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QUASI-SLIDING MODE CONTROL OF DISCRETE-TIME SYSTEMS

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A reaching law approach to the design of discrete-time quasi-sliding mode control systems is considered. First the required position of the system representative point with reference to the sliding plane is specified, and then novel control strategies, which drive the system in such a way that the position actually changes according to the specification, are proposed. The strategies are linear and consequently the undesirable chattering and high-frequency switching between different values of the control signal are avoided. Furthermore, the state of the controlled system is driven to the narrowest possible band around the sliding plane. The strategies, when compared with previously published results, are simpler and they guarantee favourable performance of the controlled systems while using essentially reduced control effort.

Keywords: sliding-mode control, discrete-time systems

1. Introduction

Continuous-time variable-structure control theory and its applications have been extensively studied since the early 1950s (DeCarlo *et al.*, 1988; Hung *et al.*, 1993; Utkin, 1977). On the other hand, due to the widespread use of digital controllers, recently many researchers have proposed discrete-time techniques of a similar nature (Chan, 1994; Furuta, 1990; Gao *et al.*, 1995; Kaynak and Denker, 1993; Pan and Furuta, 1997; Sarpturk *et al.*, 1987; Spurgeon, 1992). In an early paper in this field, Furuta (1990) proposed a variable-structure algorithm which drives the system state to an appropriately determined sector in the state space. On the other hand, Gao *et al.* (1995) presented an algorithm which drives the system state to the vicinity of a switching plane. They specified desired properties of the controlled systems and used the so-called reaching-law approach to design their control algorithm. The algorithm was further discussed in (Bartoszewicz, 1996) where an additional condition for the existence of the quasi-sliding mode, as defined in (Gao *et al.*, 1995), was established, and the vicinity of the switching surface within which the system state remains in the quasi-sliding mode was evaluated.

The reaching-law approach to the design of quasi-sliding mode control systems was further investigated in (Bartoszewicz, 1998). In that paper a new definition of

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the quasi-sliding mode was introduced and a novel time-varying sliding plane was proposed. Consequently, a more efficient control scheme employing the time-varying sliding plane was designed.

In this paper, the definition of the quasi-sliding mode introduced in (Bartoszewicz, 1998) is adopted and a modified reaching law is proposed. The reaching law helps to obtain essentially the same system performance as in (Bartoszewicz, 1998), but without resorting to the relatively complex schemes involving application of time-varying sliding planes. Since the advantageous definition of the quasi-sliding mode is used, the system state does not have to cross the sliding plane in each successive control step and consequently the control strategy proposed in the paper can be linear. As a result, the undesirable chattering does not occur in the system, which is an important advantage when compared with the results of Gao *et al.* (1995). Another feature of the strategy is that it guarantees the convergence of the system state to the vicinity of the sliding plane in a finite time, specified *a priori* by the designer. Moreover, if the disturbance change rate is limited, a modified reaching law and another control strategy which guarantees better robustness of the system are proposed.

The remainder of this paper is organised as follows. Section 2 presents the notation and assumptions used throughout the paper. The main results, i.e. the new discrete-time quasi-sliding mode control strategies are derived in Section 3. In the same section, the performance of the control strategies is verified by means of a simulation example. Finally, Section 4 presents conclusions.

2. Preliminaries

Let us consider the following discrete-time system:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \Delta\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}u(k) + \boldsymbol{f}(k), \\ y(k) &= \boldsymbol{h}^T\boldsymbol{x}(k), \end{aligned} \tag{1}$$

where $\boldsymbol{x}$ is the $n\times 1$ state vector, $\boldsymbol{A}$ is an $n\times n$ matrix, $\boldsymbol{b}$ and $\boldsymbol{h}$ are $n\times 1$ vectors, u is the system input and y is the system output. In this equation the $n\times n$ matrix $\Delta\boldsymbol{A}$ represents parameter uncertainties and the $n\times 1$ vector $\boldsymbol{f}$ denotes external disturbances. The pair $(\boldsymbol{A},\boldsymbol{b})$ is controllable and the relations

