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SINGULARITY AVOIDANCE IN REDUNDANT ROBOT KINEMATICS: A DYNAMICAL SYSTEM APPROACH

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This paper is concerned with avoidability/unavoidability of singular configurations in the redundant manipulator's kinematics with an arbitrary degree of redundancy. The dynamical system approach has been adapted as a guiding line. A self-motion distribution has been defined, spanned by the so-called Hamiltonian vector fields associated with the given kinematics. The Hamiltonian vector fields are established to be divergence-free. Around singular configurations of corank 1 a reduction procedure is applied leading to a discovery of a common constant of motion of all vector fields belonging to the self-motion distribution. By examining the stability of the Hamiltonian vector fields sufficient conditions for avoidability and unavoidability are derived, formulated in terms of the Hessian matrix of the constant of motion.

1. Introduction

It is well-known that employing of the kinematic redundancy in the robotic manipulator design improves the robot performance and increases its dexterity. In particular, the kinematic redundancy may be used as a means to overcome the problem of kinematic singularities of a manipulator. Indeed, redundant manipulators possess a property of singularity avoidance that means that a prescribed end-effector position and orientation realized by a singular configuration of the manipulator may be produced alternatively by a non-singular configuration. To exploit the advantage of redundancy towards coping with the problem of singularity we need to be able to distinguish between avoidable and unavoidable singular configurations of a redundant manipulator. Several characterizations of avoidability/unavoidability have been provided in the robotic literature, based either on the dynamical system approach (Bedrossian, 1990; Bedrossian and Flueckiger, 1991; Flueckiger and Bedrossian, 1994; Shamir, 1990; Tchoń, 1997a; Tchoń and Matuszok, 1995) or on the singularity theory approach (Seng *et al.*, 1997; Tchoń, 1997b; Tchoń and Matuszok, 1995). Most of the mentioned papers address the avoidability/unavoidability problem for the kinematics with degree of redundancy 1 and formulate avoidability conditions for singular configurations of corank 1. Towards this aim a concept of the self-motion vector field is widely applied (Bedrossian and Flueckiger, 1991; Shamir, 1990; Tchoń and Matuszok, 1995). Singular configurations of higher coranks are treated in (Tchoń, 1997a). In

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(Tchoń and Matuszok, 1995) a relationship has been discovered between avoidability and normal forms of kinematic singularities. The kinematics with the degree of redundancy greater than 1 are studied in (Seng *et al.*, 1997; Tchoń, 1997b) using the methods of singularity theory. A sufficient condition for avoidability proposed in (Seng *et al.*, 1997) holds for arbitrary redundant kinematics with corank 1 singular configurations. In (Tchoń, 1997b) the normal form approach has been employed to derive sufficient conditions for avoidability and unavoidability of corank 1 singular configurations in redundant kinematics. These conditions apply, however, only to the kinematics that are equivalent to the quadratic normal form.

In this paper, we extend the dynamical system approach to the redundant kinematics with an arbitrary degree of redundancy. Our main tool is a concept of the Hamiltonian vector field introduced in (Arnold, 1993) to deal with singularities of the so-called complete intersections. Following this approach, we introduce a concept of the self-motion distribution, spanned by Hamiltonian vector fields, whose maximal integral manifolds coincide with the self-motion manifolds of the kinematics. It is shown that all the Hamiltonian vector fields are divergence-free, a result proved in (Tchoń and Matuszok, 1995) for the self-motion vector field. A simple avoidability condition is formulated in terms of the linear approximations to the Hamiltonian vector fields. A self-motion control system is introduced as a description of all self-motion trajectories of the redundant kinematics. Around singular configurations of corank 1 the Hamiltonian vector fields are transformed to a reduced form depending on $n-m+1$ coordinate functions. A first integral (a constant of motion) for these vector fields has been revealed corresponding to the self-motion Hamilton function considered in (Tchoń and Matuszok, 1995). Sufficient conditions for avoidability and unavoidability of singular configurations have been elaborated by examining stability conditions of the Hamiltonian vector fields, and expressed in terms of the Hessian matrix of this first integral. These conditions have a form similar to those recently obtained within the singularity theory approach in (Tchoń, 1997b), reducing in the case of the kinematics with redundancy degree 1 to the avoidability/unavoidability criteria proved in (Tchoń and Matuszok, 1995). The applicability of new conditions has been tested with an example of the $4R$ planar kinematics.

