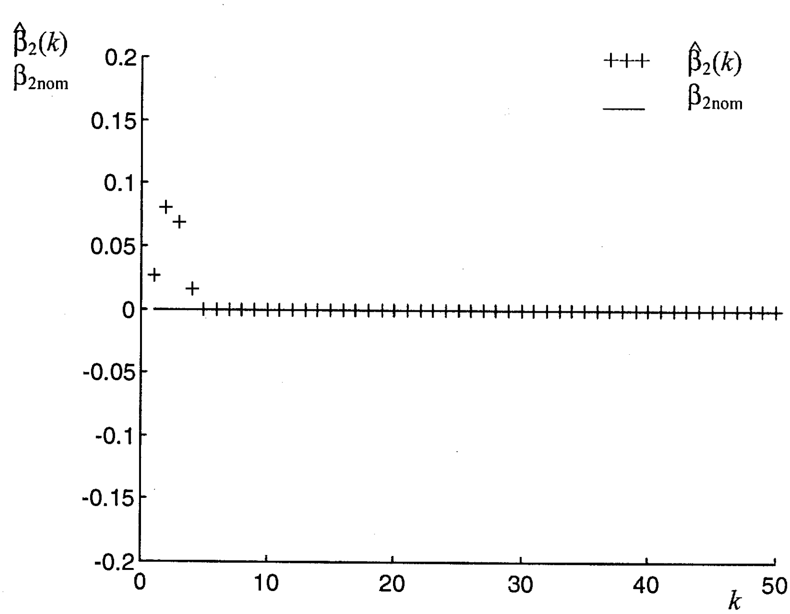

Fig. 14. Estimates of $\hat{\beta}_2(k)$ versus $\beta_{2\text{nom}}$ in the case of $l_u = 0$ (static MDOC).

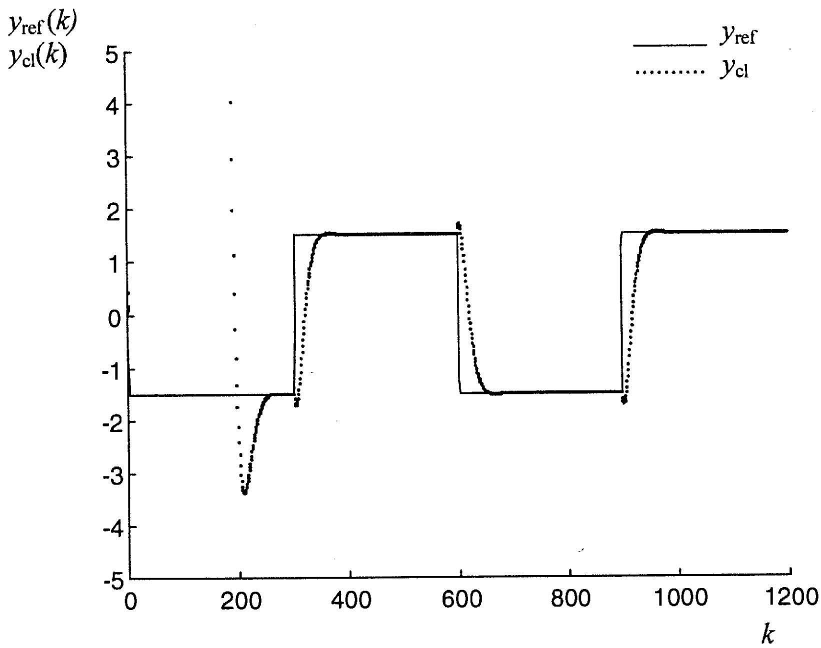

Fig. 15. Closed loop system output ycl versus reference model output y_{ref} in the case of $l_u = 0.5$ (stable dynamic MDOC).