$$\Delta\boldsymbol{A} = \boldsymbol{b}\bar{\boldsymbol{A}}, \quad \boldsymbol{f} = \boldsymbol{b}\bar{f},$$

usually referred to as matching conditions, are satisfied ($\bar{\boldsymbol{A}}$ is a row vector and $\bar{f}$ denotes a scalar). Let us define the function

$$s(k) = \boldsymbol{c}^T\boldsymbol{x}(k) \tag{2}$$

with vector $\boldsymbol{c}$ chosen in such a way that $\boldsymbol{c}^T\boldsymbol{b}\neq 0$.

The disturbances and parameter uncertainties are bounded so that the following relation holds:

$$d_l \le d(k) = \boldsymbol{c}^T\Delta\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{c}^T\boldsymbol{f}(k) \le d_u, \tag{3}$$

where d_l and d_u are known constants. Furthermore, we introduce the notation

$$d_0 = \frac{d_l + d_u}{2}, \quad \delta_d = \frac{d_u - d_l}{2}. \tag{4}$$

Definition 1. By the *quasi-sliding mode* in the ε vicinity of the sliding hyperplane $s(k) = \boldsymbol{c}^T \boldsymbol{x}(k) = 0$ we mean a system motion satisfying

$$\left|s(k)\right| \le \varepsilon, \tag{5}$$

where the positive constant ε is called the *quasi-sliding mode band width.*

This definition is essentially different from the one proposed in (Gao *et al.*, 1995), since it does not require the system state to cross the sliding plane $s(k) = 0$ in each successive control step. Consequently, it helps to eliminate chattering (i.e. after the transient period the system state and its output do not change in each successive control step), and to achieve an essential reduction in the control effort and improved quality of the quasi-sliding mode control. In the next section, control laws which guarantee the reachability of the quasi-sliding mode in a finite time specified by the designer are proposed.

3. Control Laws

In this section, the so-called reaching-law approach to the design of the discrete-time quasi-sliding mode control is adopted. First, the required evolution of the variable $s(k)$ is specified and then the control law which drives the system in such a way that the variable actually changes according to the specification is proposed. The idea behind the reaching law introduced in this paper is to make the system state converge to the sliding plane in such a way that (in the absence of disturbance $d(k)$ or when $d(k) = d_0$) the distance of the state from the plane $s(k) = 0$ decreases by the same value in each control step (i.e. in the reaching phase $s(k+1) - s(k) = \text{const}$). Thereby the control signal u can be used more economically and as a result faster convergence can be achieved.

3.1. Control Strategy Design

Let us consider the reaching law

$$s(k+1) = p(k)s(k) + d(k) - d_0, \tag{6}$$

where the unknown $d(k)$ is defined by (3) and $p(k)$ is a variable decay factor

$$p(k) = \begin{cases} \dfrac{n-k-1}{n-k} & \text{for} \quad k \le n-1 \\ 0 & \text{for} \quad k \ge n \end{cases} \tag{7}$$

The constant n above is a positive integer selected by the designer in order to achieve a trade-off between a fast convergence rate of the system and the magnitude of the control u required to achieve this convergence rate. In other words, by controlling the rate of decay (tuning n), the convergence to the sliding plane $s(k) = 0$ is tuned. In order to illustrate the effect of the variable decay factor on the convergence properties, we consider evolution of $s(k)$ for the nominal undisturbed system. For any $k \leq n$ we have

$$\begin{aligned} s(k) &= p(k-1)s(k-1) = p(k-1)p(k-2)s(k-2) \\ &= p(k-1)p(k-2)p(k-3)\cdots p(1)p(0)s(0) \\ &= \frac{n-(k-1)-1}{n-(k-1)}\frac{n-(k-2)-1}{n-(k-2)}\frac{n-(k-3)-1}{n-(k-3)}\cdots\frac{n-2}{n-1}\frac{n-1}{n}s(0) \\ &= \frac{n-k}{n}s(0) \end{aligned} \tag{8}$$