This paper is composed as follows. Section 2 presents basic concepts including an introduction of the self-motion distribution and a proof of the divergence-freeness of the Hamiltonian vector fields. In Section 3, after confining to singular configurations of corank 1, a reduction procedure of the self-motion distribution is accomplished and sufficient conditions for avoidability and unavoidability are derived. Section 4 presents an example. The paper is concluded with Section 5.

2. Basic Concepts

We shall investigate the forward kinematic map of a redundant robotic manipulator, referred to as the kinematics. In suitable coordinate systems in the joint manifold as well as in the task manifold the kinematics can be represented by an analytic map

$$k : \mathbb{R}^n \to \mathbb{R}^m, \qquad y = k(x) = \Big(k_1(x), k_2(x), \dots, k_m(x)\Big) \tag{1}$$

Since the kinematics are redundant, we shall always assume that $n > m$. A vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ denotes positions of the joints, and is called the manipulator's configuration, a vector $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$ stands for the position and orientation of the end-effector.

It is well-known that the configurations $x \in \mathbb{R}^n$ of the kinematics can be divided into *regular configurations* at which the rank of the Jacobian matrix of the kinematics is maximum,

$$\operatorname{rank} \frac{\partial k}{\partial x}(x) = m \tag{2}$$

and *singular configurations* at which the rank of the Jacobian matrix decreases, so

$$\operatorname{rank} \frac{\partial k}{\partial x}(x) < m \tag{3}$$

The difference

$$m - \operatorname{rank} \frac{\partial k}{\partial x}(x) \tag{4}$$

will be referred to as the *corank* of the configuration x. By analyticity of the map k the set of regular configurations is open and dense in $\mathbb{R}^n$. Then a set of singular configurations is analytic.

Let $x \in \mathbb{R}^n$ be a regular configuration of the kinematics k. Then the set

$$l(x) = \left\{ z \in k^{-1}(k(x)) \,\middle|\, \operatorname{rank} \frac{\partial k}{\partial x}(z) = m \right\}$$

is an $(n-m)$-dimensional analytic submanifold of $\mathbb{R}^n$ called a *leaf* passing through x. A collection of all leaves forms a foliation of the set of regular configurations. Connected components of the leaves will be referred to as the self-motion manifolds of the kinematics. Clearly, any two configurations belonging to the same self-motion manifold produce the same position and orientation of the end-effector, furthermore, they can be connected to each other by a continuous path lying entirely inside this self-motion manifold.

A singular configuration $x \in \mathbb{R}^n$ is called *avoidable*, if there exists a non-singular configuration x' included into the level set determined by $k(x)$, $x' \in k^{-1}(k(x))$, otherwise x is *unavoidable*. It turns out that avoidability of x means that the leaf $l(x') \subset k^{-1}(k(x))$, moreover this leaf is dense in the level set: $\overline{l(x')} = k^{-1}(k(x))$. Consequently, if a singular x is avoidable, then in any open neighbourhood of x one can find a regular configuration yielding the same position and orientation of the end-effector. This means that the avoidability is a local concept.

In order to characterize avoidable and unavoidable singular configurations of the kinematics k we need to introduce an infinitesimal description of the self-motion manifolds. This will be accomplished by defining a so-called *self-motion distribution* whose maximal integral manifolds coincide with the self-motion manifolds. The self-motion distribution is spanned by a collection of vector fields, named after (Arnold, 1993) *Hamiltonian vector fields*, that are defined below.

Definition 1. Given the kinematics $k(x) = (k_1(x), k_2(x), \ldots, k_m(x))$, we choose a sequence of integers $1 \le i_1 < \cdots < i_{m+1} \le n$ and define a *Hamiltonian vector field* $H_{i_1 \ldots i_{m+1}}(x)$ by a symbolic expansion with respect to the first row of the following determinant:

$$H_{i_1\ldots i_{m+1}}(x) = \det \begin{bmatrix} \frac{\partial}{\partial x_{i_1}} & \frac{\partial}{\partial x_{i_2}} & \cdots & \frac{\partial}{\partial x_{i_{m+1}}} \\ \frac{\partial k_1}{\partial x_{i_1}} & \frac{\partial k_1}{\partial x_{i_2}} & \cdots & \frac{\partial k_1}{\partial x_{i_{m+1}}} \\ \vdots & & & \vdots \\ \frac{\partial k_m}{\partial x_{i_1}} & \frac{\partial k_m}{\partial x_{i_2}} & \cdots & \frac{\partial k_m}{\partial x_{i_{m+1}}} \end{bmatrix} (x) \tag{5}$$

