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# NEW RESULTS TO THE INVERSE OPTIMAL CONTROL PROBLEM FOR DISCRETE-TIME LINEAR SYSTEMS

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The inverse optimal control problem, i.e. the problem of the optimal eigenvalue assignment for multi-input linear discrete-time systems, is considered. New necessary and sufficient conditions are proposed which lead to new simple optimal design methods. In particular, for an $n$-th order system with $m$ inputs the proposed method assigns $n-m$ dominant eigenvalues to any preselected positions while the remaining $m$ eigenvalues are accommodated to ensure stability and to satisfy optimal criteria.

## 1. Introduction

An important problem in the optimal control design with quadratic weighting cost criteria is the selection of the performance index weighting matrices which result in a state feedback gain matrix such that the closed-loop eigenvalues are at desired locations. This constitutes the inverse optimal control problem, i.e. under what conditions a given feedback gain-matrix satisfies optimal quadratic criteria (Molinari, 1973; 1977). For continuous-time, single-input systems, this problem was first raised by Kalman (1964) who posed the circle criterion as a necessary and sufficient condition. Therefore, an arbitrary eigenvalue assignment can provide a gain-matrix which is checked for optimality. This is the inverse problem of finding fixed eigenvalues from the Hamiltonian and after determining the optimal gain matrix by using e.g. Ackermann's formula (Lewis and Syrmos, 1995).

For multi-input systems where the same closed-loop eigenvalue set results in an infinite number of different state-feedback gain matrices a number of methods and techniques have been proposed. Most of these techniques, with extensive references, are included in (Johnson and Grimble, 1987) while a more detailed analysis is given in (Grimble and Johnson, 1988). Many conditions have been proposed which result in the design of optimal control systems by cost weight selection either assigning every eigenvalue to a desired position (Alexandridis and Galanos, 1987; Bar Ness, 1978; Fujii and Narazaki, 1984; Haddad and Bernstein, 1992; Jameson and Kreindler, 1973; Maki and Van de Vegte, 1974) or shifting the entire set of eigenvalues as a whole to a desired area (Amin, 1985; Kobayashi and Shimemura, 1981).

However, as is well-known, the discrete-time optimal regulator has properties essentially different from those of the continuous-time case (Alexandridis, 1996; Fujinaka and Katayama, 1988; Kim and Furuta, 1988; Sugimoto and Yamamoto, 1988).

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Therefore the conditions for the optimality of continuous-time systems cannot be directly transformed to the discrete-time case. Indeed, some different conditions have been proposed for the discrete-time case but most of the existing inverse optimal design methods seem to need complicated calculations (Alexandridis, 1996).

In this paper, some new conditions which satisfy optimality and a partial eigenvalue assignment are proposed. The main advantage of the proposed conditions is that they are easier to test and therefore they lead to simpler solutions. Using these methods one can assign the dominant eigenvalues of the discrete-time system to any preselected positions while the non-dominant eigenvalues are simply accommodated to be stable; simultaneously, the design satisfies the conditions imposed for optimality. An illustrative example demonstrates an implementation of the method.

## 2. Preliminaries

Consider the multi-input discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $A$ and $B$ are real constant matrices with dimensions $n \times n$ and $n \times m$, respectively. Let $\operatorname{rank}(B) = m$ and $\{A, B\}$ be a completely controllable pair. Then the state feedback control law

$$u(k) = Kx(k) \tag{2}$$

results in the closed-loop system

$$x(k+1) = (A + BK)x(k) \tag{3}$$

The state feedback gain matrix $K$ is optimal if it minimizes a quadratic performance index of the form (Lewis and Syrmos, 1995)

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left[x^T(k)Qx(k) + u^T(k)Ru(k)\right] = \frac{1}{2}\sum_{k=0}^{\infty} x^T(k)\left[Q + K^TRK\right]x(k) \tag{4}$$