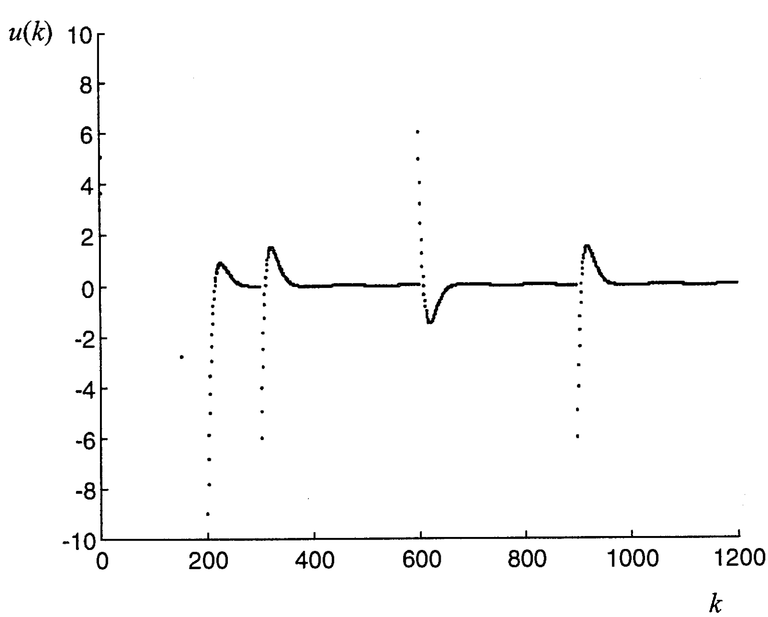

Fig. 16. Controller output $u(k)$ in the case of $l_u = 0.5$ (stable dynamic MDOC).

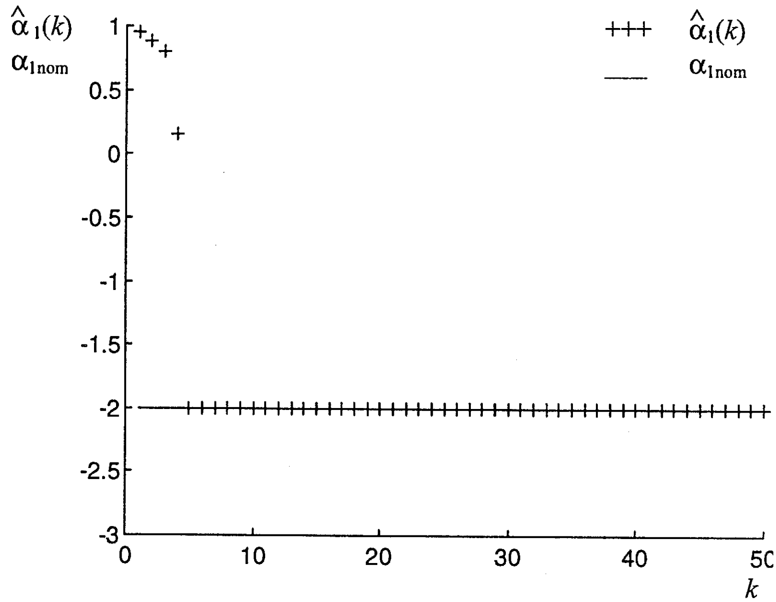

Fig. 17. Estimates of $\hat{\alpha}_1(k)$ versus $\alpha_{1\mathrm{nom}}$ in the case of $l_u = 0.5$ (stable dynamic MDOC).