Expressed in components, the j-th component of the vector field

$$H_{i_1\ldots i_{m+1}}(x)_j = 0 \quad \text{if} \quad j \neq i_1, i_2, \ldots, i_{m+1}$$

and

$$H_{i_1\ldots i_{m+1}}(x)_j = (-1)^{r+1} \det \begin{bmatrix} \frac{\partial k_1}{\partial x_{i_1}} & \cdots & \frac{\partial k_1}{\partial x_{i_{r-1}}} & \frac{\partial k_1}{\partial x_{i_{r+1}}} & \cdots & \frac{\partial k_1}{\partial x_{i_{m+1}}} \\ \vdots & & & & & \vdots \\ \frac{\partial k_m}{\partial x_{i_1}} & \cdots & \frac{\partial k_m}{\partial x_{i_{r-1}}} & \frac{\partial k_m}{\partial x_{i_{r+1}}} & \cdots & \frac{\partial k_m}{\partial x_{i_{m+1}}} \end{bmatrix} (x) \tag{6}$$

if $j = i_r$, $r = 1, 2, \ldots, m+1$.

By definition, each Hamiltonian vector field is analytic, vanishing at the singular configurations.

Definition 2. The *self-motion distribution* Δ_k associated with the kinematics k is an analytic distribution spanned by the vector fields (5):

$$\Delta_k = \operatorname{span}_{C^\omega} \left\{ H_{i_1\ldots i_{m+1}} \,\middle|\, 1 \le i_1 < \cdots < i_{m+1} \le n \right\} \tag{7}$$

To illustrate the concept of the self-motion distribution we shall compute Δ_k for the kinematics of a planar $4R$ manipulator:

$$\begin{aligned} k(x) &= \big(k_1(x),\, k_2(x)\big) \\ &= (l_1 s_1 + l_2 s_{12} + l_3 s_{123} + l_4 s_{1234},\, l_1 c_1 + l_2 c_{12} + l_3 c_{123} + l_4 c_{1234}) \end{aligned} \tag{8}$$

where l_i, $i = 1, 2, 3, 4$ denote the link lengths, while the symbols s, c stand respectively for the sine and cosine functions, e.g. $s_{ij} = \sin(x_i + x_j)$, $c_{ij} = \cos(x_i + x_j)$. Since the manipulator has $n = 4$ d.o.f., there are four Hamiltonian vector fields distinguished by the indices (1,2,3), (1,2,4), (1,3,4), (2,3,4). We shall compute $H_{123}(x)$ explicitly using the expression (6):

$$\begin{cases} H_{123}(x)_1 = \det \begin{bmatrix} \dfrac{\partial k_1}{\partial x_2} & \dfrac{\partial k_1}{\partial x_3} \\ \dfrac{\partial k_2}{\partial x_2} & \dfrac{\partial k_2}{\partial x_3} \end{bmatrix}(x) = -l_2\left(l_3 s_3 + l_4 s_{34}\right) \\ H_{123}(x)_2 = -\det \begin{bmatrix} \dfrac{\partial k_1}{\partial x_1} & \dfrac{\partial k_1}{\partial x_3} \\ \dfrac{\partial k_2}{\partial x_1} & \dfrac{\partial k_2}{\partial x_3} \end{bmatrix}(x) = l_2 l_3 s_3 + l_2 l_4 s_{34} + l_1 l_3 s_{23} + l_1 l_4 s_{234} \\ H_{123}(x)_3 = \det \begin{bmatrix} \dfrac{\partial k_1}{\partial x_1} & \dfrac{\partial k_1}{\partial x_2} \\ \dfrac{\partial k_2}{\partial x_1} & \dfrac{\partial k_2}{\partial x_2} \end{bmatrix}(x) = -l_1\left(l_2 s_2 + l_3 s_{23} + l_4 s_{234}\right) \\ H_{123}(x)_4 = 0 \end{cases} \tag{9}$$