As is well-known, the optimal feedback gain matrix $K$ must satisfy

$$K = -\left(R + B^TPB\right)^{-1}B^TPA \tag{5}$$

where $P$ is a real positive definite symmetric matrix determined by the solution to the discrete algebraic matrix Riccati equation, (DARE),

$$P = A^TPA - A^TPB\left(R + B^TPB\right)^{-1}B^TPA + Q \tag{6}$$

In the following sections, without loss of generality, we assume that the input matrix of system (1) has the following form:

$$B = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{7}$$

Let the state matrix $A$ and the feedback gain matrix $K$ be correspondingly decomposed as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad K = [K_1\ K_2] \tag{8}$$

Consequently, the closed-loop system matrix becomes

$$A + BK = \begin{bmatrix} A_{11} + K_1 & A_{12} + K_2 \\ A_{21} & A_{22} \end{bmatrix} \tag{9}$$

If $K$, given by (8), is optimal for a system with the input matrix of the form (7), then an optimal gain matrix exists for any system described by any arbitrary input matrix $B$. If $B$ has full rank, then it can be written as $B = T^{-1}\left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right]$ where $T$ is a suitable $n \times n$ invertible matrix. Then the following lemma holds.

**Lemma 1.** *(Kobayashi and Shimemura, 1981) Assume that for the modified system $TAT^{-1}$, $TB$, a matrix $K$ is an optimal state feedback gain matrix satisfying the DARE. Then the original system $A, B$ has $KT$ as an optimal state feedback gain matrix satisfying a DARE with the state weighting matrix $T^T QT$, input weighting matrix $R$ and solution matrix $T^T PT$.*

## 3. Inverse Optimal Control Problem

For the linear system (1), the inverse problem of linear optimal control is to find necessary and sufficient conditions under which a given state feedback gain matrix $K$ satisfies a performance index of the form (4). The inverse optimal control problem for continuous-time systems was originally raised by Kalman (1964). For multi-input systems several solutions have been proposed by many researchers for both the continuous and discrete-time case. In most of these cases the state weighting matrix $Q$ and the solution to the Riccati equation $P$ are obtained provided that the input weighting matrix $R$ is fixed.

In the following, we present Theorem 1 where some easier conditions for inverse optimality than the existing ones are proposed. Furthermore, Theorem 1 provides a class of positive definite solutions $P$ to the corresponding DARE.

**Theorem 1.** *The state feedback gain matrix $K$ is optimal for system (1) provided that the input weighting matrix $R > 0$ is given, if and only if*

*(i) the closed-loop system $A + BK$ is asymptotically stable,*

*(ii) the matrix*

$$F = -RK \begin{bmatrix} I \\ -A_{22}^{-1} A_{21} \end{bmatrix} \left\{ K \begin{bmatrix} I \\ -A_{22}^{-1} A_{21} \end{bmatrix} + A_{11} - A_{12} A_{22}^{-1} A_{21} \right\}^{-1}$$

*is symmetric and positive definite.*

*Proof.* (*Necessity*) Let a given gain matrix $K$ be optimal, i.e. it minimizes a performance index of the form (4). This means that there exists a symmetric constant matrix $P > 0$ such that $K$ can be expressed by (5), where $P$ satisfies the DARE (6). However, since (4) is minimized by this $K$, the second-order variation of $J$ is greater than or equal to zero, i.e.

$$\mathrm{d}^2 J = \frac{1}{2}\sum_{k=0}^{\infty}\Big[\mathrm{d}x^T(k)Q\,\mathrm{d}x(k) + \mathrm{d}u^T(k)R\,\mathrm{d}u(k)\Big] \geq 0 \tag{10}$$