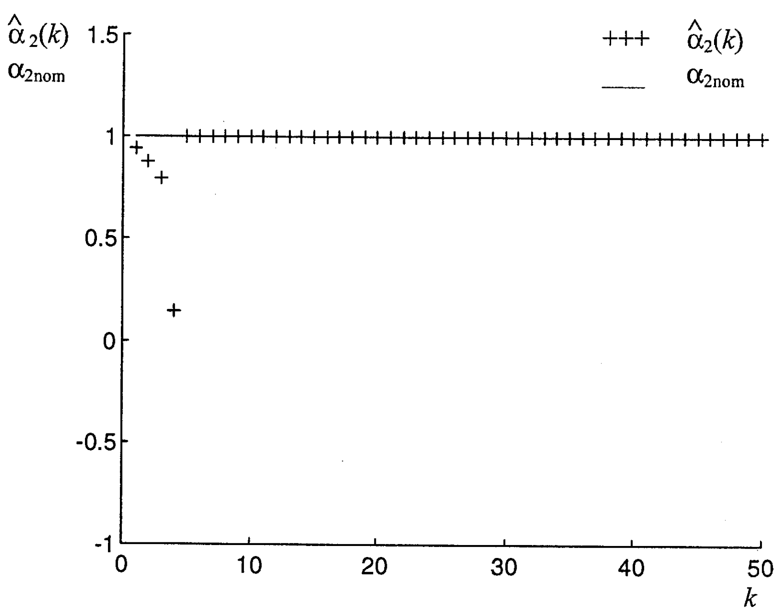

Fig. 18. Estimates of $\hat{\alpha}_2(k)$ versus $\alpha_{2\mathrm{nom}}$ in the case of $l_u = 0.5$ (stable dynamic MDOC).

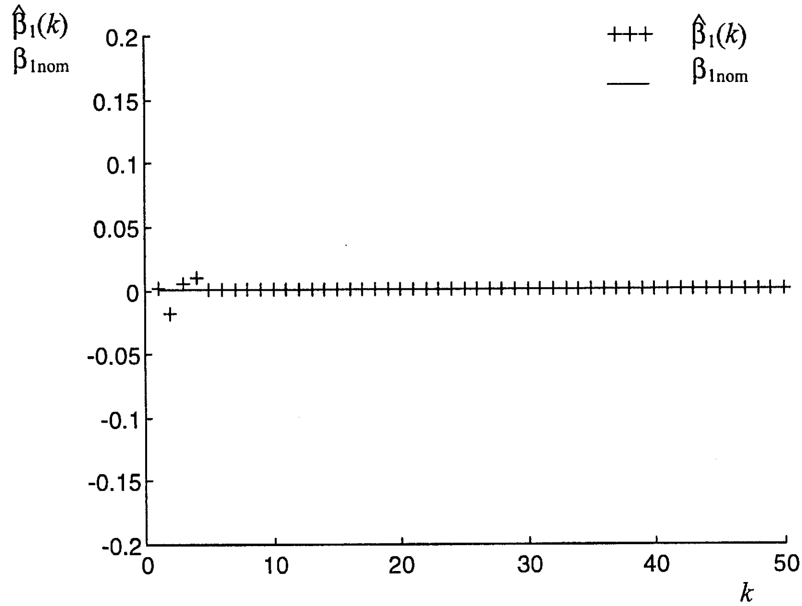

Fig. 19. Estimates of $\hat{\beta}_1(k)$ versus β_{1nom} in the case of $l_u = 0.5$ (stable dynamic MDOC).

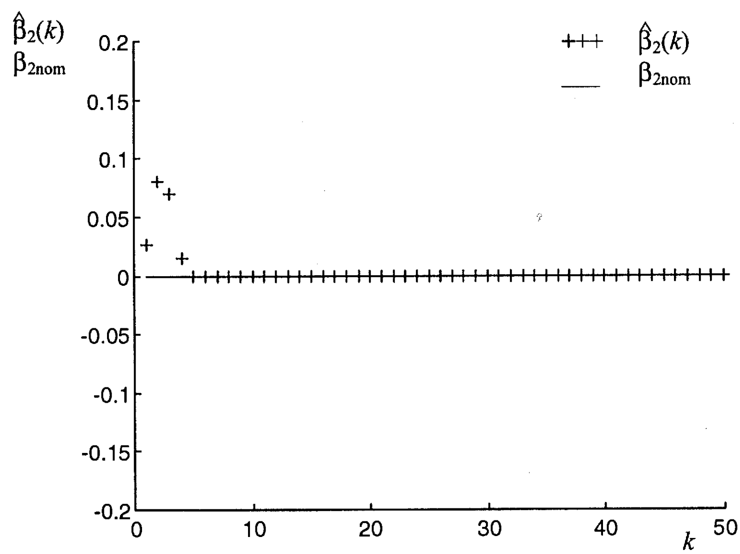

Fig. 20. Estimates of $\hat{\beta}_2(k)$ versus β_{2nom} in the case of $l_u = 0.5$ (stable dynamic MDOC).