Similarly, the other Hamiltonian vector fields are computed as follows:

$$\begin{cases} H_{124}(x)_1 = -l_4\left(l_3 s_4 + l_2 s_{34}\right) \\ H_{124}(x)_2 = l_4\left(l_3 s_4 + l_2 s_{34} + l_1 s_{234}\right) \\ H_{124}(x)_3 = 0, \; H_{124}(x)_4 = l_1\left(l_2 s_2 + l_3 s_{23} + l_4 s_{234}\right) \end{cases} \tag{10}$$

$$\begin{cases} H_{134}(x)_1 = -l_3 l_4 s_4, \; H_{134}(x)_2 = 0 \\ H_{134}(x)_3 = -l_4\left(l_3 s_4 + l_2 s_{34} + l_1 s_{234}\right) \\ H_{134}(x)_4 = \left(l_2 l_3 s_3 + l_2 l_4 s_{34} + l_1 l_3 s_{23} + l_1 l_4 s_{234}\right) \end{cases} \tag{11}$$

$$\begin{cases} H_{234}(x)_1 = 0, \; H_{234}(x)_2 = l_3 l_4 s_4 \\ H_{234}(x)_3 = -l_4\left(l_3 s_4 + l_2 s_{34}\right) \\ H_{234}(x)_4 = l_2\left(l_3 s_3 + l_4 s_{34}\right) \end{cases} \tag{12}$$

The self-motion distribution Δ_k corresponding to the kinematics (8) consists of linear combinations of Hamiltonian vector fields $H_{123}(x)$, $H_{124}(x)$, $H_{134}(x)$, $H_{234}(x)$ multiplied by any analytic functions of x.

Now let us return to the general case and examine the Hamiltonian vector fields $H_{i_1 \ldots i_{m+1}}(x)$ associated with the given kinematics k. The following result generalizes

Proposition 1 proved in (Tchoń and Matuszok, 1995):

Proposition 1. *The vector fields* $H_{i_1 \dots i_{m+1}}(x)$, $1 \le i_1 < \dots < i_{m+1} \le n$, *are divergence-free, i.e.*

$$\operatorname{div} H_{i_1 \dots i_{m+1}}(x) = \operatorname{tr} \frac{\partial H_{i_1 \dots i_{m+1}}(x)}{\partial x} = 0 \tag{13}$$

Proof. Clearly,

$$\operatorname{div} H_{i_1 \dots i_{m+1}}(x) = \sum_{r=1}^{m+1} \frac{\partial H_{i_1 \dots i_{m+1}}(x)_{i_r}}{\partial x_{i_r}}$$

so using (6) we need to compute the partial derivative

$$\frac{\partial H_{i_1 \dots i_{m+1}}(x)_{i_r}}{\partial x_{i_r}} = (-1)^{r+1} \frac{\partial}{\partial x_{i_r}} \det \left[\frac{\partial k}{\partial x_{i_1}} \cdots \frac{\partial k}{\partial x_{i_{r-1}}} \frac{\partial k}{\partial x_{i_{r+1}}} \cdots \frac{\partial k}{\partial x_{i_{m+1}}} \right](x)$$

But from the multilinearity of the determinant we deduce

$$\frac{\partial H_{i_1 \dots i_{m+1}}(x)_{i_r}}{\partial x_{i_r}} = (-1)^{r+1} \sum_{j=1, j\neq r}^{m+1} \det \left[\frac{\partial k}{\partial x_{i_1}} \cdots \frac{\partial^2 k}{\partial x_{i_j} \partial x_{i_r}} \cdots \frac{\partial k}{\partial x_{i_{m+1}}} \right](x)$$

The sum on the right-hand side can be divided into two parts yielding finally

$$\begin{aligned} \operatorname{div} H_{i_1 \dots i_{m+1}}(x) &= \sum_{r=1}^{m+1} \sum_{j=1}^{r-1} (-1)^{r+1} \det \left[\frac{\partial k}{\partial x_{i_1}} \cdots \frac{\partial^2 k}{\partial x_{i_j} \partial x_{i_r}} \cdots \frac{\partial k}{\partial x_{i_{m+1}}} \right](x) \\ &+ \sum_{r=1}^{m+1} \sum_{j=r+1}^{m+1} (-1)^{r+1} \det \left[\frac{\partial k}{\partial x_{i_1}} \cdots \frac{\partial^2 k}{\partial x_{i_j} \partial x_{i_r}} \cdots \frac{\partial k}{\partial x_{i_{m+1}}} \right](x) \end{aligned}$$