The inequality constraint (10) has $n+m$ variables corresponding to $n$ state and $m$ input variables. However, since the dynamic equation (1) constitutes an $n$-th order constraint for $J$, there are only $m$ independent variables in the performance index (4). This implies that (10) must hold for $m$ independent variables (Green and Limebeer, 1995). Therefore "completing the square" in (10) (Green and Limebeer, 1995, Ch.5; Lewis and Syrmos, 1995, Ch.4), we obtain

$$\begin{aligned}\mathrm{d}^2 J &= \frac{1}{2}\mathrm{d}x^T(0)P\,\mathrm{d}x(0) - \frac{1}{2}\mathrm{d}x^T(\infty)P\,\mathrm{d}x(\infty)\\ &\quad + \frac{1}{2}\sum_{k=0}^{\infty}\Big[\mathrm{d}u(k) - \mathrm{d}u^*(k)\Big]^T\Big[R + B^T P B\Big]\Big[\mathrm{d}u(k) - \mathrm{d}u^*(k)\Big]\end{aligned} \tag{11}$$

where $\mathrm{d}u^*(k) = [R + B^T P B]^{-1} B^T P A\,\mathrm{d}x(k)$.

Hence for $R > 0$ the condition $\mathrm{d}^2 J \geq 0$ yields for $P > 0$ that $\mathrm{d}x(\infty) = 0$, which implies for the linear system (3) that $\lim_{k\to\infty} x(k) = 0$, i.e. the closed-loop system $A + BK$ is asymptotically stable.

Let the solution to the DARE be decomposed as follows:

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \tag{12}$$

where $P_1, P_2$ and $P_3$ are constant matrices with dimensions $m \times m$, $m \times (n-m)$ and $(n-m)\times(n-m)$, respectively. Then the optimal state feedback gain matrix (5) can also be decomposed as

$$[K_1\ K_2] = -\big[R + P_1\big]^{-1}\big[P_1 A_{11} + P_2 A_{21}\ \ P_1 A_{12} + P_2 A_{22}\big] \tag{13}$$

which can be equivalently rewritten as the following set of equations:

$$-\big[R + P_1\big]K_1 = P_1 A_{11} + P_2 A_{21} \tag{14a}$$

$$-\big[R + P_1\big]K_2 = P_1 A_{12} + P_2 A_{22} \tag{14b}$$

After some algebraic manipulations, i.e. solving (14b) for $P_2$ and substituting the result in (14a), we calculate

$$P_1 = -RK\begin{bmatrix} I \\ -A_{22}^{-1}A_{21}\end{bmatrix}\left\{K\begin{bmatrix} I \\ -A_{22}^{-1}A_{21}\end{bmatrix} + A_{11} - A_{12}A_{22}^{-1}A_{21}\right\}^{-1} \tag{15}$$

Since $P > 0$, we have $P_1 > 0$, i.e.

$$-RK\begin{bmatrix} I \\ -A_{22}^{-1}A_{21} \end{bmatrix}\left\{K\begin{bmatrix} I \\ -A_{22}^{-1}A_{21} \end{bmatrix} + A_{11} - A_{12}A_{22}^{-1}A_{21}\right\}^{-1} \equiv F > 0 \quad (16)$$

(*Sufficiency*) If $F > 0$, we can construct $P$ of the form (12) with $P_1 = F$, $P_2 = [-[R + P_1]K_2 - P_1A_{12}]A_{22}^{-1}$ and $P_3 = P_2^T P_1^{-1} P_2 + P_t$, where $P_t$ is any symmetric, positive definite matrix, so that $P$ always be a symmetric, positive definite matrix (Horn and Johnson, 1991). Constructing now the matrix $-[R + B^T PB]^{-1}B^T PA$ and substituting $P_1, P_2$ and $P_3$, we confirm that

$$-\left[R + B^T PB\right]^{-1} B^T PA = -\left[R + P_1\right]^{-1}\left[P_1A_{11} + P_2A_{21} \; P_1A_{12} + P_2A_{22}\right] = K \quad (17)$$

On the other hand, for this $P$ we can find a symmetric $Q$ so that

$$\left[A + BK\right]^T P\left[A + BK\right] - P + Q + K^T RK = 0 \quad (18)$$

Substituting $K = -[R + B^T PB]^{-1}B^T PA$ in (18), we obtain the DARE (6). Simultaneously, since $A+BK$ is stable, we have $lim_{k\to\infty}x(k) = 0$ and, since $P$ is positive definite, the term $1/2\lim_{k\to\infty} x(k)^T Px(k)$ tends to zero. However, manipulating the DARE, we arrive at the closed-loop Lyapunov equation

$$\left[A + BK\right]^T P\left[A + BK\right] - P = -\left[Q + K^T RK\right] \quad (19)$$

