PERIODICITY OF THE POPULATION SIZE OF AN AGE-DEPENDENT MODEL WITH A DOMINANT AGE CLASS BY MEANS OF IMPULSIVE PERTURBATIONS[†]

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For an age-dependent model with a dominant age class an ω-periodic regime of the population size is sought by means of impulsive perturbations. For both noncritical and critical cases of first order the problem is reduced to operator systems solvable by a convergent simple iteration method.

Keywords: population dynamics, age-dependent model, periodic solutions, nonlinear delay equations

1. Model Description

The following model is described in (Kostova, 1995; Kostova and Milner, 1995), where the existence of oscillatory solutions is proved. For two fixed ages σ_1, σ_2 such that $0 \le \sigma_1 < \sigma_2 < \infty$ the age distribution $u(a,t)$ of a population is considered, where a is the age and t stands for time, with dynamics described by the following integro-differential equation with an age-boundary condition in integral form:

$$\begin{cases} \dfrac{\partial u}{\partial a} + \dfrac{\partial u}{\partial t} = -\delta(a,Q)u(a,t), & a,t > 0, \\ u(0,t) = \displaystyle\int_0^\infty \beta(a,Q)u(a,t)\,\mathrm{d}a, & t \ge 0, \\ u(a,0) = u_0(a), & a \ge 0, \end{cases} \tag{1}$$

where

$$Q = Q(t) = \int_{\sigma_1}^{\sigma_2} u(a,t)\,\mathrm{d}a$$

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is the *dominant age cohort* size and $\delta(a, Q)$ and $\beta(a, Q)$ are an age-specific death rate and the birth modulus when the dominant age group is of size Q, respectively. It is assumed that δ, β and u_0 are nonnegative, and that u_0 is integrable (so that the initial population is finite). This model is a generalization of the classical one of Gurtin and MacCamy (1974), which is obtained by setting $\sigma_1 = 0$ and $\sigma_2 = \infty$.

Further on in (Kostova, 1995; Kostova and Milner, 1995) the special case

$$\beta(a, Q) = \begin{cases} \beta(Q), & a \in [\sigma_1, \sigma_2], \\ 0, & \text{otherwise}, \end{cases}$$

is considered. This means that the dominant age class is the only one capable of having offspring, i.e. births are possible only in the age interval $[\sigma_1, \sigma_2]$ and the fertility rate depends just on the size of the dominant age group itself (and not on the age within the group). Moreover, $\beta(Q) \in C^1(\mathbb{R}_+; \mathbb{R}_+)$ and the mortality rate $\delta > 0$ is assumed constant. Then for the total population size,

$$P(t) = \int_0^\infty u(a, t)\,\mathrm{d}a,$$

the equation

$$\dot{P} + \delta P = \beta(Q)Q \tag{2}$$

is derived, where

$$Q(t) = P(t - \sigma_1)e^{-\sigma_1\delta} - P(t - \sigma_2)e^{-\sigma_2\delta} \quad \text{for} \quad t > \sigma_2, \tag{3}$$

$$Q(t) = e^{-\delta t}\int_{\sigma_1 - t}^{\sigma_2 - t} u_0(a)\,\mathrm{d}a \quad \text{for} \quad t \le \sigma_1,$$

while, for $\sigma_1 < t \le \sigma_2$, $Q(t)$ satisfies the integral equation (assumed to be uniquely solvable)

$$e^{\delta t}Q(t) = \int_0^{t-\sigma_1} e^{\delta a}\beta\big(Q(a)\big)Q(a)\,\mathrm{d}a + \int_0^{\sigma_2 - t} u_0(a)\,\mathrm{d}a. \tag{4}$$

Thus, for $t > \sigma_2$, $P(t)$ satisfies a nonlinear scalar delay equation (2) with Q given by (3), while for $t \in [0, \sigma_2]$, $Q(t)$ and eventually $P(t)$ can be expressed in terms of the initial function $u_0(a)$ of the age-dependent model (1):

$$P(t) = e^{-\delta t}\left[\int_0^\infty u_0(a)\,\mathrm{d}a + \int_0^t e^{\delta a}\beta\big(Q(a)\big)Q(a)\,\mathrm{d}a\right]. \tag{5}$$

Thus we find the initial function $P_0(t)$, $t \in [0, \sigma_2]$ of the above-mentioned delay equation.