Denoting respective terms in the sums above by M_{jr} and N_{jr}, we conclude that the divergence is equal to

$$\operatorname{div} H_{i_1 \dots i_{m+1}}(x) = \sum_{r=1}^{m+1} \sum_{j=1}^{r-1} (-1)^{r+1} M_{jr} + \sum_{r=1}^{m+1} \sum_{j=r+1}^{m+1} (-1)^{r+1} N_{jr}$$

Now we use the identity

$$\sum_{r=1}^{m+1} \sum_{j=1}^{r-1} = \sum_{j=1}^{m+1} \sum_{r=j+1}^{m+1}$$

swap indices of summation, note that by the equality of mixed partial derivatives $M_{rj} = (-1)^{r-j-1} N_{jr}$, and conclude that

$$\begin{aligned}
\operatorname{div} H_{i_1\dots i_{m+1}}(x) &= \sum_{j=1}^{m+1} \sum_{r=j+1}^{m+1} (-1)^{r+1} M_{jr} + \sum_{r=1}^{m+1} \sum_{j=r+1}^{m+1} (-1)^{r+1} N_{jr} \\
&= \sum_{r=1}^{m+1} \sum_{j=r+1}^{m+1} (-1)^{j+1} M_{rj} + \sum_{r=1}^{m+1} \sum_{j=r+1}^{m+1} (-1)^{r+1} N_{jr} \\
&= \sum_{r=1}^{m+1} \sum_{j=r+1}^{m+1} (-1)^{r} (N_{jr} - N_{jr}) = 0
\end{aligned}$$

■

We have proved that the Hamiltonian vector fields are divergence-free which yields that the self-motion distribution consists of divergence-free vector fields exclusively. The self-motion distribution generates all self-motions of the kinematics that can be treated as trajectories of the following self-motion control system:

$$\dot{x} = \sum_{1 \le i_1 < \dots < i_{m+1} \le n} H_{i_1\dots i_{m+1}}(x)\, u_{i_1\dots i_{m+1}} \tag{14}$$

Locally, the number of controls in (14) can be reduced to $n - m$ by a suitable choice of a local basis of Δ_k.

Having determined the generators of the self-motion distribution we shall examine invariant manifolds of every Hamiltonian vector field. Being divergence-free these vector fields have either the stable, the unstable and the centre invariant manifold or only the centre manifold (Guckenheimer and Holmes, 1983). In the former case, the following sufficient condition for avoidability of singular configurations holds:

Theorem 1. *Let x_0 be a singular configuration of the kinematics k (x_0 is then an equilibrium point of every Hamiltonian vector field). Suppose that there exists a vector field $H_{i_1\dots i_{m+1}}(x)$ whose linear approximation at x_0, $\partial H_{i_1\dots i_{m+1}}(x_0)/\partial x$, has an eigenvalue with a non-zero real part. Then x_0 is avoidable.*

3. Corank 1 Singular Configurations

Suppose that $x_0 \in \mathbb{R}^n$ is a corank 1 singular configuration of the kinematics $k = (k_1, \dots, k_m)$. Without any loss of generality, we may assume that the upper left submatrix of the Jacobian matrix of k has full rank at x_0, i.e.

$$\operatorname{rank} \begin{bmatrix} \dfrac{\partial k_1}{\partial x_1} & \cdots & \dfrac{\partial k_1}{\partial x_{m-1}} \\ \vdots & & \vdots \\ \dfrac{\partial k_{m-1}}{\partial x_1} & \cdots & \dfrac{\partial k_{m-1}}{\partial x_{m-1}} \end{bmatrix} (x_0) = m - 1 \tag{15}$$

By (15) the following map is clearly a local diffeomorphism around x_0

$$\xi = \varphi(x) = \Big(k_1(x),\ \ldots,\ k_{m-1}(x),\ x_m,\ \ldots,\ x_n \Big) \tag{16}$$