Substituting (19) in (4), we obviously conclude that

$$J = \frac{1}{2}\sum_{k=0}^{\infty} x(k)^T\left[Q + K^T RK\right]x(k) = \frac{1}{2}x^T(0)Px(0) \quad (20)$$

Therefore the performance index $J$ converges to the positive optimal value $J = (1/2)\, x^T(0)Px(0)$. ■

**Remark 1.** Let us note a main restriction, i.e. that the inverse of $A_{22}$ must exist. Also, the inverse of $K\left[\begin{smallmatrix} I \\ -A_{22}^{-1}A_{21} \end{smallmatrix}\right] + A_{11} - A_{12}A_{22}^{-1}A_{21}$ is required. If this is not the case, we apply an initial optimal state feedback gain matrix $K_0$ to the system subject to an arbitrary state weighting matrix $Q_0$ and an input weighting matrix $R$. At the second stage, a new $K$ is determined by using the proposed method on the resulting system, i.e. on the system with the plant matrix $A + BK_0$, input matrix $B$ and input weighting matrix $R + B^T P_0 B$, where $P_0$ is the solution to the DARE at the initial stage. Then the gain matrix $K_0 + K$ assigns the closed-loop eigenvalues to the desired locations (Alexandridis, 1996).

**Remark 2.** It is well-known that main assumptions in the optimal control design are $R > 0$ and $Q \geq 0$. However, it has been proved many times in applications that the performance of a system is further improved if the optimal gain matrix satisfies a quadratic criterion $J$ with $Q$ indefinite (Alexandridis, 1996; Grimble and Johnson, 1988; Jameson and Kreindler, 1973; Lewis and Syrmos, 1995; Shih and

Chen, 1974). This is a result due to the inverse optimal control problem where, as indicated by (18), the definiteness of $Q$ is not guaranteed. As discussed in the proof of Theorem 1, inverse optimality requires that the minimum of $J$ exist in the space of at least $m$ dimensions. Therefore the requirements for $R > 0$ and $F > 0$ (which lead to $P > 0$) are adequate. It is easily seen that in this case $J$ is certainly minimized in the space of $m$ independent dimensions (Theorem 1). Other methods (Alexandridis, 1996) lead to the minimization of $J$ in the space of $n$ dimensions (where $n \geq m$), i.e. in the case where $Q + K^T RK > 0$, and therefore $J = (1/2)\sum_{k=0}^{\infty} x(k)^T[Q + K^T RK]x(k) \geq 0$. In the case of inverse optimal control, we start from the assumption that an optimal $K$ is given. Therefore the formulated conditions are sufficient conditions with respect to the conventional optimal control which uses the definiteness of $Q$ and $R$ as necessary conditions in the space of $n+m$ dimensions in order to obtain an optimal $K$. Therefore, in inverse optimal control the *a-posteriori* calculation of $Q$ is meaningless.

## 4. Optimal Eigenvalue Assignment

In this section, the conditions for optimality from Theorem 1 are used to achieve a partial eigenvalue assignment, i.e. the assignment of $n - m$ dominant closed-loop eigenvalues, while the remaining $m$ closed-loop eigenvalues of system (3) are assigned to stable non-dominant positions.

To this end, we first prove the following lemma which provides an alternative method for the eigenvalue assignment in two sequential stages.

**Lemma 2.** *Let a linear time-invariant, discrete-time system be of the form (1). There exists a state feedback gain-matrix $K$ which assigns the entire set of $n$ eigenvalues of the closed-loop system (3) exactly to the values where:*

*(i) an arbitrary $m \times (n-m)$ matrix $X$ assigns $n-m$ eigenvalues of the matrix $A_{22} + A_{21}X$, and*

*(ii) the square part of the feedback gain-matrix, i.e. the $m \times m$ submatrix $K_1$, assigns $m$ eigenvalues of $A_1 + K_1$, where $A_1 = A_{11} - XA_{21}$.*

*Proof.* Let $M$ be an $n \times n$ matrix of the form

$$M = \begin{bmatrix} I_m & -X \\ 0 & I_{n-m} \end{bmatrix}$$

where

$$M^{-1} = \begin{bmatrix} I_m & X \\ 0 & I_{n-m} \end{bmatrix}$$

$I_m$ and $I_{n-m}$ are the $m \times m$ and $(n-m) \times (n-m)$ identity matrices, respectively.

