Int. J. Appl. Math. and Comp. Sci., *2000*, *Vol.10*, *No.2*, *205–225*

CRITERIA FOR STABILITY OF UNCERTAIN LINEAR SYSTEMS WITH TIME-VARYING DELAY[†]

QING-LONG HAN*, KEQIN GU*

This paper focuses on the stability problem for uncertain linear systems with *time-varying* delays. Based on a *discretized Lyapunov functional* approach, delay-dependent criteria that are formulated in terms of linear matrix inequalities are proposed to guarantee asymptotic stability for such systems. Numerical examples are also included to show the effectiveness of the method.

Keywords: stability, time-varying delay, uncertainty, linear matrix inequalities, Lyapunov functional

Notation

$\mathbb{R}$	– real number field
$\mathbb{R}^n$	– n-dimensional real vector space
$\mathbb{R}^{n\times n}$	– $(n\times n)$-dimensional real vector space
I	– identity matrix of appropriate dimensions
W^T	– transpose of W
$W>0$ $(W<0)$	– symmetric positive (negative) definite matrix
$W\geq 0$ $(W\leq 0)$	– symmetric positive (negative) semi-definite matrix
$\dot{W}$	– derivative of W with respect to time t
$\dot{W}(\alpha)$	– derivative of W evaluated at α
C	– set of continuous $\mathbb{R}^n$-valued functions on $[-r_M, 0]$
$r(t)$	– time delay
r_m	– lower bound on $r(t)$
r_M	– upper bound on $r(t)$
β	– upper bound on the rate of increase in the delay

[†] This work was partly supported by the U.S. National Science Foundation under Grant INT-9818312.

* Department of Mechanical and Industrial Engineering, Southern Illinois University at Edwardsville, Edwardsville, Illinois 62026-1805, U.S.A., e-mail: {qhan,kgu}@siue.edu. The first author is on leave from LAII-ESIP, Université de Poitiers, France and Institute of Automation, East China University of Science and Technology, Shanghai, P.R. China

1. Introduction

Stability analysis and stabilization of time-delay systems have received much attention in the past decade. This is due to theoretical interests as well as a need for practical system analysis and design. Delays are frequently encountered in various engineering systems, and their existence is often the source of instability, the generation of oscilation, and poor performance (Malek-Zavarei and Jamshidi, 1987). Numerous works on this topic have been reported in the recent few years. Depending on whether the stability criterion itself contains the delay argument as a parameter, stability criteria for time-delay systems can be classified into two categories (Mori, 1985), namely *delay-dependent* stability criteria (Gu, 1997; 1999a; Han and Mehdi, 1999a; Li and de Souza, 1997; Niculescu *et al.*, 1995; 1996) and *delay-independent* ones (Chen and Latchman, 1995; Han and Mehdi, 1998; 1999b; Phoojaruenchanachai and Furuta, 1992).

Many delay-dependent results have been pursued based on the Razumikhin theorem (Han and Mehdi, 1999a; Li and de Souza, 1997; Niculescu *et al.*, 1995; 1996). Although these results are usually less conservative than the delay-independent results, they can still be rather conservative. Moreover, there are no obvious ways to obtain less conservative results even if one is willing to spend more computational time on the problem. Furthermore, most criteria do not reduce to a necessary and sufficient condition when applied to uncertainty-free systems. Gu (1997; 1999a) proposed a *discretized Lyapunov functional* approach to check the stability of uncertain linear systems with *constant* time delays. The criteria showed significant improvements over the existing results even under very coarse discretizations. For uncertainty-free systems, the analytical results can be approached with fine discretization. The results have also been generalized to systems with multiple delays (Gu, 1999b) and distributed delays (Gu *et al.*, 1999).

In this paper, we deal with the stability problem for uncertain linear time-delay systems. The delay is assumed to be a single, *time-varying*, bounded function. Based on the *discretized Lyapunov functional* approach, we develop delay-dependent criteria for stability analysis. The proposed results are generalizations of the results derived in (Gu, 1997) to the case with a *time-varying* delay. Numerical examples are also included to show the effectiveness of the method.

2. Problem Statement

Consider the uncertain linear system with time-varying delay

$$\dot{x}(t) = A(t)x(t) + B(t)x\big(t - r(t)\big), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times n}$ are uncertain matrices, which are unknown and possibly time-varying, but known to be bounded by some compact set Ω, i.e.

$$\begin{bmatrix} A(t) & B(t) \end{bmatrix} \in \Omega \subset \mathbb{R}^{n\times 2n} \quad \text{for all } \ t \in [0, \infty),$$

and the delay $r(t)$ is a time-varying bounded function satisfying

$$0 \le r_m \le r(t) \le r_M, \quad \dot{r}(t) \le \beta < 1, \tag{2}$$

where r_m, r_M and β are constant. For $t \in [0, \infty)$ define

$$x_t \in C, \quad x_t(\theta) = x(t+\theta), \quad \theta \in [-r_M, 0]. \tag{3}$$

In this paper, we will develop a practical *delay-dependent* criterion to check the stability of the above system. More specifically, given three scalars r_m, r_M and β, our objective is to determine if system (1) is asymptotically stable for any $r(t)$ satisfying (2).

Since $x(t)$ is continuously differentiable for $t \ge 0$, one can write (Hale and Lunel, 1993)

$$\begin{aligned} x\big(t - r(t)\big) &= x(t - r_M) + \int_{-r_M}^{-r(t)} \dot{x}(t+\theta)\,\mathrm{d}\theta \\ &= x(t - r_M) + \int_{-r_M}^{-r(t)} \Big[A(t+\theta)x(t+\theta) + B(t+\theta)x\big(t+\theta - r(t+\theta)\big)\Big]\mathrm{d}\theta. \end{aligned}$$

Using this expression for $x(t - r(t))$ in (1), we obtain

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)x(t - r_M) \\ &\quad + B(t)\int_{-r_M}^{-r(t)} \Big[A(t+\theta)x(t+\theta) + B(t+\theta)x\big(t+\theta - r(t+\theta)\big)\Big]\mathrm{d}\theta. \end{aligned} \tag{4}$$