In new coordinates the kinematics are expressed as

$$K(\xi) = \Big(\xi_1,\ \xi_2,\ \ldots,\ \xi_{m-1},\ h(\xi) \Big) \tag{17}$$

where $h(\xi)$ is analytic and verifies the condition

$$h \circ \varphi(x) = k_m(x) \tag{18}$$

Let us compute the Hamiltonian vector fields for the transformed kinematics (17). Because the Jacobian matrix of K takes the form

$$\frac{\partial K}{\partial \xi}(\xi) = \begin{bmatrix} I_{m-1} & 0 & \cdots & 0 \\ \dfrac{\partial h}{\partial \xi^{m-1}} & \dfrac{\partial h}{\partial \xi_m} & \cdots & \dfrac{\partial h}{\partial \xi_n} \end{bmatrix}(\xi) \tag{19}$$

it is easily seen that according to the expression (6) we obtain for $i = 1, 2, \ldots, n-m$

$$H_{1\,2\ldots m\,m+i}(\xi) = (-1)^{m+1} \frac{\partial h}{\partial \xi_{m+i}} \frac{\partial}{\partial \xi_m} + (-1)^{m+2} \frac{\partial h}{\partial \xi_m} \frac{\partial}{\partial \xi_{m+i}} \tag{20}$$

i.e. only the components number m and $m+i$ of $H_{1\,2\ldots m\,m+i}(\xi)$ are non-zero. The Hamiltonian vector fields corresponding to other choices of indices $1 \le i_1 < \cdots < i_{m+1} \le n$ are zero vector fields. We assert that locally, in a certain neighbourhood of $\xi_0 = \varphi(x_0)$, the self-motion distribution is spanned by $(n-m)$ vector fields (20). Obviously, these vector fields are locally conjugate to the Hamiltonian vector fields (5). Furthermore, since all the vector fields $H_{1\,2\ldots m\,m+i}(\xi)$ have zero components along $\xi_1, \xi_2, \ldots, \xi_{m-1}$ coordinates, these coordinates must remain constant on the maximal integral manifolds of the self-motion distribution. Consequently, we can accomplish a reduction of the vector fields (20) to the remaining $(n-m+1)$ coordinates $\xi_m, \xi_{m+1}, \ldots, \xi_n$, cf. (Tchoń, 1997a; Tchoń and Matuszok, 1995). To this aim, we define new variables $\theta_i = \xi_{m+i}$, $i = 0, 1, \ldots, n-m$ and introduce a function

$$H(\theta_0, \theta_1, \ldots, \theta_{n-m}) = h\Big(k_1(x_0),\ k_2(x_0),\ \ldots,\ k_{m-1}(x_0),\ \theta_0,\ \ldots,\ \theta_{n-m}\Big) \tag{21}$$

After this restriction, the Hamiltonian vector fields (20) assume the following form for $i = 1, 2, \ldots, n-m$:

$$H_{1\,2\ldots m\,m+i}(\theta) = (-1)^{m+1} \frac{\partial H}{\partial \theta_i} \frac{\partial}{\partial \theta_0} + (-1)^{m+2} \frac{\partial H}{\partial \theta_0} \frac{\partial}{\partial \theta_i} \tag{22}$$

The vector fields (22) span the reduced self-motion distribution Δ_k^r. In the case of the kinematics with degree of redundancy 1, we have shown in (Tchoń and Matuszok, 1995) that the corresponding vector field (22) is indeed Hamiltonian with (21) playing the role of a Hamilton function. The reduced Hamiltonian vector fields enjoy the following remarkable property:

Proposition 2. *The function $H(\theta_0, \theta_1, \ldots, \theta_{n-m})$ is the first integral (a constant of motion) of each vector field from the reduced self-motion distribution, i.e. for any $V \in \Delta_k^r$ the directional derivative*

$$\mathrm{d}H\, V = 0 \tag{23}$$

Proof. It suffices to compute (23) for the vector fields (22) spanning Δ_k^r. Indeed,

$$\begin{aligned} \mathrm{d}H\, H_{1\,2\ldots m\,m+i} &= (-1)^{m+1} \frac{\partial H}{\partial \theta_i} \frac{\partial H}{\partial \theta_0} + (-1)^{m+2} \frac{\partial H}{\partial \theta_0} \frac{\partial H}{\partial \theta_i} \\ &= (-1)^{m+1} \left(\frac{\partial H}{\partial \theta_i} \frac{\partial H}{\partial \theta_0} - \frac{\partial H}{\partial \theta_0} \frac{\partial H}{\partial \theta_i} \right) = 0 \end{aligned}$$