The present approach may be extended to solve other important problems in the area of adaptive control, such as adaptive pole placement, adaptive LQ optimal regulation, decentralized adaptive control, etc., and for other types of systems, such as time-varying periodic and non-periodic linear multivariable ones. Adaptive schemes including robustness considerations are currently under investigation.

Appendices

A. Proof of Theorem 2

We will first prove that the matrix pair (35) is observable if the pair $\left(\boldsymbol{A},\ \boldsymbol{c}^T \right)$ is observable and the relation (34) simultaneously holds. To this end, let

$$\boldsymbol{A}^* = \left[\begin{array}{cc} \boldsymbol{A} & \boldsymbol{b} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right], \qquad \boldsymbol{c}^{*T} = \left[\begin{array}{cc} \boldsymbol{c}^T & 0 \end{array} \right] \tag{A1}$$

We next build the observability matrix $\mathbb{Q}$ of the pair $\left(\boldsymbol{A}^*,\ \boldsymbol{c}^{*T} \right)$ which, as can be shown, takes on the form

$$\mathbb{Q} = \left[\begin{array}{c|c|c|c} \boldsymbol{c} & \boldsymbol{A}^T\boldsymbol{c} & \cdots & (\boldsymbol{A}^T)^n\boldsymbol{c} \\ \hline 0 & \boldsymbol{b}^T\boldsymbol{c} & \cdots & \boldsymbol{b}^T(\boldsymbol{A}^T)^{n-1}\boldsymbol{c} \end{array} \right]$$

The matrix $\mathbb{Q}$ may further be written down as a product of two matrices $\mathbb{Q}_1$ and $\mathbb{Q}_2$, i.e. $\mathbb{Q} = \mathbb{Q}_1\mathbb{Q}_2$, where

$$\mathbb{Q}_1 = \left[\begin{array}{c|c} \boldsymbol{A}^T & \boldsymbol{c} \\ \hline \boldsymbol{b}^T & 0 \end{array} \right]$$

$$\mathbb{Q}_2 = \left[\begin{array}{c|c|c|c|c} 0 & \boldsymbol{c} & \boldsymbol{A}^T\boldsymbol{c} & \cdots & (\boldsymbol{A}^T)^{n-1}\boldsymbol{c} \\ \hline 1 & 0 & 0 & \cdots & 0 \end{array} \right]$$

Let us now check the rank of the matrix $\mathbb{Q}$. As is shown in (Gantmacher, 1959), the following matrix rank inequality holds:

$$\operatorname{rank}\mathbb{Q}_1 + \operatorname{rank}\mathbb{Q}_2 - q \le \operatorname{rank}\left(\mathbb{Q}_1\mathbb{Q}_2\right) \le \min\left\{\operatorname{rank}\mathbb{Q}_1, \operatorname{rank}\mathbb{Q}_2\right\}$$

where q is the common dimension of $\mathbb{Q}_1$ and $\mathbb{Q}_2$. Observe now that in our case

$$\operatorname{rank}\mathbb{Q}_1 = n+1, \quad \operatorname{rank}\mathbb{Q}_2 = n+1, \quad \min\left\{\operatorname{rank}\mathbb{Q}_1, \operatorname{rank}\mathbb{Q}_2\right\} = n+1, \quad q = n+1$$

Hence, $\operatorname{rank}\mathbb{Q} = \operatorname{rank}(\mathbb{Q}_1\,\mathbb{Q}_2) = n+1$ and the matrix pair (35) is observable.