Then the stability of system (1) can be investigated by choosing a quadratic Lyapunov functional candidate

$$V(x_t) : C \to \mathbb{R}$$

$$\begin{aligned} V(x_t) &= \frac{1}{2}x^T(t)Px(t) + x^T(t)\int_{-r_M}^{0} Q(\xi)x(t+\xi)\,\mathrm{d}\xi \\ &\quad + \frac{1}{2}\int_{-r_M}^{0}\left[\int_{-r_M}^{0} x^T(t+\xi)R(\xi-\eta)x(t+\eta)\mathrm{d}\eta\right]\mathrm{d}\xi \\ &\quad + \frac{1}{2}\int_{-r_M}^{0} x^T(t+\xi)S(\xi)x(t+\xi)\,\mathrm{d}\xi \\ &\quad + \frac{1}{2}\int_{-r_M}^{-r_m}\left[\int_{\theta}^{0} x^T(t+\zeta)K_1x(t+\zeta)\,\mathrm{d}\zeta\right]\mathrm{d}\theta \\ &\quad + \frac{1}{2}\int_{-r_M}^{-r_m}\left[\int_{\theta-r(t+\theta)}^{0} x^T(t+\zeta)K_2x(t+\zeta)\,\mathrm{d}\zeta\right]\mathrm{d}\theta, \end{aligned} \tag{5}$$

where

$$\begin{cases} P \in \mathbb{R}^{n\times n}, \quad P = P^T > 0, \\ Q : [-r_M, 0] \to \mathbb{R}^{n\times n}, \\ S : [-r_M, 0] \to \mathbb{R}^{n\times n}, \quad S^T(\xi) = S(\xi) > 0, \\ R : [-r_M, r_M] \to \mathbb{R}^{n\times n}, \quad R(-\xi) = R^T(\xi), \\ K_i \in \mathbb{R}^{n\times n}, \quad K_i = K_i^T \geq 0, \ i = 1, 2. \end{cases} \tag{6}$$

The following theorem is well-known (Hale and Lunel, 1993):

Theorem 1. *System (1) is asymptotically stable if there exists a quadratic Lyapunov functional $V(x_t)$ of the form (5) such that for some $\varepsilon > 0$ and arbitrary $x_t \in C$*

$$V(x_t) \geq \varepsilon x^T(t)x(t) \tag{7}$$

and its derivative along the solution of (1) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) \leq -\varepsilon x^T(t)x(t). \tag{8}$$

Using (4), it can be easily verified that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V(x_t) = {} & x^T(t)P\big[A(t)x(t) + B(t)x(t-r_M)\big] \\ & + x^T(t)PB(t)\int_{-r_M}^{-r(t)} \Big[A(t+\theta)x(t+\theta) + B(t+\theta)x\big(t+\theta-r(t+\theta)\big)\Big]\mathrm{d}\theta \\ & + \big[A(t)x(t) + B(t)x(t-r_M)\big]^T \int_{-r_M}^{0} Q(\xi)x(t+\xi)\,\mathrm{d}\xi \\ & + \left[\int_{-r_M}^{0} x^T(t+\xi)Q^T(\xi)\,\mathrm{d}\xi\right] \\ & \times B(t)\int_{-r_M}^{-r(t)} \Big[A(t+\theta)x(t+\theta) + B(t+\theta)x\big(t+\theta-r(t+\theta)\big)\Big]\mathrm{d}\theta \\ & + x^T(t)\int_{-r_M}^{0} Q(\xi)\dot{x}(t+\xi)\,\mathrm{d}\xi + \int_{-r_M}^{0}\left[\int_{-r_M}^{0} x^T(t+\xi)R(\xi-\eta)\dot{x}(t+\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi \\ & + \int_{-r_M}^{0} x^T(t+\xi)S(\xi)\dot{x}(t+\xi)\,\mathrm{d}\xi + \frac{1}{2}(r_M - r_m)x^T(t)(K_1+K_2)x(t) \\ & - \frac{1}{2}\int_{-r_M}^{-r_m} \Big[x^T(t+\theta)K_1x(t+\theta) + \big(1-\dot{r}(t+\theta)\big)x^T\big(t+\theta-r(t+\theta)\big) \\ & \times K_2x\big(t+\theta-r(t+\theta)\big)\Big]\mathrm{d}\theta. \end{aligned} \tag{9}$$

Noting that (Gu, 1997)

$$\int_{-r_M}^{0}\left[\int_{-r_M}^{0} x^T(t+\xi)\dot{R}(\xi-\eta)x(t+\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi = 0 \tag{10}$$

and integrating by parts in (9), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) &= \frac{1}{2}x^T(t)\big[PA(t)+A^T(t)P+S(0)+Q(0)+Q^T(0)+(r_M-r_m)(K_1+K_2)\big]x(t)\\
&\quad + x^T(t)\big[PB(t)-Q(-r_M)\big]x(t-r_M) - \frac{1}{2}x^T(t-r_M)S(-r_M)x(t-r_M)\\
&\quad + x^T(t)\int_{-r_M}^{0}\big[A^T(t)Q(\xi)-\dot{Q}(\xi)+R^T(\xi)\big]x(t+\xi)\,\mathrm{d}\xi\\
&\quad + x^T(t-r_M)\int_{-r_M}^{0}\big[B^T(t)Q(\xi)-R^T(\xi+r_M)\big]x(t+\xi)\,\mathrm{d}\xi\\
&\quad - \frac{1}{2}\int_{-r_M}^{0} x^T(t+\xi)\dot{S}(\xi)x(t+\xi)\,\mathrm{d}\xi\\
&\quad + x^T(t)PB(t)\int_{-r_M}^{-r(t)}\Big[A(t+\theta)x(t+\theta)+B(t+\theta)x\big(t+\theta-r(t+\theta)\big)\Big]\mathrm{d}\theta\\
&\quad + \left[\int_{-r_M}^{0} x^T(t+\xi)Q^T(\xi)\,\mathrm{d}\xi\right]\\
&\quad \times B(t)\int_{-r_M}^{-r(t)}\Big[A(t+\theta)x(t+\theta)+B(t+\theta)x\big(t+\theta-r(t+\theta)\big)\Big]\mathrm{d}\theta\\
&\quad - \frac{1}{2}\int_{-r_M}^{-r_m}\Big[x^T(t+\theta)K_1x(t+\theta)+\big(1-\dot{r}(t+\theta)\big)x^T\big(t+\theta-r(t+\theta)\big)\\
&\quad \times K_2 x\big(t+\theta-r(t+\theta)\big)\Big]\mathrm{d}\theta.
\end{aligned}\tag{11}$$

3. Discretization

Equations (5) and (11) indicate that the Lyapunov functional and its derivative are both quadratic functions. Choosing Q, R, S, T and $\Im$ to be piecewise linear, we can write (7) and (8) as regular linear matrix inequalities. More specifically, the interval $[-r_M, 0]$ is partitioned into N segments $[\delta_{i-1}, \delta_i]$ such that

$$\delta_i = -r_M + ih, \quad i = 0, 1, 2, \ldots, N, \quad h = r_M/N. \tag{12}$$

Write

$$\begin{cases} Q_i = Q(\delta_i), & S_i = S(\delta_i), \qquad T_i = T(\delta_i),\\ T_{kj,i} = T_{kj}(\delta_i), & \Im_{kj,i} = \Im_{kj}(\delta_i). \end{cases}\tag{13}$$