■

The knowledge of $H(\theta_0, \theta_1, \ldots, \theta_{n-m})$ being the first integral of all the vector fields $V \in \Delta_k^r$ allows us to employ a stability theorem of Dirichlet (Siegel and Moser, 1971) to derive the following sufficient condition for avoidability:

Theorem 2. *Let $x_0 = (x_{01}, \ldots, x_{0m-1}, x_{0m}, \ldots, x_{0n}) \in \mathbb{R}^n$ denote a singular configuration of the kinematics k. Define $\overline{\theta}_0 = x_{0m}, \ldots, \overline{\theta}_{n-m} = x_{0n}$. Suppose that the Hessian matrix*

$$\frac{\partial^2 H}{\partial \theta^2} \left(\overline{\theta}_0, \ldots, \overline{\theta}_{n-m}\right) \tag{24}$$

is definite (positive or negative). Then x_0 is unavoidable. The matrix (24) can be computed effectively in the original coordinates of the kinematics k in the following manner:

$$\begin{aligned} \frac{\partial^2 H}{\partial \theta^2} \left(\overline{\theta}_0, \ldots, \overline{\theta}_{n-m}\right) &= -M^T (X - P) M \\ &\quad - \left(M^T (Y - Q) + \left(M^T (Y - Q) \right)^T \right) + (W - S) \end{aligned} \tag{25}$$

where

$$\begin{cases} M = \left(\dfrac{\partial \underline{k}}{\partial \underline{x}} \right)^{-1} \dfrac{\partial \underline{k}}{\partial \overline{x}}, \quad X = \dfrac{\partial^2 k_m}{\partial \underline{x}^2}, \quad Y = \dfrac{\partial^2 k_m}{\partial \overline{x} \partial \underline{x}}, \quad W = \dfrac{\partial^2 k_m}{\partial \overline{x}^2} \\ P_{ij} = \dfrac{\partial k_m}{\partial \underline{x}} \left(\dfrac{\partial \underline{k}}{\partial \underline{x}} \right)^{-1} \dfrac{\partial^2 \underline{k}}{\partial x_i \partial x_j}, \quad i, j \le m-1 \\ Q_{ij} = \dfrac{\partial k_m}{\partial \underline{x}} \left(\dfrac{\partial \underline{k}}{\partial \underline{x}} \right)^{-1} \dfrac{\partial^2 \underline{k}}{\partial x_i \partial x_j}, \quad i \le m-1,\ m \le j \le n \\ S_{ij} = \dfrac{\partial k_m}{\partial \underline{x}} \left(\dfrac{\partial \underline{k}}{\partial \underline{x}} \right)^{-1} \dfrac{\partial^2 \underline{k}}{\partial x_i \partial x_j}, \quad m \le i, j \le n \end{cases} \tag{26}$$

In expressions (26) $\underline{k} = (k_1, \ldots, k_{m-1})$, $\underline{x} = (x_1, \ldots, x_{m-1})$, $\overline{x} = (x_m, \ldots, x_n)$ and all the partial derivatives should be computed at the singular configuration x_0.

Remark 1. The function $H(\theta_0, \theta_1, \ldots, \theta_{n-m})$ may serve as a starting point to define a Lyapunov function for $V \in \Delta_k^r$. It is worth noting that, although derived here in a different way, the condition (24) coincides with that proposed in (Tchoń, 1997b).