Let us note that on the basis of (A1), the matrix $\left[\,\boldsymbol{H} \;\vdots\; \boldsymbol{d}\,\right]$ can be written as

$$\left[\,\boldsymbol{H} \;\vdots\; \boldsymbol{d}\,\right] = \begin{bmatrix} \boldsymbol{c}^{*T}\exp\left(-\boldsymbol{A}^{*}NT^{*}\right) \\ \boldsymbol{c}^{*T}\exp\left(-\boldsymbol{A}^{*}(N-1)T^{*}\right) \\ \vdots \\ \boldsymbol{c}^{*T}\exp\left(-\boldsymbol{A}^{*}T^{*}\right) \end{bmatrix} \tag{A2}$$

Multiplying both sides of (A2) by $\exp\left(\boldsymbol{A}^{*}T_0\right)$ yields

$$\left[\,\boldsymbol{H} \;\vdots\; \boldsymbol{d}\,\right]\exp\left(\boldsymbol{A}^{*}T_0\right) = \begin{bmatrix} \boldsymbol{c}^{*T} \\ \boldsymbol{c}^{*T}\exp\left(\boldsymbol{A}^{*}T^{*}\right) \\ \vdots \\ \boldsymbol{c}^{*T}\exp\left(\boldsymbol{A}^{*}(N-1)T^{*}\right) \end{bmatrix}$$

It is now easy to find a series of elementary transformations Ξ (with $\det\Xi \neq 0$) defined as follows:

$$\Xi\left[\,\boldsymbol{H} \;\vdots\; \boldsymbol{d}\,\right]\exp\left(\boldsymbol{A}^{*}T_0\right) = \begin{bmatrix} \boldsymbol{c}^{*T} \\ \boldsymbol{c}^{*T}\left\{\exp\left(\boldsymbol{A}^{*}T^{*}\right) - I\right\} \\ \vdots \\ \boldsymbol{c}^{*T}\left\{\exp\left(\boldsymbol{A}^{*}T^{*}\right) - I\right\}^{N-1} \end{bmatrix}$$

Due to the fact that the matrices Ξ and $\exp\left(\boldsymbol{A}^{*}T_0\right)$ are nonsingular, in order to prove Theorem 2, it is now sufficient to prove that the matrix

$$\Xi\left[\,\boldsymbol{H} \;\vdots\; \boldsymbol{d}\,\right]\exp\left(\boldsymbol{A}^{*}T_0\right) \tag{A3}$$

has full column rank if we choose $N \geq n+1$. To this end, dropping appropriate rows of the matrix $\Xi[\,\boldsymbol{H} \;\vdots\; \boldsymbol{d}\,]\exp\left(\boldsymbol{A}^{*}T_0\right)$, we obtain the following square matrix:

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{c}^{*T} \\ \boldsymbol{c}^{*T}\left\{\exp\left(\boldsymbol{A}^{*}T^{*}\right) - I\right\} \\ \vdots \\ \boldsymbol{c}^{*T}\left\{\exp\left(\boldsymbol{A}^{*}T^{*}\right) - I\right\}^{n} \end{bmatrix}$$

The matrix $\boldsymbol{\Omega}$ has the same rank as the matrix

$$\boldsymbol{\Omega}^{*} = \begin{bmatrix} \boldsymbol{c}^{*T} \\ \boldsymbol{c}^{*T}\left\{\exp\left(\boldsymbol{A}^{*}T^{*}\right) - I\right\}/T^{*} \\ \vdots \\ \boldsymbol{c}^{*T}\left\{\exp\left(\boldsymbol{A}^{*}T^{*}\right) - I\right\}^{n}/T^{*n} \end{bmatrix}$$