Define

$$\tau_i = ih, \quad i = 0, \pm 1, \pm 2, \ldots, \pm N, \quad R_i = R_{-i}^T = R(\tau_i). \tag{14}$$

For $0 \leq \alpha \leq 1$, set

$$\begin{cases} Q^i(\alpha) \triangleq Q(\delta_{i-1} + \alpha h) = (1-\alpha)Q_{i-1} + \alpha Q_i, \\ S^i(\alpha) \triangleq S(\delta_{i-1} + \alpha h) = (1-\alpha)S_{i-1} + \alpha S_i, \\ T^i(\alpha) \triangleq T(\delta_{i-1} + \alpha h) = (1-\alpha)T_{i-1} + \alpha T_i, \\ T_{kj}^i(\alpha) \triangleq T_{kj}(\delta_{i-1} + \alpha h) = (1-\alpha)T_{kj,i-1} + \alpha T_{kj,i}, \\ \Im_{kj}^i(\alpha) \triangleq \Im_{kj}(\delta_{i-1} + \alpha h) = (1-\alpha)\Im_{kj,i-1} + \alpha \Im_{kj,i}, \\ R^i(\alpha) \triangleq R(\tau_{i-1} + \alpha h) = (1-\alpha)R_{i-1} + \alpha R_i. \end{cases} \tag{15}$$

4. Characterization of the Lyapunov Functional

Using the discretization of Q, S and R, we can write (7) in the form of a linear matrix inequality.

Theorem 2. *Given piecewise linear Q, S and R, the Lyapunov functional (5) satisfies (7) if*

$$S_i \geq 0, \quad i = 0, 1, 2, \ldots, N, \tag{16}$$

$$\begin{bmatrix} \hat{R} & \hat{Q}^T \\ \hat{Q} & P \end{bmatrix} > 0, \tag{17}$$

$$K_i = K_i^T \geq 0, \quad i = 1, 2, \tag{18}$$

where

$$\hat{R} = \begin{bmatrix} R_0 & R_{-1} & \cdots & R_{-N} \\ R_1 & R_0 & \cdots & R_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ R_N & R_{N-1} & \cdots & R_0 \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_N \end{bmatrix} \tag{19}$$

Proof. We see that

$$\begin{aligned} V(x_t) \geq & \frac{1}{2}x^T(t)Px(t) + x^T(t)\int_{-r_m}^0 Q(\xi)x(t+\xi)\,\mathrm{d}\xi \\ & + \frac{1}{2}\int_{-r_m}^0 \Big[\int_{-r_m}^0 x^T(t+\xi)R(\xi-\eta)x(t+\eta)\,\mathrm{d}\eta\Big]\mathrm{d}\xi \\ & + \frac{1}{2}\int_{-r_m}^0 x^T(t+\xi)S(\xi)x(t+\xi)\,\mathrm{d}\xi \geq \varepsilon x^T(t)x(t) \end{aligned}$$

The last step was proved in (Gu, 1997). ■

5. Characterization of the Lyapunov Derivative

Now we first give the following key lemma that plays an important role in constructing the stability criterion for system (1).

Lemma 1. *Condition (8) is satisfied if there exist symmetric matrix functions*

$$T(\xi) = \begin{bmatrix} T_{11}(\xi) & T_{12}(\xi) \\ T_{12}^T(\xi) & T_{22}(\xi) \end{bmatrix}, \quad \Im(\xi) = \begin{bmatrix} \Im_{33}(\xi) & \Im_{34}(\xi) \\ \Im_{34}^T(\xi) & \Im_{44}(\xi) \end{bmatrix}$$

satisfying

$$\int_{-r_M}^{0} T(\xi)\,\mathrm{d}\xi = 0 \tag{20}$$

together with

$$\Im(\xi) \geq 0 \tag{21}$$

and a symmetric matrix

$$L = \begin{bmatrix} L_1 & L_{12} \\ L_{12}^T & L_2 \end{bmatrix}$$

such that

$$H_{\Delta YT} = \begin{bmatrix} \Delta^1 + Y^1 + T_{11}(\xi) & -\Delta^2 + T_{12}(\xi) & \Delta^3(\xi) + Y^3 \\ -\Delta^{2T} + T_{12}^T(\xi) & S(-r_M) + T_{22}(\xi) & \Delta^4(\xi) \\ \Delta^{3T}(\xi) + Y^{3T} & \Delta^{4T}(\xi) & \frac{1}{r_M}(\dot{S}(\xi) + Y^2) \end{bmatrix} > 0 \tag{22}$$

$$H_{LGK} = \begin{bmatrix} L_1 & L_{12} & \Gamma^1(\theta) & \Gamma^2(\theta) \\ L_{12}^T & L_2 & \Gamma^3(\xi,\theta) & \Gamma^4(\xi,\theta) \\ \Gamma^{1T}(\xi) & \Gamma^{3T}(\xi,\theta) & K_1 - \Im_{33}(\xi) & -\Im_{34}(\xi) \\ \Gamma^{2T}(\xi) & \Gamma^{4T}(\xi,\theta) & -\Im_{34}^T(\xi) & (1-\beta)K_2 - \Im_{44}(\xi) \end{bmatrix} \geq 0 \tag{23}$$

for all

$$\xi \in [-r_M, 0], \quad \theta \in [-r_M, -r_m], \quad \begin{bmatrix} A(t) & B(t) \end{bmatrix} \in \Omega, \quad \begin{bmatrix} A(t+\theta) & B(t+\theta) \end{bmatrix} \in \Omega,$$

where

$$\begin{cases} \Delta^1 = -\big[PA(t) + A^T(t)P + S(0) + Q(0) + Q^T(0)\big], \\ \Delta^2 = PB(t) - Q(-r_M), \\ \Delta^3(\xi) = A^T(t)Q(\xi) - \dot{Q}(\xi) + R^T(\xi), \\ \Delta^4(\xi) = B^T(t)Q(\xi) - R^T(\xi + r_M), \\ Y^1 = -(r_M - r_m)(L_1 + K_1 + K_2), \\ Y^2 = -\dfrac{r_M - r_m}{r_M} L_2, \\ Y^3 = \dfrac{r_M - r_m}{r_M} L_{12}, \\ \Gamma^1(\theta) = -PB(t)A(t+\theta), \\ \Gamma^2(\theta) = -PB(t)B(t+\theta), \\ \Gamma^3(\xi,\theta) = -r_M Q^T(\xi)B(t)A(t+\theta), \\ \Gamma^4(\xi,\theta) = -r_M Q^T(\xi)B(t)B(t+\theta). \end{cases} \tag{24}$$

