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# AN EXTENSION OF THE CAYLEY-HAMILTON THEOREM FOR A STANDARD PAIR OF BLOCK MATRICES

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The Cayley-Hamilton theorem is extended for a standard pair of matrices partitioned into blocks that commute in pairs. The Victoria theorem (Victoria, 1982) is a particular case for $E = I$ of the extended Cayley-Hamilton theorem. The new theorem is illustrated by an example. Some remarks on extension of the theorem for non-square block matrices are also given.

## 1. Introduction

The Cayley-Hamilton theorem plays an important role in linear algebra, linear networks and automatic control systems (Chang and Chen, 1992; Fragulis, 1995; Gantmacher, 1974; Kaczorek, 1992; 1993; Lancaster, 1969; Lewis, 1982; 1986; Mertzios, 1989). The theorem says that every square matrix satisfies its own characteristic equation (Gantmacher, 1974; Kaczorek, 1992; 1993; Lancaster, 1969). The classical Cayley-Hamilton theorem was extended for pairs of square matrices (Chang and Chen, 1992; Lewis, 1982; 1986), square block matrices (Victoria, 1982) and for two-dimensional (2D) and $n$D ($n > 2$) linear systems described by the Roesser model or by the general model (Kaczorek, 1992; 1993; Mertzios, 1989; Mertzios and Christodoulou, 1986; Smart and Barnett, 1989; Theodoru, 1989). Recently in (Kaczorek, 1994; 1995a; 1995b) the Cayley-Hamilton theorem was extended for non-square matrices, non-square block matrices and for singular 2D linear systems with non-square matrices. In (Fragulis, 1995) the Cayley-Hamilton theorem was extended for polynomial matrices of an arbitrary degree. In the analysis and synthesis of generalized control systems we deal with standard pairs of block matrices (Kaczorek, 1992; 1993).

In this paper, the Cayley-Hamilton theorem will be extended for a standard pair of matrices partitioned into blocks that commute in pairs. The Victoria theorem (Victoria, 1982) is a particular case for $E = I$ of the theorem given in this paper. The extended Cayley-Hamilton theorem can be used e.g. for computing the inverse matrix of a block matrix and in the analysis of linear systems consisting of subsystems.

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## 2. Preliminaries

Let $P_n(\mathbb{C})$ be the set of $n$-order square complex matrices that commute in pairs and $M_m(P_n)$ be the set of square matrices partitioned in $m^2$ blocks belonging to $P_n(\mathbb{C})$. The Kronecker product $\otimes$ of the block matrix

$$A = \begin{bmatrix} A_{11} \dots A_{1m} \\ \dots\dots\dots\dots \\ A_{m1} \dots A_{mm} \end{bmatrix}, \quad A_{ij} \in P_n(\mathbb{C}) \tag{1}$$

and a matrix $B \in \mathbb{C}^{n \times n}$ is defined by

$$A \otimes B := \begin{bmatrix} A_{11}B \dots A_{1m}B \\ \dots\dots\dots\dots\dots \\ A_{m1}B \dots A_{mm}B \end{bmatrix} \tag{2}$$

where $\mathbb{C}^{n \times n}$ is the set of $n \times n$ complex matrices.

**Definition 1.** A pair of block matrices $(E, A)$ is called *standard* if there exist scalars $\alpha$ and $\beta$ such that

$$E\alpha + A\beta = I \quad \text{(the identity matrix)} \tag{3}$$

**Lemma 1.** *If the pair $(E, A)$ is standard, then it is also commutative, i.e.*

$$EA = AE \tag{4}$$

*Proof.* Let $\beta \neq 0$. From (3) we have

$$A = \frac{1}{\beta} I - \frac{\alpha}{\beta} E$$

and

$$EA = E\left(\frac{1}{\beta} I - \frac{\alpha}{\beta} E\right) = \left(\frac{1}{\beta} I - \frac{\alpha}{\beta} E\right) E = AE$$

If $\alpha \neq 0$, the proof is similar. ■

**Lemma 2.** *Let the pair $(E, A)$ be standard.*

*(i) If $E \in M_m(P_n)$ and $\beta \neq 0$, then $A \in M_m(P_n)$.*

*(ii) If $E$ is symmetric and $\beta \neq 0$, then $A$ is also symmetric.*

*Proof.* Let $\beta \neq 0$. From (3) we have $A = (1/\beta)I - (\alpha/\beta)E$ and $A \in M_n(P_n)$, since by assumption $E \in M_m(P_n)$. ■

In Lemma 2 the roles of $E$ and $A$ ($\beta$ and $\alpha$, respectively) can be interchanged.

**Definition 2.** The matrix polynomial

$$\Delta(\Lambda, M) = \det\left[E \otimes \Lambda - A \otimes M\right] = \sum_{i=0}^{m} D_{i,m-i}\Lambda^i M^{m-i}, \quad D_{ij} \in \mathbb{C}^{n\times n} \tag{5}$$

is called the (matrix) *characteristic polynomial* of the pair $E, A \in M_m(P_n)$. $\Lambda$ and $M$ constitute the block indeterminate pair of $(E, A)$. The pair $(\Lambda, M)$ is called the block-eigenvalue pair of $(E, A)$.