As the sampling period T_0 tends to zero, the matrix $\boldsymbol{\Omega}^{*}$ tends to the matrix

$$\hat{\boldsymbol{\Omega}} = \begin{bmatrix} \boldsymbol{c}^{*T} \\ \boldsymbol{c}^{*T}\boldsymbol{A}^{*} \\ \vdots \\ \boldsymbol{c}^{*T}\boldsymbol{A}^{*n} \end{bmatrix}$$

and the determinant of $\boldsymbol{\Omega}^{*}$ tends to the determinant of $\hat{\boldsymbol{\Omega}}$, which, due to the fact that the pair $\left(\boldsymbol{A}^{*},\ \boldsymbol{c}^{*T}\right)$ is observable, takes a non-zero value. This means that the determinant of the matrix of the form (A3) takes also a non-zero value for sufficiently small values of T_0. Due to the fact that the determinant of the matrix (A3) is an analytic function of T_0, we conclude that this determinant takes non-zero values for almost all sampling periods T_0. Consequently, the matrix $[\,\boldsymbol{H} \;\vdots\; \boldsymbol{d}\,]$ has full column rank for almost all sampling periods T_0.

B. Proof of Theorem 4

Observe that

$$\hat{\boldsymbol{A}} = \exp\left(\boldsymbol{A}NT^{*}\right), \quad \hat{\boldsymbol{b}} = \int_0^{NT^{*}} \exp\left(\boldsymbol{A}\lambda\right)\boldsymbol{b}\,\mathrm{d}\lambda$$

and the matrices $\boldsymbol{S}^{*}$ and

$$\boldsymbol{U} = \left[\,\hat{\boldsymbol{b}} \;\vdots\; \hat{\boldsymbol{A}}\hat{\boldsymbol{b}} \;\vdots\; \cdots \;\vdots\; \hat{\boldsymbol{A}}^{n-1}\hat{\boldsymbol{b}}\,\right]$$

are interrelated according to the following relationship:

$$\boldsymbol{S}^* = \boldsymbol{U}\Xi, \qquad \Xi = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Hence, if $\boldsymbol{U}$ is nonsingular for almost every T_0, so does $\boldsymbol{S}^*$. Now, define

$$\gamma(T^*) = \det \boldsymbol{U}$$

Since $\gamma(T^*)$ is an analytic function of T^*, in order to prove that $\gamma(T^*) \neq 0$, it is sufficient to prove that $\gamma^{(k)}(0) \neq (0)$ holds for some positive integer k. To this end, we proceed as follows:

First, observe that, by the formula about the differential of a determinant, we obtain

$$\gamma^{(k)}(T^*) = \sum \frac{k!}{\prod_{i=1}^{n} \lambda_i!} \det \left[\hat{\boldsymbol{b}}^{(\lambda_1)} \;\vdots\; \left(\hat{\boldsymbol{A}}\hat{\boldsymbol{b}}\right)^{(\lambda_2)} \;\vdots\; \cdots \;\vdots\; \left(\hat{\boldsymbol{A}}^{n-1}\hat{\boldsymbol{b}}\right)^{(\lambda_n)} \right] \tag{B1}$$

where the summation is carried out for $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = k$. Since

$$\left(\hat{\boldsymbol{A}}^{i-1}\hat{\boldsymbol{b}}\right)^{(k)}\Big|_{T^*=0} = \begin{cases} \boldsymbol{0} & \text{if} \quad k = 0 \\ \left\{ i^k - (i-1)^k \right\} \boldsymbol{A}^{k-1}\boldsymbol{b} & \text{if} \quad k \geq 1 \end{cases} \tag{B2}$$

we obtain

$$\gamma^{(k)}(0) = \sum \frac{k!}{\prod_{i=1}^{n} \lambda_i!} \prod_{i=1}^{n} \left\{ i^{\lambda_i} - (i-1)^{\lambda_i} \right\}$$
$$\times \det \left[\boldsymbol{A}^{\lambda_1 - 1}\boldsymbol{b} \;\vdots\; \boldsymbol{A}^{\lambda_2 - 1}\boldsymbol{b} \;\vdots\; \cdots \;\vdots\; \boldsymbol{A}^{\lambda_n - 1}\boldsymbol{b} \right] \tag{B3}$$