Proof. Using (2) and the fact that

$$-\int_{-r_M}^{-r_m} x^T(\theta)Wx(\theta)\,\mathrm{d}\theta \le -\int_{-r_M}^{-r(t)} x^T(\theta)Wx(\theta)\,\mathrm{d}\theta$$

for any $W \ge 0$, we obtain

$$\begin{aligned} -\frac{1}{2}\int_{-r_M}^{-r_m} \Big[&x^T(t+\theta)K_1x(t+\theta) \\ &+ \big(1-\dot{r}(t+\theta)\big)x^T\big(t+\theta-r(t+\theta)\big)K_2x\big(t+\theta-r(t+\theta)\big)\Big]\mathrm{d}\theta \\ \le -\frac{1}{2}\int_{-r_M}^{-r(t)} \Big[&x^T(t+\theta)K_1x(t+\theta) \\ &+ (1-\beta)x^T\big(t+\theta-r(t+\theta)\big)K_2x\big(t+\theta-r(t+\theta)\big)\Big]\mathrm{d}\theta. \end{aligned} \tag{25}$$

Let

$$\vartheta^T(t,\xi,\theta,r) = \Big[x^T(t), x^T(t+\xi), x^T(t+\theta), x^T\big(t+\theta-r(t+\theta)\big)\Big],$$

$$\sigma^T(\theta,r) = \Big[x^T(t+\theta), x^T\big(t+\theta-r(t+\theta)\big)\Big].$$

Then we have

$$
\begin{aligned}
& x^T(t)PB(t)\int_{-r_M}^{-r(t)} \Big[A(t+\theta)x(t+\theta) + B(t+\theta)x\big(t+\theta-r(t+\theta)\big)\Big]\,\mathrm{d}\theta \\
&\quad + \left[\int_{-r_M}^{0} x^T(t+\xi)Q^T(\xi)\,\mathrm{d}\xi\right] \\
&\qquad \times B(t)\int_{-r_M}^{-r(t)} \Big[A(t+\theta)x(t+\theta) + B(t+\theta)x\big(t+\theta-r(t+\theta)\big)\Big]\,\mathrm{d}\theta \\
&\quad - \frac{1}{2}\int_{-r_M}^{-r(t)} \Big[x^T(t+\theta)K_1x(t+\theta) + (1-\beta)x^T\big(t+\theta-r(t+\theta)\big) \\
&\qquad \times K_2x\big(t+\theta-r(t+\theta)\big)\Big]\,\mathrm{d}\theta \\
&= \frac{1}{r_M}\int_{-r_M}^{0}\left\{\int_{-r_M}^{-r(t)} x^T(t)PB(t)\Big[A(t+\theta)x(t+\theta) + B(t+\theta)\right. \\
&\qquad \left. \times x\big(t+\theta-r(t+\theta)\big)\Big]\,\mathrm{d}\theta\right\}\mathrm{d}\xi + \int_{-r_M}^{0}\left\{\int_{-r_M}^{-r(t)} x^T(t+\xi)Q^T(\xi)\right. \\
&\quad \left. \times B(t)\Big[A(t+\theta)x(t+\theta) + B(t+\theta)x\big(t+\theta-r(t+\theta)\big)\Big]\,\mathrm{d}\theta\right\}\mathrm{d}\xi \\
&\quad - \frac{1}{2r_M}\int_{-r_M}^{0}\left\{\int_{-r_M}^{-r(t)} \Big[x^T(t+\theta)K_1x(t+\theta) + (1-\beta)x^T\big(t+\theta-r(t+\theta)\big)\right. \\
&\qquad \left. \times K_2x\big(t+\theta-r(t+\theta)\big)\Big]\,\mathrm{d}\theta\right\}\mathrm{d}\xi \\
&= -\frac{1}{2r_M}\int_{-r_M}^{0}\left\{\int_{-r_M}^{-r(t)} \vartheta^T(t,\xi,\theta,r)H_{L\Gamma K}\vartheta(t,\xi,\theta,r)\,\mathrm{d}\theta\right\}\mathrm{d}\xi \\
&\quad + \frac{1}{2r_M}\int_{-r_M}^{0}\left[\int_{-r_M}^{-r(t)} x^T(t)L_1x(t)\,\mathrm{d}\theta\right]\mathrm{d}\xi \\
&\quad + \frac{1}{2r_M}\int_{-r_M}^{0}\left[\int_{-r_M}^{-r(t)} x^T(t+\xi)L_2x(t+\xi)\,\mathrm{d}\theta\right]\mathrm{d}\xi \\
&\quad + \frac{1}{r_M}\int_{-r_M}^{0}\left[\int_{-r_M}^{-r(t)} x^T(t)L_{12}x(t+\xi)\,\mathrm{d}\theta\right]\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2r_M}\int_{-r_M}^{0}\left[\int_{-r_M}^{-r(t)}\sigma^T(\theta,r)\Im(\xi)\sigma(\theta,r)\,\mathrm{d}\theta\right]\mathrm{d}\xi \\
\leq\ &-\frac{1}{2r_M}\int_{-r_M}^{0}\left\{\int_{-r_M}^{-r(t)}\vartheta^T(t,\xi,\theta,r)H_{L\Gamma K}\vartheta(t,\xi,\theta,r)\,\mathrm{d}\theta\right\}\mathrm{d}\xi \\
&+\frac{1}{2}(r_M-r_m)x^T(t)L_1x(t)+\frac{r_M-r_m}{2r_M}\int_{-r_M}^{0}x^T(t+\xi)L_2x(t+\xi)\,\mathrm{d}\xi \\
&+\frac{r_M-r_m}{r_M}x^T(t)\int_{-r_M}^{0}L_{12}x(t+\xi)\,\mathrm{d}\xi \\
&-\frac{1}{2r_M}\int_{-r_M}^{0}\left[\int_{-r_M}^{-r(t)}\sigma^T(\theta,r)\Im(\xi)\sigma(\theta,r)\,\mathrm{d}\theta\right]\mathrm{d}\xi. \qquad (26)
\end{aligned}
$$