We apply the similarity transformation $M$ on the closed-loop matrix $A+BK$ given by (9). Thus we have

$$M\big[A+BK\big]M^{-1}$$

$$= \begin{bmatrix} A_{11} - XA_{21} + K_1 & (A_{11} + K_1)X - XA_{21}X - XA_{22} + A_{12} + K_2 \\ A_{21} & A_{22} + A_{21}X \end{bmatrix} \quad (21)$$

By selecting $K_2 = S - K_1X$ with

$$S = -A_{11}X + XA_{21}X + XA_{22} - A_{12} \quad (22)$$

the similar of $A+BK$ becomes

$$M\big[A+BK\big]M^{-1} = \begin{bmatrix} A_1 + K_1 & 0 \\ A_{21} & A_{22} + A_{21}X \end{bmatrix} \quad (23)$$

where $A_1 = A_{11} - XA_{21}$. Equation (23) yields that there exists $K = [K_1\ K_2]$ such that the eigenvalues of the closed-loop matrix $A+BK$ are placed at the values where

(i) an arbitrary matrix $X$ locates $n-m$ eigenvalues of $A_{22} + A_{21}X$, and

(ii) the matrix $K_1$ locates the $m$ eigenvalues of $A_1 + K_1$. ■

We note that the assignment of the $n-m$ closed-loop eigenvalues by selecting an appropriate $X$ of $A_{22} + A_{21}X$ is possible if the pair $\{A_{22}, A_{21}\}$ is completely controllable. This is true for system (1) in accordance with the following lemma.

**Lemma 3.** *The pair $\{A_{22}, A_{21}\}$ is completely controllable if and only if the pair $\{A, B\}$ is completely controllable.*

*Proof.* If $\{A, B\}$ is a controllable pair, then $\operatorname{rank}[A - \lambda I\ B] = n$ which can be equivalently written as

$$\operatorname{rank} \begin{bmatrix} A_{11} - \lambda I_m & A_{12} & I_m \\ A_{21} & A_{22} - \lambda I_{n-m} & 0 \end{bmatrix} = n$$

This yields

$$\operatorname{rank}\big[A_{21}\ A_{22} - \lambda I\big] = \operatorname{rank}\big[A_{22} - \lambda I\ A_{21}\big] = n - m$$

which implies that $\{A_{22}, A_{21}\}$ is a controllable pair and vice versa. ■

Similarly, the assignment of the $m$ closed-loop eigenvalues by selecting an appropriate $K_1$ of $A_1 + K_1$ is always possible since in this case the input matrix is the identity matrix $I_m$ and therefore the pair $\{A_1, I_m\}$ is completely controllable for every $A_1$.

Now, to proceed with our approach, we combine Theorem 1 with Lemma 2 to establish the following theorem.

**Theorem 2.** *The state feedback gain matrix $K = [K_1\ K_2]$ with $K_1$ and $K_2$ determined as*

$$K_1 = \left[SA_{22}^{-1}A_{21} - \left[A_{11} - A_{12}A_{22}^{-1}A_{21}\right]\frac{a}{a+1}\right]\left[I + XA_{22}^{-1}A_{21}\right]^{-1} \tag{24}$$

*and*

$$K_2 = S - K_1X \tag{25}$$

*is an optimal state feedback gain-matrix in the sense of Theorem 1, which simultaneously assigns $n-m$ eigenvalues of the closed-loop system (3) where*

*(i) $X$ and $S$ are selected in accordance with Lemma 2, and*

*(ii) $a$ is a positive real scalar selected in such a manner that $A_1 + K_1$ is stable.*

*Proof.* Let $X$ be selected by assigning $n-m$ eigenvalues of $A_{22}+A_{21}X$ to any desired positions. Then these eigenvalues are $n-m$ closed-loop eigenvalues of $A+BK$ in accordance with Lemma 2. Calculating $S$ from (22) and selecting $a > 0$ so that $A_1 + K_1$ be stable, we ensure the stability of $A+BK$ (condition (i) of Theorem 1).