and consequently

$$s(k+1) - s(k) = \frac{-s(0)}{n} = \text{const}. \tag{9}$$

The convergence of the nominal system state to the sliding plane $s(k) = 0$ is shown in Fig. 1. The figure presents evolution of $s(k)$ when the variable decay factor introduced in this paper is applied, and when the decay factor is constant. In order to assure a fair comparison, the maximum difference between $s(k+1)$ and $s(k)$ is equal in both the cases. One can see from the figure that the variable decay factor guarantees faster and finite-time convergence, while the constant decay factor assures only asymptotic convergence to the sliding plane.

Furthermore, when the system is subject to disturbance $d(k)$, it can be easily verified that the reaching law (6) together with definition (7) of the decay factor, imply the existence of a quasi-sliding mode in the δ_d-vicinity of the sliding plane $s(k) = \boldsymbol{c}^T\boldsymbol{x}(k) = 0$ for any $k \geq n$. Therefore, the n-th control step could also be regarded as a boundary point between two different parts of the control process: the reaching phase and the quasi-sliding phase.

In order to determine a control u which drives the system in such a way that the reaching law (6) is satisfied, we use (1) to calculate $s(k+1)$:

$$\begin{aligned} s(k+1) &= \boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{c}^T\Delta\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{c}^T\boldsymbol{b}u(k) + \boldsymbol{c}^T\boldsymbol{f}(k) \\ &= \boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(k) + d(k) + \boldsymbol{c}^T\boldsymbol{b}u(k). \end{aligned} \tag{10}$$

Comparing this equation with the reaching law (6), we get

$$p(k)s(k) - d_0 = \boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{c}^T\boldsymbol{b}u(k) \tag{11}$$

and consequently

$$\begin{aligned} u(k) &= -\left(\boldsymbol{c}^T\boldsymbol{b}\right)^{-1}\left[\boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(k) + d_0 - p(k)s(k)\right] \\ &= -\left(\boldsymbol{c}^T\boldsymbol{b}\right)^{-1}\left\{\boldsymbol{c}^T\left[\boldsymbol{A} - p(k)\boldsymbol{I}\right]\boldsymbol{x}(k) + d_0\right\}, \end{aligned} \tag{12}$$

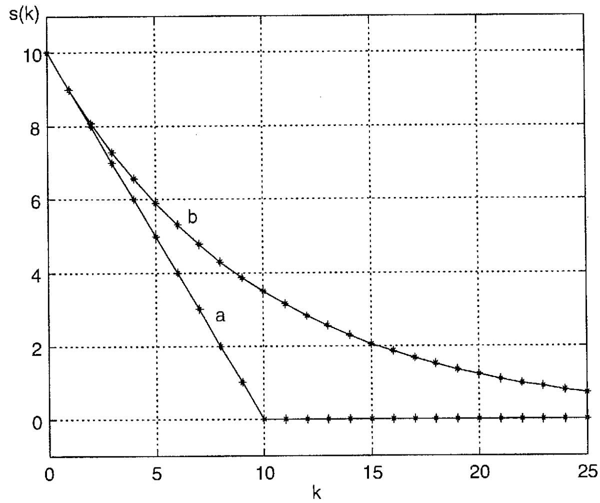

Fig. 1. Evolution of $s(k)$: a variable decay factor (a), and a constant decay factor (b).

where $\boldsymbol{I}$ is the $n \times n$ identity matrix. Thereby we have designed a control law which guarantees that for any $k \geq n$ the system state satisfies the following inequality:

$$|s(k)| = |d(k-1) - d_0| \leq \delta_d. \tag{13}$$

This means that for the system (1) controlled according to strategy (12) the maximum distance of the system state from the sliding plane $s(k) = 0$ is less than half of the same distance ($|s(k)| \leq 2\delta_d + \varepsilon T$, where $T > 0$ is the discretization period and ε is a positive constant) for the system controlled according to the algorithm introduced in (Gao *et al.*, 1995). Consequently, the robustness of the strategy is essentially better.