Noting that (25)–(26) and using the notation of (24), from (11) we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(x_t)\ \leq\ &-\frac{1}{2}\int_{-r_M}^{0}x^T(t+\xi)\big[\dot{S}(\xi)+Y^2\big]x(t+\xi)\,\mathrm{d}\xi \\
&+x^T(t)\int_{-r_M}^{0}\big[\Delta^3(\xi)+Y^3\big]x(t+\xi)\,\mathrm{d}\xi \\
&-\frac{1}{2}x^T(t)(\Delta^1+Y^1)x(t)+x^T(t-r_M)\int_{-r_M}^{0}\Delta^4(\xi)x(t+\xi)\,\mathrm{d}\xi \\
&-\frac{1}{2}x^T(t-r_M)S(-r_M)x(t-r_M)+x^T(t)\Delta^2x(t-r_M) \\
&-\frac{1}{2r_M}\int_{-r_M}^{0}\left\{\int_{-r_M}^{-r(t)}\vartheta^T(t,\xi,\theta,r)H_{L\Gamma K}\vartheta(t,\xi,\theta,r)\,\mathrm{d}\theta\right\}\mathrm{d}\xi \\
&-\frac{1}{2r_M}\int_{-r_M}^{0}\left[\int_{-r_M}^{-r(t)}\sigma^T(\theta,r)\Im(\xi)\sigma(\theta,r)\,\mathrm{d}\theta\right]\mathrm{d}\xi \qquad (27)
\end{aligned}
$$

For

$$
\dot{S}(\xi)+Y^2>0 \qquad (28)
$$

which is guaranteed by (22), we can write

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(x_t)\ \leq\ &-\frac{1}{2}\int_{-r_M}^{0}\omega^T(\xi)\big[\dot{S}(\xi)+Y^2\big]^{-1}\omega(\xi)\,\mathrm{d}\xi \\
&-\frac{1}{2}\begin{bmatrix}x^T(t) & x^T(t-r_M)\end{bmatrix}\Delta\begin{bmatrix}x(t)\\ x(t-r_M)\end{bmatrix}
\end{aligned}
$$

$$-\frac{1}{2r_M}\int_{-r_M}^{0}\left\{\int_{-r_M}^{-r(t)}\vartheta^T(t,\xi,\theta,r)H_{L\Gamma K}\vartheta(t,\xi,\theta,r)\,\mathrm{d}\theta\right\}\mathrm{d}\xi$$

$$-\frac{1}{2r_M}\int_{-r_M}^{0}\left[\int_{-r_M}^{-r(t)}\sigma^T(\theta,r)\Im(\xi)\sigma(\theta,r)\,\mathrm{d}\theta\right]\mathrm{d}\xi, \tag{29}$$

where

$$\omega(\xi) = \left[\dot{S}(\xi)+Y^2\right]x^T(t+\xi) - (\Delta^3(\xi)+Y^3)^T x(t) - \Delta^{4T}(\xi)x(t-r_M),$$

$$\begin{aligned}\Delta &= \begin{bmatrix}\Delta^1+Y^1 & -\Delta^2\\ -\Delta^{2T} & S(-r_M)\end{bmatrix}\\ &\quad -\int_{-r_M}^{0}\begin{bmatrix}\Delta^3(\xi)+Y^3\\ \Delta^4(\xi)\end{bmatrix}\left[\dot{S}(\xi)+Y^2\right]^{-1}\left[\Delta^{3T}(\xi)+Y^{3T}\quad \Delta^{4T}(\xi)\right]\mathrm{d}\xi\\ &= \int_{-r_M}^{0}\left\{\frac{1}{r_M}\begin{bmatrix}\Delta^1+Y^1 & -\Delta^2\\ -\Delta^{2T} & S(-r_M)\end{bmatrix}+\frac{1}{r_M}T(\xi)\right.\\ &\quad\left.-\begin{bmatrix}\Delta^3(\xi)+Y^3\\ \Delta^4(\xi)\end{bmatrix}\left[\dot{S}(\xi)+Y^2\right]^{-1}\left[\Delta^{3T}(\xi)+Y^{3T}\quad \Delta^{4T}(\xi)\right]\right\}\mathrm{d}\xi.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{r_M}\begin{bmatrix}\Delta^1+Y^1 & -\Delta^2\\ -\Delta^{2T} & S(-r_M)\end{bmatrix}+\frac{1}{r_M}T(\xi)\\ &\quad-\begin{bmatrix}\Delta^3(\xi)+Y^3\\ \Delta^4(\xi)\end{bmatrix}\left[\dot{S}(\xi)+Y^2\right]^{-1}\left[\Delta^{3T}(\xi)+Y^{3T}\quad \Delta^{4T}(\xi)\right]>0.\end{aligned} \tag{30}$$

and (21) and (23) for all $\xi\in[-r_M,0]$ and $\theta\in[-r_M,-r_m]$ are sufficient conditions for (8). Expression (30), together with (28), is equivalent to (22) in view of Schur's complement. ■

Remark 1. If $r_M-r_m\to 0^+$, then $Y^1\to 0^-$, $Y^2\to 0^-$ and $Y^3\to 0^+$. Therefore inequality (22) approaches inequality (12) in (Gu, 1997).

Remark 2. If we choose the symmetric matrix function $T(\xi)$ such that

$$\int_{-r_M}^{0}T(\xi)\,\mathrm{d}\xi\le 0, \tag{31}$$

it is easy to see that the result of Lemma 1 is also true.

Choosing piecewise linear functions as described in Section 3, conditions (21)–(23) can be written as linear matrix inequalities. The result is as follows.

Theorem 3. *If the functions Q, R, S, T and $\Im$ are chosen as piecewise linear as described by (13)–(15), then conditions (21)–(23) are equivalent to*

$$\Im_i^{\rm Disc} = \begin{bmatrix} \Im_{33,i+k-1} & \Im_{34,i+k-1} \\ \Im_{34,i+k-1}^T & \Im_{44,i+k-1} \end{bmatrix} \geq 0, \tag{32}$$

$$H_{\Delta YT}^{\rm Disc} = \begin{bmatrix} \Delta^1 + Y_0^1 + T_{11,i+k-1} & -\Delta^2 + T_{12,i+k-1} & \Delta_{ik}^3 + Y_0^3 \\ -\Delta^{2T} + T_{12,i+k-1}^T & S_0 + T_{22,i+k-1} & \Delta_{ik}^4 \\ \Delta_{ik}^{3T} + Y_0^{3T} & \Delta_{ik}^{4T} & \frac{1}{N}(S_i - S_{i-1} + Y_0^2) \end{bmatrix} > 0, \tag{33}$$

$$H_{L\Gamma K}^{\rm Disc} = \begin{bmatrix} L_1 & L_{12} & \Gamma^1(\theta) & \Gamma^2(\theta) \\ L_{12}^T & L_2 & \Gamma_{ik}^3(\theta) & \Gamma_{ik}^4(\theta) \\ \Gamma^{1T}(\theta) & \Gamma_{ik}^{3T}(\theta) & K_1 - \Im_{33,i+k-1} & -\Im_{34,i+k-1} \\ \Gamma^{2T}(\theta) & \Gamma_{ik}^{4T}(\theta) & -\Im_{34,i+k-1}^T & (1-\beta)K_2 - \Im_{44,i+k-1} \end{bmatrix} \geq 0, \tag{34}$$