On the other hand, condition (ii) of Theorem 1, after some simple manipulations, leads to

$$F = -R\left[I+\left[A_{11}-A_{12}A_{22}^{-1}A_{21}\right]\left[K_1[I+XA_{22}^{-1}A_{21}]-SA_{22}^{-1}A_{21}\right]^{-1}\right]^{-1} > 0 \tag{26}$$

Substituting $K_1$ and $S$ in (26) by (24) and (22) we arrive at

$$F = aR > 0 \tag{27}$$

which is evidently true for $a > 0$ and $R > 0$. Therefore $K = [K_1\ K_2]$ is an optimal gain matrix according to Theorem 1. ■

Let us note that $[I_m + XA_{22}^{-1}A_{21}]^{-1}$ is supposed to exist. This is the usual case provided that $A_{22}^{-1}$ exists, since the first term is the $m$-th order identity matrix.

**Remark 3.** In practical applications, the $n-m$ eigenvalues which are assigned exactly to any desired stable positions are dominant eigenvalues of the closed-loop system. The $m$ eigenvalues which are manipulated to provide optimality are constrained to be located near the origin on the complex plane. Usually these eigenvalues are selected to be non-dominant closed-loop eigenvalues.

## 5. Illustrative Example

Let us consider a discrete-time system with the following system and input matrices ($n = 4, m = 2$):

$$A = \begin{bmatrix} -0.2612 & -1.7358 & 1.1061 & -1.5287 \\ 1.4910 & -0.2495 & 2.7318 & -1.4463 \\ -0.9076 & -2.5249 & -0.4374 & -2.0431 \\ 1.8779 & 1.5460 & 1.6391 & -0.4281 \end{bmatrix}, \quad B = \begin{bmatrix} 1.20 & 0.80 \\ 0.10 & -0.10 \\ 0.20 & -0.20 \\ 0.00 & 0.10 \end{bmatrix} \tag{28}$$

In order to transform the input matrix of the system into the standard form (7), we use the following coordinate transformation:

$$T = \left[B\ B_c\right]^{-1} = \begin{bmatrix} 0.50 & 4.00 & 0.00 & 0.00 \\ 0.50 & -6.00 & 0.00 & 0.00 \\ 0.00 & -2.00 & 1.00 & 0.00 \\ -0.05 & 0.60 & 0.00 & 1.00 \end{bmatrix} \tag{29}$$

where $B_c$ is any $n \times (n-m)$ matrix such that $[B\ Bc]^{-1}$ exists, i.e. in this case

$$B_c = \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}$$

Then the system and input matrices become respectively

$$TAT^{-1} = \begin{bmatrix} 9.1095 & 1.9023 & 11.4802 & -6.5496 \\ -13.9966 & -3.3653 & -15.8377 & 7.9134 \\ -6.0503 & -1.6439 & -5.9010 & 0.8495 \\ 4.1356 & 1.3136 & 3.2229 & -1.2194 \end{bmatrix}, \quad TB = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix} \tag{30}$$

The open-loop eigenvalues are $-0.2615 \pm 4.3651i$, $-0.0928$ and $-0.7605$, so they are unstable. Using the proposed method for the modified system, we first determine $n-m=2$ dominant eigenvalues at the positions $-0.60 \pm 0.20i$. To this end, we select $X$ in such a way that $A_{22} + A_{21}X$ has also eigenvalues $-0.60 \pm 0.20i$ (Lemma 2). Selection of $X$ constitutes a reduced order state feedback $(n-m)$-eigenvalue assignment problem. Thus $X$ can be determined by any well-known state feedback technique (Jamshidi *et al.*, 1992) as follows:

$$X = \begin{bmatrix} -2.1472 & 0.1191 \\ 4.7023 & 0.1269 \end{bmatrix} \tag{31}$$