Now let us examine the property of avoidability. By analyzing the linear approximation to the reduced Hamiltonian vector fields $H_{1\,2\ldots m\,m+i}(\theta)$ defined by (22) and after taking into consideration Theorem 1 we arrive at the following conclusion:

Theorem 3. *Suppose that* $x_0 = (x_{01}, \ldots, x_{0m-1}, x_{0m}, \ldots, x_{0n}) \in \mathbb{R}^n$ *denotes a singular configuration of the kinematics* k. *Let for a certain* $i = 1, 2, \ldots, n-m$

$$\frac{\partial^2 H}{\partial \theta_0^2} \frac{\partial^2 H}{\partial \theta_i^2} - \left(\frac{\partial^2 H}{\partial \theta_0 \partial \theta_i} \right)^2 < 0 \tag{27}$$

Then the singular configuration x_0 *is avoidable. The partial derivatives in (27) are taken at* $\overline{\theta}_0 = x_{0m}, \ldots, \overline{\theta}_{n-m} = x_{0n}$. *Suitable expressions in the original coordinates are easily derived from (25), (26).*

Remark 2. The condition (27) for the kinematics with redundancy degree 1 has been proved in (Tchoń and Matuszok, 1995).

4. Example

We shall apply the sufficient conditions provided by Theorems 2 and 3 in order to establish avoidability or unavoidability of singular configurations of the $4R$ planar kinematics defined by (8). It is immediate to observe that singular configurations of the kinematics (8) take the form $x_0 = (x_{01}, x_{02}, x_{03}, x_{04})$, where $x_{0i} = 0, \pm\pi$. In what follows, we shall concentrate on a pair of singular configurations $x_0^1 = (0,0,0,0)$ (all four links stretched out) and $x_0^2 = (0,0,0,\pi)$ (the links 1–3 stretched out, the 4-th link folded down). Assisted moderately by the symbolic computation in MATHEMATICA we have found from the expressions (26)

$$\frac{\partial^2 H}{\partial \theta^2}\left(x_0^1\right) = -l_{1234}^{-1} \begin{bmatrix} l_1 l_{234} & l_1 l_{34} & l_1 l_4 \\ l_1 l_{34} & l_{12} l_{34} & l_{12} l_4 \\ l_1 l_4 & l_{12} l_4 & l_{123} l_4 \end{bmatrix} \tag{28}$$

with notations $l_{ij} = l_i + l_j$, etc. The matrix (28) is negative definite, therefore by Theorem 2 the configuration x_0^1 is unavoidable. An analogous computation for x_0^2 has provided

$$\frac{\partial^2 H}{\partial \theta^2}\left(x_0^2\right) = (l_{123} - l_4)^{-1} \begin{bmatrix} -l_1(l_{23} - l_4) & -l_1(l_3 - l_4) & l_1 l_4 \\ -l_1(l_3 - l_4) & -l_{12}(l_3 - l_4) & l_{12} l_4 \\ l_1 l_4 & l_{12} l_4 & l_{123} l_4 \end{bmatrix} \tag{29}$$

valid under assumption that $l_{123} \neq l_4$. An application of Theorem 3 results in the following avoidability conditions ($H_{ij} = \partial^2 H / \partial\theta_i \partial\theta_j$):

$$H_{00}H_{11} - H_{01}^2 = l_1 l_2 (l_3 - l_4) < 0 \implies l_3 < l_4$$

$$H_{00}H_{22} - H_{02}^2 = -l_1 l_4 l_{23} < 0, \quad \text{that is always satisfied}$$

Eventually, we conclude that x_0^2 is avoidable independently of any particular values assumed by geometric parameters of the manipulator. Both the avoidability and the unavoidability result obtained above can be verified immediately by inspection of the $4R$ kinematics under consideration.

5. Conclusions

Starting from the concept of the Hamiltonian vector fields we have extended the dynamical system approach to deriving avoidability/unavoidability conditions of singular configurations in the redundant manipulator's kinematics with an arbitrary degree of redundancy. The self-motion distribution has been introduced as an infinitesimal characterization of the self-motion manifolds of the kinematics. Self motions generated by the kinematics have been described by the self-motion control system. Sufficient conditions for avoidability and unavoidability have been delivered, expressed in terms of the Hessian matrix of a constant of motion of all the vector fields belonging to the self-motion distribution. The results obtained in (Tchoń and Matuszok, 1995) for the case of the degree of redundancy 1 have been generalized to the kinematics with redundancy degree ≥ 2. They are also comparable to the avoidability/unavoidability conditions found in (Tchoń, 1997b) using the normal form approach of the singularity theory. We believe that by a cross-section of the present approach with the differential form approach elaborated in (Tchoń, 1997a) one should be able to establish avoidability/unavoidability conditions for arbitrary redundant kinematics with singular configurations of arbitrary coranks.

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