2. Problem Statement

We fix $\omega > 0$ much larger than the age σ_2, and try to obtain an ω-periodic regime of the population size by means of impulsive perturbations for a suitably chosen initial function u_0. More precisely, suppose that at certain moments t_i such that $t_{i+2} = t_i + \omega$ for all $i \in \mathbb{Z}$, the population size $P(t)$ is abruptly changed while for the moment equation (2) with (3) is assumed to hold for all $t \in \mathbb{R}$, $t \neq t_i$. We normalize the quantities in eqn. (2) as follows:

$$s = t/\omega, \quad \Pi(s) = P(\omega s), \quad D = \omega\delta, \quad B(Q) = \omega\beta(Q).$$

Henceforth we write again t, δ and β instead of s, D and B, respectively, x instead of Π, and $h = \sigma_2/\omega$ will be a small parameter, while the small quantity σ_1/ω will be neglected for the sake of simplicity. We suppose that the time interval between two successive abrupt changes (impulse effects) $t_{i+1} - t_i$ is large in comparison with the 'age' h for all $i \in \mathbb{Z}$, and look for 1-periodic solutions to the problem

$$\begin{cases} \dot{x} = -\delta + H(h, x, \bar{x}), & t \neq t_i, \\ \Delta x(t_i) = B_i x_i + a_i, & i \in \mathbb{Z}, \end{cases} \tag{6}$$

where $\bar{x}(t) \equiv x(t-h)$, $\Delta x(t_i) \equiv x(t_i + 0) - x(t_i - 0)$ is the magnitude of the impulse effect at the moment t_i, $x_i \equiv x(t_i) \equiv x(t_i - 0)$, $H(h, x, \bar{x}) = \beta(Q)Q$, $Q = Q(h, x, \bar{x}) = x - \bar{x}e^{-h\delta}$, $0 < t_1 < t_2 < 1$.

Remark 1. The assumption that $\sigma_1/\omega = 0$ is of a technical character. It leads to system (6) with just one small delay of the argument. Similar systems were studied in (Bainov and Covachev, 1994; 1996; Boichuk and Covachev, 1997). In a subsequent paper the case of several small delays of the argument is considered.

Remark 2. We see that the nonlinearity $H(h, x, \bar{x})$ is not precisely of the form studied in (Boichuk and Covachev, 1997). However, its particular form much simplifies our calculations.

Suppose that $x(t)$ is a 1-periodic solution to the problem (6). Thus we find an ω-periodic solution $P(t)$ of eqn. (2) with (3) satisfying the respective impulse conditions. However, if $P(t)$ is the population size of the age-dependent model (1), for $0 \leq t \leq \sigma_2$ it must satisfy (2) with Q on the right-hand side determined from (4), i.e.

$$P(t)e^{\delta t} = \int_0^\infty u_0(a)\,\mathrm{d}a + \int_0^t e^{\delta a}\beta\big(Q(a)\big)Q(a)\,\mathrm{d}a,$$

or, by virtue of (4),

$$\int_{\sigma_2 - t}^\infty u_0(a)\,\mathrm{d}a + Q(t)e^{\delta t} = P(t)e^{\delta t}. \tag{7}$$

Since $Q(t)$ can be expressed from (4) in terms of $u_0(a)$ by means of an integral operator, the initial function $u_0(a)$ of (1) must satisfy (7). This is a counterpart of the well-known fact that if a delay differential equation has a periodic solution, then its initial function must satisfy an integral equation.