In (5) 'det' means the formal determinant of a block matrix $F \in M_m(P_n)$ which we obtain by developing the determinant of $F$ and considering its commuting blocks as elements (Victoria, 1982). Denoting by $\operatorname{Det} F$ the usual determinant of $F$, we have the well-known relation (Victoria, 1982)

$$\operatorname{Det} F = \operatorname{Det}(\det F) \tag{6}$$

## 3. Main Result

Consider a standard pair of block matrices $E, A \in M_m(P_n)$.

**Theorem 1.** *Let (5) be the characteristic polynomial of $(E, A)$. Then*

$$\Delta(A, E) = \sum_{i=0}^{m} (I \otimes D_{i,m-i})\, A^i E^{m-i} = 0 \tag{7}$$

*Proof.* Let

$$\begin{aligned} B(\Lambda, M) &= B_{m-1,0} \otimes \Lambda^{m-1} + B_{m-2,1} \otimes \Lambda^{m-2}M + \cdots \\ &\quad + B_{1,m-2} \otimes \Lambda M^{m-2} + B_{0,m-1} \otimes M^{m-1} \end{aligned} \tag{8}$$

be the block-adjoint matrix of $[E \otimes \Lambda - A \otimes M]$. By using (6) it can be shown that (Victoria, 1982)

$$B(\Lambda, M)\left[E \otimes \Lambda - A \otimes M\right] = I \otimes \Delta(\Lambda, M) \tag{9}$$

Substituting (5) and (8) into (9), comparing the matrix coefficients of the some powers of $\Lambda$ and $M$ and using the well-known property of the Kronecker product (Lancaster, 1969) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, we obtain

$$\begin{aligned} B_{m-1,0}E &= I \otimes D_{m,0} \\ -B_{m-1,0}A + B_{m-2,1}E &= I \otimes D_{m-1,1} \\ -B_{m-2,1}A + B_{m-3,2}E &= I \otimes D_{m-2,2} \\ -B_{1,m-1}A + B_{0,m-1}E &= I \otimes D_{1,m-1} \\ -B_{0,m-1}A &= I \otimes D_{0,m} \end{aligned} \tag{10}$$

Postmultiplying (10) successively by $A^m, A^{m-1}E, \dots AE^{m-1}, E^m$ and adding them, we obtain (7). ■

Note that in the particular case $E = I$ the Victoria theorem (Victoria, 1982) can be obtained from Theorem 1.

## 4. Example

Consider the pair of block matrices

$$
\begin{aligned}
E &= \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} = \left[\begin{array}{cc:cc} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ \hdashline 3 & 0 & 2 & 2 \\ 0 & 3 & 0 & 2 \end{array}\right] \in M_2(P_2) \\
A &= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \left[\begin{array}{cc:cc} 2 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ \hdashline 3 & 0 & 3 & 2 \\ 0 & 3 & 0 & 3 \end{array}\right] \in M_2(P_2)
\end{aligned}
\tag{11}
$$

The pair (11) is standard since it satisfies (3) for $\alpha = -1$ and $\beta = 1$. The characteristic polynomial of (11) has the form

$$
\begin{aligned}
\Delta(\Lambda, M) &= \det\left[E \otimes \Lambda - A \otimes M\right] = \begin{vmatrix} E_1\Lambda - A_1 M & E_2\Lambda - A_2 M \\ E_3\Lambda - A_3 M & E_4\Lambda - A_4 M \end{vmatrix} \\
&= D_{20}\Lambda^2 + D_{11}\Lambda M + D_{02}M^2
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
D_{20} &= E_1E_4 - E_3E_2 = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \\
D_{11} &= E_3A_2 + A_3E_2 - E_1A_4 - A_1E_4 = \begin{bmatrix} 5 & -5 \\ 0 & 5 \end{bmatrix} \\
D_{02} &= A_1A_4 - A_3A_2 = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}
\end{aligned}
\tag{13}
$$

Using (7), (12) and (13), we obtain

$$\begin{bmatrix} D_{20} & 0 \\ 0 & D_{20} \end{bmatrix} A^2 + \begin{bmatrix} D_{11} & 0 \\ 0 & D_{11} \end{bmatrix} AE + \begin{bmatrix} D_{02} & 0 \\ 0 & D_{02} \end{bmatrix} E^2$$

$$= \begin{bmatrix} -4 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 10 & 7 & 10 & 11 \\ 0 & 10 & 0 & 10 \\ 15 & 9 & 15 & 15 \\ 0 & 15 & 0 & 15 \end{bmatrix}$$

$$+ \begin{bmatrix} 5 & -5 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 8 & 6 & 8 & 10 \\ 0 & 8 & 0 & 8 \\ 12 & 9 & 12 & 13 \\ 0 & 12 & 0 & 12 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 & 6 & 9 \\ 0 & 7 & 0 & 6 \\ 9 & 9 & 10 & 11 \\ 0 & 9 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the standard pair (11) is a zero of its characteristic polynomial (12).

## 5. Concluding Remarks

In (Kaczorek, 1995a), the Victoria theorem was only extended for non-square block matrices with square commutative blocks. In this paper, the Cayley-Hamilton theorem was extended for a standard pair of matrices partitioned into blocks that commute in pairs. The Victoria theorem in (Victoria, 1982) is a particular case of the proved theorem for $E = I$.

In a similar way as in (Kaczorek, 1994; 1995a; 1995b), the presented theorem can be extended for non-square block matrices and can be used for the computation of the left and right inverses of block matrices. Another example of application of the extended Cayley-Hamilton theorem is the analysis and synthesis of large-scale linear systems consisting of a number of subsystems.

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