for all $i = 1, 2, \ldots, N$, $k = 0, 1$, $\theta \in [-r_M, -r_m]$, $[A(t), B(t)] \in \Omega$, $[A(t+\theta), B(t+\theta)] \in \Omega$, where

$$\begin{cases} \Delta^1 = -\left[PA(t) + A^T(t)P + S_N + Q_N + Q_N^T\right], \\ \Delta^2 = PB(t) - Q_0, \\ \Delta_{i0}^3 = hA^T(t)Q_{i-1} - (Q_i - Q_{i-1}) + hR_{i-N-1}^T, \\ \Delta_{i0}^4 = hB^T(t)Q_{i-1} - hR_{i-1}^T, \\ \Delta_{i1}^3 = hA^T(t)Q_i - (Q_i - Q_{i-1}) + hR_{i-N}^T, \\ \Delta_{i1}^4 = hB^T(t)Q_i - hR_i^T, \\ Y_0^1 = -(Nh - r_m)(L_1 + K_1 + K_2), \\ Y_0^2 = -\dfrac{Nh - r_m}{N} L_2, \\ Y_0^3 = \dfrac{r_M - Nh}{N} L_{12}, \end{cases} \tag{35a}$$

$$\begin{cases} \Gamma^1(\theta) = -PB(t)A(t+\theta), \\ \Gamma^2(\theta) = -PB(t)B(t+\theta) \\ \Gamma^3_{i0}(\theta) = -NhQ^T_{i-1}B(t)A(t+\theta), \\ \Gamma^3_{i1}(\theta) = -NhQ^T_iB(t)A(t+\theta), \\ \Gamma^4_{i0}(\theta) = -NhQ^T_{i-1}B(t)B(t+\theta), \\ \Gamma^4_{i1}(\theta) = -NhQ^T_iB(t)B(t+\theta) \end{cases} \tag{35b}$$

and (20) is equivalent to

$$T_0 + T_N + 2\sum_{i=1}^{N-1} T_i = 0. \tag{36}$$

Proof. For any $i = 1, 2, \ldots, N$, we have

$$\xi \in [\delta_{i-1}, \delta_i], \quad \xi = \delta_{i-1} + \alpha h.$$

Then we can write (21)–(23) as

$$\begin{bmatrix} \Im^i_{33}(\alpha) & \Im^i_{34}(\alpha) \\ \Im^{iT}_{34}(\alpha) & \Im^i_{44}(\alpha) \end{bmatrix} \geq 0, \tag{37}$$

$$\begin{bmatrix} \Delta^1 + Y^1_0 + T^i_{11}(\alpha) & -\Delta^2 + T^i_{12}(\alpha) & \hat{\Delta}^3(\alpha) + \frac{1}{h}Y^3_0 \\ -\Delta^{2T} + T^{iT}_{12}(\alpha) & S_0 + T^i_{22}(\alpha) & \hat{\Delta}^4(\alpha) \\ \hat{\Delta}^{3T}(\alpha) + \frac{1}{h}Y^{3T}_0 & \hat{\Delta}^{4T}(\alpha) & \frac{1}{Nh^2}(S_i - S_{i-1} + Y^2_0) \end{bmatrix} > 0 \tag{38}$$

and

$$\begin{bmatrix} L_1 & L_{12} & \Gamma^1(\theta) & \Gamma^2(\theta) \\ L^T_{12} & L_2 & \hat{\Gamma}^3(\alpha,\theta) & \hat{\Gamma}^4(\alpha,\theta) \\ \Gamma^{1T}(\theta) & \hat{\Gamma}^{3T}(\alpha,\theta) & K_1 - \Im^i_{33}(\alpha) & -\Im^i_{34}(\alpha) \\ \Gamma^{2T}(\theta) & \hat{\Gamma}^{4T}(\alpha,\theta) & -\Im^{iT}_{34}(\alpha) & (1-\beta)K_2 - \Im^i_{44}(\alpha) \end{bmatrix} \geq 0, \tag{39}$$

where

$$\hat{\Delta}^3(\alpha) = A^T(t)Q^i(\alpha) - \frac{1}{h}(Q_i - Q_{i-1}) + [R^{i-N}(\alpha)]^T,$$

$$\hat{\Delta}^4(\alpha) = B^T(t)Q^i(\alpha) - R^{iT}(\alpha),$$

$$\hat{\Gamma}^3(\alpha,\theta) = -NhQ^{iT}(\alpha)B(t)A(t+\theta),$$

$$\hat{\Gamma}^4(\alpha,\theta) = -NhQ^{iT}(\alpha)B(t)B(t+\theta).$$

Multiplying the left- and right-hand sides of (38) by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & hI \end{bmatrix},$$

we see that

$$\begin{bmatrix} \Delta^1 + Y_0^1 + T_{11}^i(\alpha) & -\Delta^2 + T_{12}^i(\alpha) & h\hat{\Delta}^3(\alpha) + Y_0^3 \\ -\Delta^{2T} + T_{12}^{iT}(\alpha) & S_0 + T_{22}^i(\alpha) & h\hat{\Delta}^4(\alpha) \\ h\hat{\Delta}^{3T}(\alpha) + Y_0^{3T} & h\hat{\Delta}^{4T}(\alpha) & \frac{1}{N}(S_i - S_{i-1} + Y_0^2) \end{bmatrix} > 0. \quad (40)$$

Note that all the terms in (37), (39) and (40) are linear in α. Therefore, (37), (39) and (40) are valid for all α if and only if they are valid for $\alpha = 0$ and $\alpha = 1$, which are (32), (33) and (34) for $k = 0, 1$. The equivalence of (20) and (36) is proved in (Gu, 1997). ■

6. Stability Criterion

Based on the above discussion, now we state and establish the main theorem in this paper.

Theorem 4. *The time-varying delay system (20) is asymptotically stable if there exist real matrices P, $\hat{Q}$ and $\hat{R}$ satisfying (17), $S_i \geq 0$, $i = 0, 1, 2, \ldots, N$, $K_1^T = K_1 \geq 0$, $K_2^T = K_2 \geq 0$, T_i, $i = 0, 1, 2, \ldots, N$ satisfying (36), $\Im_i^{\mathrm{Disc}}$, $i = 1, 2, \ldots, N$ and a symmetric matrix L such that (32)–(34) are satisfied.*

Remark 3. Theorem 4 provides a *delay-dependent* condition for asymptotic stability of uncertain linear systems with a single *time-varying* delay in terms of linear matrix inequalities. The criterion does not require any parameter tuning and can be tested numerically very efficiently using interior-point algorithms which have been developed for solving linear matrix inequalities (Boyd *et al.*, 1994). It is interesting to note that h appears linearly. Therefore, a generalized eigenvalue problem (GEVP) as defined in Boyd *et al.* (1994), can be formulated to find a minimum acceptable $1/h$ and therefore, for a fixed number N of discretization segments, the maximum time-varying delay $r_M = Nh$ can be achieved to maintain asymptotic stability.