3. Main Result

3.1. Preliminaries

We can easily find that the fundamental solution $X(t)$ to the homogeneous system

$$\begin{cases} \dot{x} = -\delta x, & t \neq t_i, \\ \Delta x(t_i) = B_i x_i, & i \in \mathbb{Z}, \end{cases} \tag{8}$$

is 1-periodic if and only if

$$(1 + B_1)(1 + B_2) = e^{\delta}. \tag{9}$$

If (9) is violated, we have to do with the so-called *noncritical case* considered in (Bainov and Covachev, 1994; 1996) Green's function $G(t,\tau)$ of the periodic problem for a nonhomogeneous system corresponding to (8) is given by

$$G(t,\tau) = \begin{cases} X(t)\big(1 - X(1)\big)^{-1} X^{-1}(\tau), & 0 \leq \tau \leq t \leq 1, \\ X(1+t)\big(1 - X(1)\big)^{-1} X^{-1}(\tau), & 0 \leq t < \tau \leq 1. \end{cases}$$

A 1-periodic solution to system (6) must satisfy

$$x(t) = \int_0^1 G(t,\tau) H\big(h, x(\tau), x(\tau - h)\big)\,\mathrm{d}\tau + \sum_{i=1}^{2} G(t, t_i + 0) a_i.$$

For h small enough this equation has a unique solution provided by the Implicit Function Theorem (Bainov and Covachev, 1994) (or the Contraction Mapping Principle (Bainov and Covachev, 1996)). This yields the existence of a unique ω-periodic solution to (2).

Next, we consider the *critical case* when (9) holds. Then

$$X(t) = \begin{cases} \exp(-\delta t), & 0 \leq t \leq t_1, \\ (1 + B_1)\exp(-\delta t), & t_1 < t \leq t_2, \\ \exp\big(\delta(1 - t)\big), & t_2 < t \leq 1. \end{cases}$$

Since $X_1 \equiv X(t_1 + 0) = (1 + B_1)e^{-\delta t_1}$, $X_2 \equiv X(t_2 + 0) = e^{\delta(1 - t_2)}$, the nonhomogeneous system

$$\begin{cases} \dot{x} = -\delta x, & t \neq t_i, \\ \Delta x(t_i) = B_i x_i + a_i, & i \in \mathbb{Z}, \end{cases}$$

has a 1-parametric family of 1-periodic solutions $x_0(t,c)$ if and only if the nonhomogeneities a_1 and a_2 satisfy

$$e^{\delta t_1} a_1 + e^{\delta(t_2 - 1)}(1 + B_1) a_2 = 0, \tag{10}$$

and

$$x_0(t,c) = \begin{cases} c\exp(-\delta t), & 0 \le t \le t_1, \\ c(1+B_1)\exp(-\delta t) + a_1 \exp\big(\delta(t_1 - t)\big), & t_1 < t \le t_2, \\ c\exp\big(\delta(1-t)\big), & t_2 < t \le 1. \end{cases}$$

3.2. Equation for the Generating Amplitudes

Let us find conditions for the existence of 1-periodic solutions $x(t,h)$ of (6) depending continuously on h and such that for some $c \in \mathbb{R}$ we have $x(t,0) = x_0(t,c)$. In (6) we change the variables according to the formula

$$x(t,h) = x_0(t,c) + y(t,h) \tag{11}$$

and are led to the problem of finding 1-periodic solutions $y = y(t,h)$ to the impulsive system of functional differential equations

$$\begin{cases} \dot{y} = -\delta y + H\big(h, x(t,h), x(t-h,h)\big), & t \ne t_i, \\ \Delta y(t_i) = B_i y_i, & i \in \mathbb{Z}, \end{cases} \tag{12}$$

such that $y(t,h) \to 0$ as $h \to 0$.

We can formally consider (12) as a linear nonhomogeneous system. Then it has a 1-periodic solution y if and only if

$$\int_0^1 X^{-1}(\tau) H\big(h, x(\tau,h), x(\tau-h,h)\big)\, \mathrm{d}\tau = 0. \tag{13}$$

We divide the left-hand side of (13) by h and then study its behaviour as $h \to 0$. We can represent the integral in (13) by a sum of integrals over intervals containing no points of discontinuity of the integrand. It is obvious that for $\tau \in (t_i, t_i + h)$, $i = 1,2$, the interval $(\tau - h, \tau)$ contains the point of discontinuity t_i while for τ inside the remaining intervals the interval $(\tau - h, \tau)$ contains no such points. We write $\Delta_1^h = (t_1, t_1 + h) \cup (t_2, t_2 + h)$, $\Delta_2^h = [0,1] \setminus \Delta_1^h$ and make use of the representation $\int_0^1 = \int_{\Delta_1^h} + \int_{\Delta_2^h}$.