To calculate $K_1$ and $K_2$, we use (24) and (25), where $S$ is given from (22) and the parameter $a$ is selected in such a way that $A_1 + K_1$ is stable with eigenvalues near the origin (non-dominant eigenvalues). To this end, we plot the spectral radius of $A_1 + K_1$, i.e. $\rho(A_1 + K_1) = \max\{|\lambda[A_1 + K_1]|\}$ versus the parameter $a$ (Fig. 1). From Fig. 1 we observe that for $a > 7$ we obtain stable eigenvalues. Selecting $a = 75$, we get $\rho(A_1 + K_1)$ which is less than 0.15 (i.e. we get non-dominant eigenvalues). Hence

$$K_1 = \begin{bmatrix} 4.4744 & 1.7824 \\ -14.0122 & -4.1911 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1.7970 & 4.5685 \\ -11.7142 & -4.0646 \end{bmatrix} \tag{32}$$

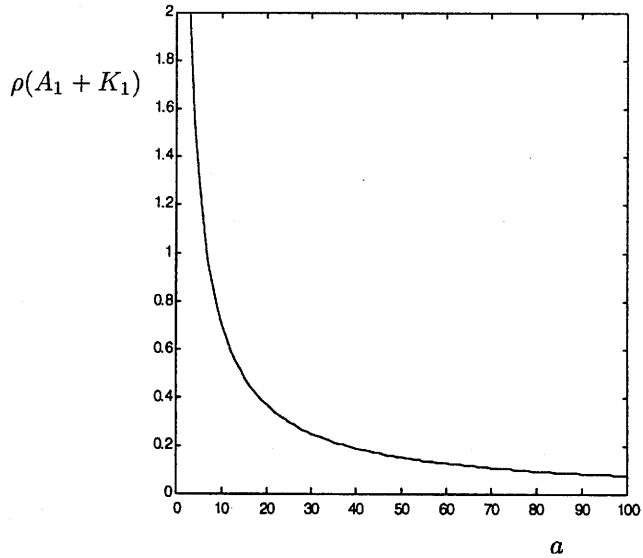


Fig. 1. The spectral radius $\rho(A_1+K_1)$ versus the parameter $a$.

However, (32) provides an optimal gain matrix for the modified system (30). According to Lemma 1, we calculate the optimal gain matrix $K$ for the original system:

$$K=\begin{bmatrix} 2.8999 & 6.3504 & 1.7970 & 4.5685 \\ -8.8984 & -9.9127 & -11.7142 & -4.0646 \end{bmatrix} \tag{33}$$

This leads to the closed-loop system $A+BK$ with the eigenvalues $-0.6000\pm0.2000i$, 0.0058 and 0.1013. Selecting

$$R=\begin{bmatrix} 0.3 & 0 \\ 0 & 0.5 \end{bmatrix}$$

we obtain $P=T^T\bar{P}T$, where $\bar{P}$ is calculated from (12) with $P_1$ given by (27) and

$$P_t=\begin{bmatrix} 1020.9 & 46.3 \\ 46.3 & 32.5 \end{bmatrix} \tag{34}$$

in order to ensure the definiteness of the matrix

$$P=\begin{bmatrix} 15.6 & 61.1 & -67.7 & -6.5 \\ 61.1 & 600.5 & -723.4 & -15.8 \\ -67.7 & -723.4 & 1020.9 & 46.3 \\ -6.5 & -15.8 & 46.3 & 32.5 \end{bmatrix} \tag{35}$$

which satisfies the DARE with a given $R$ and some symmetric $Q$.

## 6. Conclusion

Necessary and sufficient conditions for the inverse optimal control problem of discrete-time systems have been established. Based on these conditions, a simple procedure for implementing an optimal partial eigenvalue assignment has been presented. An illustrative example demonstrates the application of the method.

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