Some LMI solvers (Gahinet *et al.*, 1995) may not allow for equality constraints. Thus (36) may not be directly entered into the software. By Remark 2, condition (20) can be replaced with (31). The corresponding discretization condition is

$$T_0 + T_N + 2\sum_{i=1}^{N-1} T_i \leq 0. \quad (41)$$

The LMI's derived must hold for all possible uncertain system matrices. Here, we assume that the uncertainty set is polytopic, i.e.

$$\Omega = \mathrm{Co}\{(A_j, B_j) \mid j = 1, 2, \ldots, n_v\}. \tag{42}$$

Then it is easy to see that we have to check on the satisfaction of LMI's only at the vertices. By Theorem 4 the following corollary is easily obtained.

Corollary 1. *The time-varying delay system (1) with polytopic uncertainty (42) is asymptotically stable if there exist real matrices P, $\hat{Q}$ and $\hat{R}$ satisfying (17), $S_i \geq 0$, $i = 0, 1, 2, \ldots, N$, $K_1^T = K_1 \geq 0$, $K_2^T = K_2 \geq 0$, T_i, $i = 0, 1, 2, \ldots, N$ satisfying (41), $\Im_i^{\mathrm{Disc}}$, $i = 1, 2, \ldots, N$ satisfying (32) and a symmetric matrix L such that the following LMI's are satisfied:*

$$\Im_i^{\mathrm{Disc}} = \begin{bmatrix} \Im_{33,i+k-1} & \Im_{34,i+k-1} \\ \Im_{34,i+k-1}^T & \Im_{44,i+k-1} \end{bmatrix} \geq 0, \tag{43}$$

$$H_{\Delta\aleph T}^{\mathrm{Disc}-ijk} = \begin{bmatrix} \Delta_j^1 + Y_0^1 + T_{11,i+k-1} & -\Delta_j^2 + T_{12,i+k-1} & \Delta_{j,ik}^3 + Y_0^3 \\ -\Delta_j^{2T} + T_{12,i+k-1}^T & S_0 + T_{22,i+k-1} & \Delta_{j,ik}^4 \\ \Delta_{j,ik}^{3T} + Y_0^{3T} & \Delta_{j,ik}^{4T} & \frac{1}{N}(S_i - S_{i-1} + Y_0^2) \end{bmatrix} > 0, \tag{44}$$

$$H_{L\Gamma K}^{\mathrm{Disc}-ijkl} = \begin{bmatrix} L_1 & L_{12} & \Gamma_{jl}^5 & \Gamma_{jl}^6 \\ L_{12}^T & L_2 & \Gamma_{jl,ik}^7 & \Gamma_{jl,ik}^8 \\ \Gamma_{jl}^{5T} & \Gamma_{jl,ik}^{7T} & K_1 - \Im_{33,i+k-1} & -\Im_{34,i+k-1} \\ \Gamma_{jl}^{6T} & \Gamma_{jl,ik}^8 & -\Im_{34,i+k-1}^T & (1-\beta)K_2 - \Im_{44,i+k-1} \end{bmatrix} \geq 0, \tag{45}$$

for all $i = 1, 2, \ldots, N$, $k = 0, 1$, $j = 1, 2, \ldots, n_v$, $l = 1, 2, \ldots, n_v$ *where*

$$\begin{cases} \Delta_j^1 = -[PA_j + A_j^T P + S_N + Q_N + Q_N^T], \\ \Delta_j^2 = PB_j - Q_0, \\ \Delta_{j,i0}^3 = hA_j^T Q_{i-1} - (Q_i - Q_{i-1}) + hR_{i-N-1}^T, \\ \Delta_{j,i0}^4 = hB_j^T Q_{i-1} - hR_{i-1}^T, \\ \Delta_{j,i1}^3 = hA_j^T Q_i - (Q_i - Q_{i-1}) + hR_{i-N}^T, \\ \Delta_{j,i1}^4 = hB_j^T Q_i - hR_i^T, \end{cases} \tag{46a}$$

$$\begin{cases} Y_0^1 = -(Nh - r_m)(L_1 + K_1 + K_2), \\ Y_0^2 = -\dfrac{Nh - r_m}{N} L_2, \\ Y_0^3 = \dfrac{Nh - r_m}{N} L_{12}, \\ \Gamma_{jl}^1 = -PB_j A_l, \\ \Gamma_{jl}^2 = -PB_j B_l, \\ \Gamma_{jl,i0}^3 = -NhQ_{i-1}^T B_j A_l, \\ \Gamma_{jl,i1}^3 = -NhQ_i^T B_j A_l, \\ \Gamma_{jl,i0}^4 = -NhQ_{i-1}^T B_j B_l, \\ \Gamma_{jl,i1}^4 = -NhQ_i^T B_j B_l. \end{cases} \tag{46b}$$

Remark 4. It should be pointed out that the results of this paper can be improved by choosing a quadratic Lyapunov functional whose kernel is a function of two variables and using a generalized discretization scheme proposed in (Gu, 1999).

We present two examples to illustrate the effectiveness of the approach.

Example 1. Consider the following linear system with time-varying delay which was studied in (Li and de Souza, 1997; Niculescu *et al.*, 1995):

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x\big(t - r(t)\big). \tag{47}$$

By the criteria given in both the papers system (47) is asymptotically stable for any $r(t)$ satisfying $0 \le r(t) \le r_M = 0.8571$ and $0 \le r(t) \le r_M = 0.7433$, respectively. Those criteria allow for a fast time-varying delay and impose no requirements on the derivative of the delay.

Applying our approach, we obtain the maximum values of r_M listed in Tables 1 and 2 (Figs. 1 and 2) for $r_m = 0.5r_M$, $\beta = 0.1$ and $r_m = 0.9r_M$, $\beta = 0.1$, respectively. When compared with the results in (Li and de Souza, 1997a; Niculescu *et al.*, 1995), it is clear that we can obtain less conservative stability bounds by taking advantage of the additional information on the delay.

When $r_m = r_M$, Gu (1997) also studied this example and obtained the maximum time delay which approached the analytical limit.

For $N = 1$, Tab. 3 (Fig. 3) indicates that r_M approaches the result of Gu (1997) as $r_m \to r_M$ even though $\beta \neq 0$. For other N's, we also can give a similar result.

Tab. 1. Bound r_M calculated for $r_m = 0.5r_M$, $\beta = 0.1$.