As in (Boichuk and Covachev, 1997), denote by $\varepsilon(h,x)$ the expressions tending to 0 as $h \to 0$, and satisfying a Lipschitz condition with respect to x with a constant tending to 0 as $h \to 0$. Making use of the particular form of the nonlinearity $H(h,x,\bar{x})$ and of the linear part of the equation, for τ in the 'good' set Δ_2^h we find

$$Q\big(h, x(\tau,h), x(\tau-h,h)\big)/h = \varepsilon(h,x).$$

Thus

$$h^{-1} \int_{\Delta_2^h} X^{-1}(\tau) H\big(h, x(\tau,h), x(\tau-h,h)\big)\, \mathrm{d}\tau = \varepsilon(h,x).$$

On the other hand, for $i=1,2$ we have

$$h^{-1}\int_{t_i}^{t_i+h} X^{-1}(\tau)H\big(h,x(\tau,h),x(\tau-h,h)\big)\,\mathrm{d}\tau$$

$$= X_i^{-1}\varphi_i(c,h)\beta\big(\varphi_i(c,h)\big) + \varepsilon(h,x),$$

were $\varphi_i(c,h)$ are linear terms with respect to c,

$$\varphi_i(c,h) = B_i x(t_i,h) + a_i.$$

Thus (13) takes on the form

$$\sum_{i=1}^{2} X_i^{-1}\varphi_i(c,h)\beta\big(\varphi_i(c,h)\big) + \varepsilon(h,x) = 0. \tag{14}$$

We have

$$\varphi_i(c,h) = \varphi_i(c,0) + B_i y_i, \quad i=1,2.$$

For $\varphi_i(c) \equiv \varphi_i(c,0)$, $i=1,2$, we find

$$\varphi_1(c) = B_1 e^{-\delta t_1}c + a_1,$$

$$\varphi_2(c) = B_2 e^{-\delta t_2}(1+B_1)c + B_2 e^{\delta(t_1-t_2)}a_1 + a_2.$$

Passing to the limit $h \to 0$ in (14), we derive the so-called *equation for the generating amplitudes* (Grebennikov and Ryabov, 1979)

$$e^{\delta t_1}\phi_1(c)\beta\big(\phi_1(c)\big) + e^{\delta(t_2-1)}(1+B_1)\varphi_2(c)\beta\big(\varphi_2(c)\big) = 0. \tag{15}$$

If in the solvability condition (10) a_1 and a_2 are not zero, then (15) can also be written as

$$\varphi_1(c)\beta\big(\varphi_1(c)\big) \big/ \big(\varphi_2(c)\beta(\varphi_2(c))\big) = a_1/a_2. \tag{16}$$

On the other hand, if $a_1 = a_2 = 0$, neglecting the solution $c=0$ of (15) which yields the trivial solution $x=0$ of (6), by virtue of (9) we get

$$B_1(1+B_2)\beta\big(B_1 e^{-\delta t_1}c\big) + (1+B_1)B_2\beta\big(B_2 e^{-\delta t_2}(1+B_1)c\big) = 0. \tag{17}$$

The precise form of (16) or (17) depends on the form of the function β.

3.3. Critical Case of First Order

Now suppose that c^* is a real root of (15). Then the 1-periodic solution $y(t,h)$ of system (12) such that $y(t,0) \equiv 0$ can be represented in the form

$$y(t,h) = X(t)c + hy^{(1)}(t,h), \tag{18}$$

where the unknown real constant $c = c(h)$ must satisfy an equation derived below from (14), while the unknown 1-periodic function $y^{(1)}(t,h)$ can be represented as

$$y^{(1)}(t,h) = h^{-1} \int_0^1 G(t,\tau) H\big(h, x_0(\tau, c^*) + y(\tau,h),$$

$$x_0(\tau - h, c^*) + y(\tau - h, h)\big)\, \mathrm{d}\tau \tag{19}$$

in terms of the *generalized Green's function*

$$G(t,\tau) = \begin{cases} e^{\delta(\tau - t)} g(t,\tau), & 0 \le \tau \le t \le 1, \\ 0, & 0 \le t < \tau \le 1, \end{cases}$$

and $g(t,\tau)$ is a piecewise constant function:

$$g(t,\tau) = \begin{cases} 1, & [\tau, t] \subset [0, t_1] \cup (t_1, t_2] \cup (t_2, 1], \\ 1 + B_1, & 0 \le \tau \le t_1 < t \le t_2, \\ e^{\delta}, & 0 \le \tau \le t_1 < t_2 < t \le 1, \\ 1 + B_2, & t_1 \le \tau \le t_2 < t \le 1. \end{cases}$$