In order to assure a proper work of the controlled system, the vector $\boldsymbol{c}$ must be chosen in such a way that the resulting quasi-sliding motion is stable. This is equivalent to the stability of the closed-loop system. On the other hand, substituting (12) into (1) and taking into account that in the sliding mode $p(k) = 0$, we obtain

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \Delta\boldsymbol{A}\boldsymbol{x}(k) - \boldsymbol{b}(\boldsymbol{c}^T\boldsymbol{b})^{-1}\boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(k) - \boldsymbol{b}(\boldsymbol{c}^T\boldsymbol{b})^{-1}\boldsymbol{d}_0 + \boldsymbol{f}(k) \\ &= \left[\boldsymbol{A} + \Delta\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}^T\boldsymbol{b})^{-1}\boldsymbol{c}^T\boldsymbol{A}\right]\boldsymbol{x}(k) + \boldsymbol{b}\left[\bar{f}(k) - (\boldsymbol{c}^T\boldsymbol{b})^{-1}\boldsymbol{d}_0\right]. \end{aligned} \tag{14}$$

Since the last term on the right-hand side does not depend on the state vector $\boldsymbol{x}(k)$, we conclude that the quasi-sliding motion is stable if and only if all the eigenvalues of the matrix $\boldsymbol{Q} = \boldsymbol{A} + \Delta\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}^T\boldsymbol{b})^{-1}\boldsymbol{c}^T\boldsymbol{A}$ are placed strictly within the unit circle in the complex plane. Furthermore, for any stable system, its steady state accuracy can easily be evaluated. Suppose that the system is subject to a constant disturbance $\boldsymbol{f}(k) = \boldsymbol{f}_{SS} = \boldsymbol{b}\bar{f}_{SS} = \text{const}$, and let us denote by $\boldsymbol{x}_{SS}$ the steady state vector of the system. Since $\boldsymbol{Q}$ is a stable matrix, the difference $\boldsymbol{I}-\boldsymbol{Q}$ is invertible, and consequently one has

$$x_{SS} = \left(\boldsymbol{I}-\boldsymbol{Q}\right)^{-1}\boldsymbol{b}\Big[\bar{f}_{SS} - \left(\boldsymbol{c}^T\boldsymbol{b}\right)^{-1}d_0\Big], \tag{15}$$

where $\boldsymbol{I}$ is the $n\times n$ identity matrix. Equation (15) shows that strategy (12) (in the presence of constant disturbance) assures convergence of the system to the vicinity of the state-space origin, rather than to the origin itself. Therefore, in the next section a modified control algorithm which can effectively compensate for slowly time-varying disturbances and drive the system error to zero is introduced.

3.2. Disturbance Compensation

In this section, a modified reaching-law control strategy which can guarantee a further reduction in the distance between the system state and the sliding plane is proposed. Application of this strategy is a favourable option if the disturbance rate and the parameter change rate are limited. Suppose that for any $k\geq 0$, $|d(k+1)-d(k)|\leq \Delta_d$, where Δ_d is a known constant such that $\Delta_d<\delta_d$. Then we replace the reaching law (6) with the relation

$$s(k+1) = d(k) - d_0 + p(k)s(k) - \sum_{i=1}^{k}\big[s(i)-p(i-1)s(i-1)\big]. \tag{16}$$