where the summation is made for

$$\lambda_i \geq 1, \qquad \sum_{i=1}^{n} \lambda_i = k \tag{B4}$$

We next focus our attention on the k_0-th derivative, where

$$k_0 = \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \tag{B5}$$

The determinant in (B3) does not reduce to zero if the λ_i's satisfy

$$\lambda_i \neq \lambda_j \quad (i \neq j) \tag{B6}$$

and from (B4) and (B5) we have

$$\sum_{i=1}^{n} \lambda_i = \frac{n(n+1)}{2} \tag{B7}$$

From (B6) and (B7) we obtain the following set

$$\left\{\lambda_1, \lambda_1, \ldots, \lambda_n\right\} \equiv \left\{1, 2, \ldots, n\right\} \tag{B8}$$

This means that, for $k = k_0 = n(n+1)/2$, the summation in (B3) needs to be made for λ_i, $i = 1, 2, \ldots, n$, given by (B8). Next, denote by σ the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$$

and let $\lambda_i = \sigma(i)$. Let also $\mathbb{S}$ be the symmetric group of permutations σ. Then the summation in (B3), for $k = k = n(n+1)/2$, can be written as

$$\gamma^{(k_0)}(0) = \sum \frac{k_0!}{\prod_{i=1}^{n} \sigma(i)!} \prod_{i=1}^{n} \left\{i^{\sigma(i)} - (i-1)^{\sigma(i)}\right\}$$

$$\times \det \left[A^{\sigma(1)-1}b \;\vdots\; A^{\sigma(2)-1}b \;\vdots\; \cdots \;\vdots\; A^{\sigma(n)-1}b \right] \tag{B9}$$

The relation (B9) can be rewritten as

$$\gamma^{(k_0)}(0) = \frac{k_0!}{\prod_{i=1}^{n} i!} \left\{ \sum_{\sigma \in \mathbb{S}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left\{i^{\sigma(i)} - (i-1)^{\sigma(i)}\right\} \right\}$$

$$\times \det \left[b \;\vdots\; Ab \;\vdots\; \cdots \;\vdots\; A^{n-1}b \right] \tag{B10}$$

Observe now that

$$\sum_{\sigma \in \mathbb{S}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left\{i^{\sigma(i)} - (i-1)^{\sigma(i)}\right\} = \det \boldsymbol{\Theta} \tag{B11}$$

where

$$\boldsymbol{\Theta} = \begin{bmatrix} 1^1 - 0^1 & \cdots & 1^n - 0^n \\ 2^1 - 1^1 & \cdots & 2^n - 1^n \\ \vdots & & \vdots \\ n^1 - (n-1)^1 & \cdots & n^n - (n-1)^n \end{bmatrix} \tag{B12}$$

Introducing (B11) in (B10) yields

$$\gamma^{(k_0)}(0) = \frac{k_0!}{\prod_{i=1}^{n} i!} \det \boldsymbol{\Theta} \det \left[\boldsymbol{b} \;\vdots\; \boldsymbol{A}\boldsymbol{b} \;\vdots\; \cdots \;\vdots\; \boldsymbol{A}^{n-1}\boldsymbol{b} \right] \tag{B13}$$

Observe also that

$$\det \boldsymbol{\Theta} = \det \begin{bmatrix} 1^1 & 1^2 & \cdots & 1^n \\ 2^1 & 2^2 & \cdots & 2^n \\ \vdots & & & \vdots \\ n^1 & n^2 & \cdots & n^n \end{bmatrix} = n! \prod_{1 \le \rho < q \le n} (q - \rho) = \prod_{i=1}^{n} i! \tag{B14}$$

Introducing (B14) in (B13), we obtain

$$\gamma^{(k_0)}(0) = k_0! \det \left[\boldsymbol{b} \;\vdots\; \boldsymbol{A}\boldsymbol{b} \;\vdots\; \cdots \;\vdots\; \boldsymbol{A}^{n-1}\boldsymbol{b} \right] \tag{B15}$$

Since the system (1) is assumed to be controllable, we finally obtain

$$\gamma^{(k_0)}(0) \neq 0 \tag{B16}$$

This completes the proof.

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