By arguments similar to those above we find

$$y^{(1)}(t,h) = \sum_{i=1}^{2} G(t, t_i + 0) \varphi_i(c^*, h) \beta\big(\varphi_i(c^*, h)\big) + \varepsilon\big(h, x(t,h)\big).$$

We perform linearization with respect to y making use of the expansions

$$\varphi_i(c^*, h)\beta\big(\varphi_i(c^*, h)\big) \equiv \big(\varphi_i(c^*) + B_i y_i\big)\beta\big(\varphi_i(c^*) + B_i y_i\big)$$

$$= \varphi_i(c^*)\beta\big(\varphi_i(c^*)\big) + \beta_{1i} B_i y_i \; + \; \beta_{2i}(y_i),$$

where

$$\beta_{1i} = \beta\big(\varphi_i(c^*)\big) + \beta\big(\varphi_i(c^*)\big)\varphi_i(c^*),$$

β' is the derivative of β, while $\beta_{2i}(y)$ is such that

$$\beta_{2i}(0) = 0, \quad \frac{\partial}{\partial y}\beta_{2i}(0) = 0.$$

Now, since c^* satisfies (15), eqn. (14) becomes

$$\sum_{i=1}^{2} X_i^{-1}\Big\{\beta_{1i}B_i y(t_i,h) + \beta_{2i}\big(y(t_i,h)\big)\Big\} + \varepsilon(h,x) = 0.$$

In view of the representation (18), let us write

$$\mathcal{B}_0 = \sum_{i=1}^{2} \beta_{1i}\frac{B_i}{1+B_i}.$$

Then we have

$$\mathcal{B}_0 c = -\sum_{i=1}^{2} X_i^{-1}\Big\{h\beta_{1i}B_i y^{(1)}(t_i,h) + \beta_{2i}\big(y(t_i,h)\big)\Big\} + \varepsilon(h,x) \tag{20}$$

and

$$y^{(1)}(t,h) = \sum_{i=1}^{2} G(t,t_i+0)\Big\{\varphi_i(c^*)\beta\big(\varphi_i(c^*)\big) + \beta_{1i}B_i\big[X(t_i)c + hy^{(1)}(t_i,h)\big] + \beta_{2i}\big(y(t_i,h)\big)\Big\} + \varepsilon\big(h,x(t,h)\big). \tag{21}$$

Thus problem (6) is reduced to the equivalent operator system $(11|_{c=c^*-\text{a root of (15)}})$, (18), (20), (21).

It is easy to see that $\mathcal{B}_0 \neq 0$ is equivalent to the simplicity of the root c^* of eqn. (15). This is the so-called *critical case of first order.* Then eqn. (20) can be solved with respect to c and we obtain a Fredholm operator system of the second type to which a convergent simple iteration method can be applied (Grebennikov and Ryabov, 1979). Thus to any simple real root of eqn. (15) for h small enough (i.e. for σ_2 small enough with respect to ω) there corresponds a 1-periodic solution to problem (6) tending to $x_0(t,c^*)$ as $h \to 0$, i.e. an ω-periodic solution to eqn. (2).

We could also apply the same method to the 2- and 3-dimensional systems obtained in (Swick, 1980; Yuan, 1988).

4. Conclusions

In this paper we considered an age-dependent model for which the dominant age class is the only one capable of having offspring, and the upper bound σ_2 of the age of this class is much larger than its lower bound σ_1. For any $\omega > 0$ much larger than σ_2 we showed that an ω-periodic regime of the population size can be obtained by means of impulsive perturbations for a suitably chosen initial function. The assumption that just two impulses take place during a period was made for the sake of convenience. It allowed us to simplify some equations and at the same time to show the application of our general methods in the noncritical and critical cases.

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