The sum on the right-hand side is introduced to reduce the effect of the disturbance $d(k)$. By measuring the variable $s(k)$, useful information about previous disturbances is extracted, and since the disturbance change rate is limited, this information can further be used in the control process. In order to determine the performance of the system driven according to the modified reaching law, we use (16) to evaluate $s(k)$. For any $k\geq 1$ we obtain

$$s(k) = d(k-1)-d_0+p(k-1)s(k-1)-\sum_{i=1}^{k-1}\big[s(i)-p(i-1)s(i-1)\big], \tag{17}$$

where by definition for $k=1$ we have $\sum_{i=1}^{k-1}[s(i)-p(i-1)s(i-1)]=0$. From (17) it follows that

$$\sum_{i=1}^{k-1}\big[s(i)-p(i-1)s(i-1)\big] = -s(k)+p(k-1)s(k-1)+d(k-1)-d_0. \tag{18}$$

Substituting (18) into (16), we obtain

$$\begin{aligned} s(k+1) &= d(k) - d_0 + p(k)s(k) - \big[d(k-1)-d_0\big] \\ &= p(k)s(k) + d(k) - d(k-1) \end{aligned} \tag{19}$$

Relation (19) implies that the modified reaching law (16) drives the system (1) in such a way that in the sliding phase, i.e. for any $k\geq\max(n,2)$ we have

$$|s(k)| = \big|d(k-1)-d(k-2)\big| \leq \Delta_d < \delta_d. \tag{20}$$

This conclusion means that the distance of the state of the system (1) from the sliding plane $s(k)=0$ is further reduced and consequently it implies improved system robustness.

Similarly to the procedure applied before, comparing (10) with the reaching law (16), we get the control law which helps to achieve the favourable performance described in this section:

$$u(k) = -\left(\boldsymbol{c}^T\boldsymbol{b}\right)^{-1}\left[\boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(k)+d_0-p(k)s(k)+\sum_{i=1}^{k}\left[s(i)-p(i-1)s(i-1)\right]\right]. \quad (21)$$

If we set $s(0)=p(-1)s(-1)$, the control law (21) can also be expressed as

$$u(k) = -\left(\boldsymbol{c}^T\boldsymbol{b}\right)^{-1}\left[\boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(k)+d_0-p(k)s(k)+\sum_{i=0}^{k}\left[s(i)-p(i-1)s(i-1)\right]\right]. \quad (22)$$

3.3. Simulation Example

Let us consider the system described by (1) with the parameters

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad \Delta\boldsymbol{A} = 0, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{h}^T = [1\ 0].$$

The initial conditions of the system are $x_1 = 500$ and $x_2 = 0$. The sliding line $\boldsymbol{c}^T\boldsymbol{x} = x_1 + 2x_2 = 0$ is chosen, the convergence time is set to $n = 10$, and it is assumed that the system is affected by the constant disturbance $\boldsymbol{f}(k) = [f_1(k)\ f_2(k)]^T = [0\ 0.1]^T$. Figures 2 and 3 show the performance of the system controlled according to the strategy with the disturbance compensation proposed in this paper. The strategy is compared with Gao's control law discussed in (Gao *et al.*, 1995). Figure 2 presents evolution of the switching variable $s(k)$, and Fig. 3 illustrates the control signal $u(k)$.

It can be seen from Fig. 2 that our strategy guarantees faster convergence of the system state to the quasi-sliding mode band around the line $\boldsymbol{c}^T\boldsymbol{x}(k) = 0$. Furthermore, Fig. 3 shows that the convergence rate in our strategy is faster even though the maximum value of the control signal $u(k)$ for the strategy is smaller than for the algorithm proposed by Gao *et al.* Both the figures demonstrate that our strategy does not exhibit chattering, which is an important advantage when compared with Gao's results. Favourable performance of the strategy is also demonstrated in Fig. 4 which shows the phase trajectories of the system controlled according to our strategy and according to the algorithm proposed by Gao *et al.*

4. Conclusions

In this paper, new computationally efficient discrete time quasi-sliding mode control strategies based on the so-called reaching-law approach have been proposed. The strategies are linear and consequently they eliminate chattering and guarantee a favourable performance of the controlled systems. The feasibility of the strategies is verified both theoretically and by means of a simulation example.

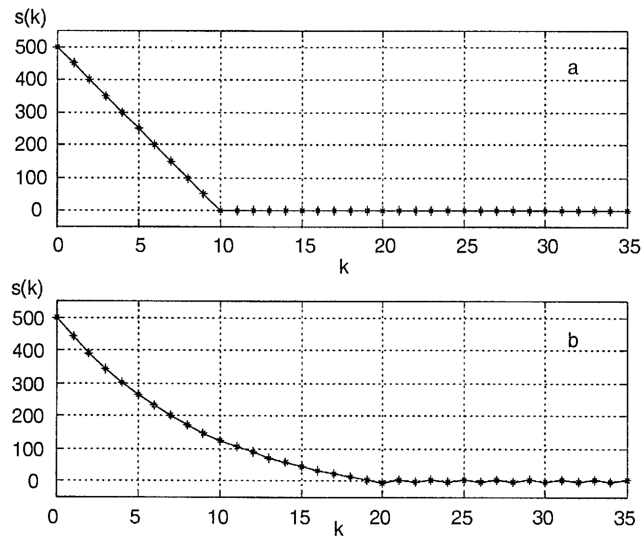

Fig. 2. Evolution of $s(k)$: the proposed strategy with disturbance compensation (a), and Gao's algorithm (b).

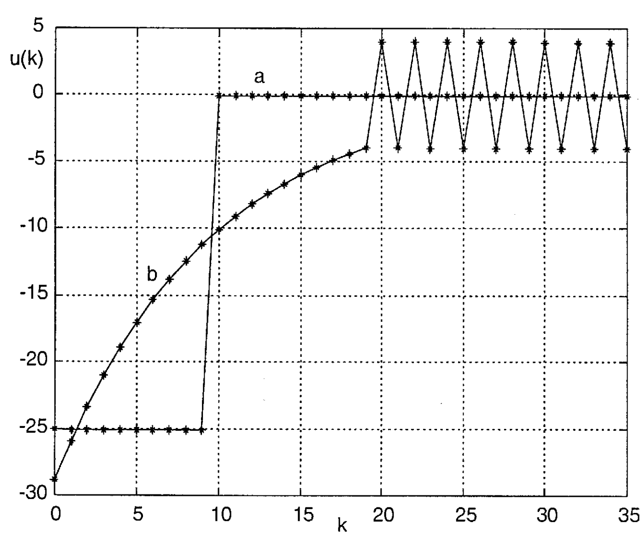

Fig. 3. Control signal: the proposed strategy with disturbance compensation (a), and Gao's algorithm (b).

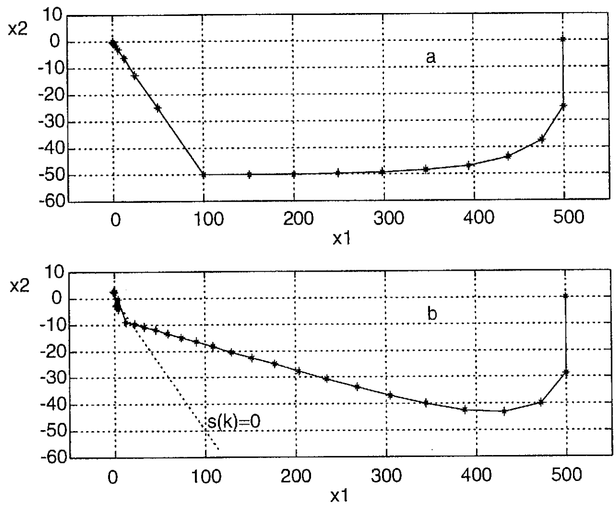

Fig. 4. Phase plane trajectory: the proposed strategy with disturbance compensation (a), and Gao's algorithm (b).

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