N	1	2	3	4	5	6	7	8	9	10
r_M	0.942	0.949	0.953	0.956	0.958	0.960	0.961	0.962	0.963	0.964

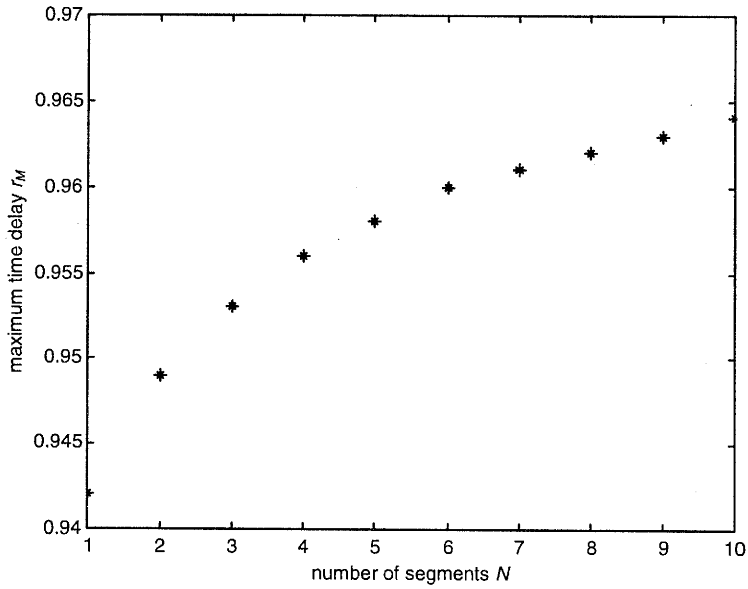

Fig. 1. Variability of r_M for $r_m = 0.5r_M$, $\beta = 0.1$.

Tab. 2. Bound r_M calculated for $r_m = 0.9r_M$, $\beta = 0.1$.

N	1	2	3	4	5	6	7	8	9	10
r_M	1.771	1.822	1.847	1.858	1.864	1.867	1.869	1.870	1.872	1.874

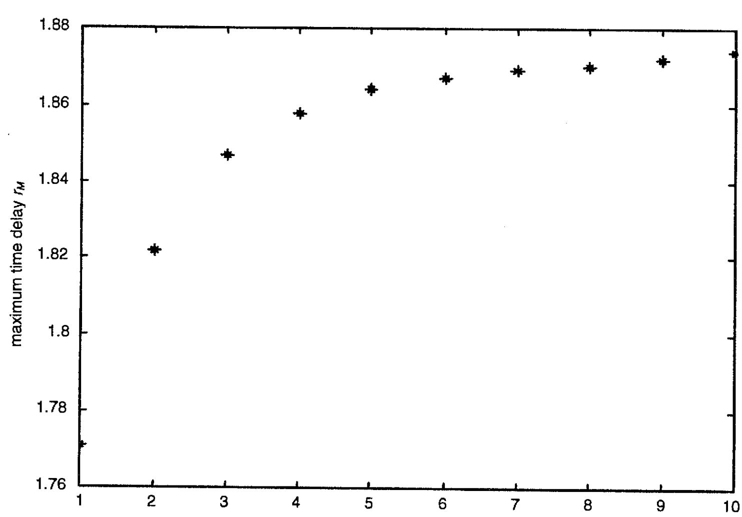

Fig. 2. Variability of r_M for $r_m = 0.9r_M$, $\beta = 0.1$.

Tab. 3. Bound r_M calculated for $r_m = \delta r_M$, $\beta = 0.1$, $N = 1$.

δ	0.90	0.91	0.92	0.93	0.94	0.95	0.96
r_M	1.771	1.840	1.919	2.010	2.118	2.250	2.415
δ	0.97	0.98	0.99	0.999	0.9999	0.99999	1.00
r_M	2.638	2.972	3.556	4.938	5.258	5.296	5.30

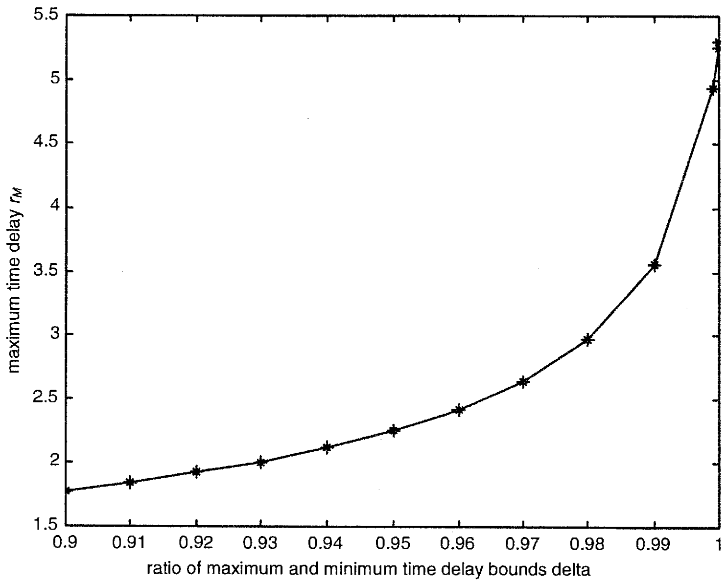

Fig. 3. Variability of r_M for $r_m = \delta r_M$, $\beta = 0.1$, $N = 1$.

Example 2. Consider the following uncertain time-varying delay system studied by Gu (1997):

$$\dot{x}(t) = \begin{bmatrix} -2+\rho(t) & \rho(t) \\ \rho(t) & -0.9+\rho(t) \end{bmatrix} x(t) + \begin{bmatrix} -1+\rho(t) & 0 \\ -1 & -1-\rho(t) \end{bmatrix} x\big(t-r(t)\big), \tag{48}$$

where

$$|\rho(t)| \le 0.1.$$

It can be modelled as a polytopic system, with $n_v = 2$ and

$$A_1 = \begin{bmatrix} -2.1 & -0.1 \\ -0.1 & -1.0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1.9 & 0.1 \\ 0.1 & -0.8 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -1.1 & 0 \\ -1 & -0.9 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -0.9 & 0 \\ -1 & -1.1 \end{bmatrix}.$$

The maximum time-varying delays r_M are listed in Tab. 4 (Fig. 4) for $r_m = 0.9r_M$ and $\beta = 0.1$.

For $N = 1$, as $r_m \to r_M$, Tab. 5 (Fig. 5) shows that the maximum time-varying delay r_M approaches Gu's result even though $\beta \neq 0$.

Tab. 4. Bound r_M for $r_m = 0.9r_M$, $\beta = 0.1$.

N	1	2	3	4	5	6	7	8	9	10
r_M	1.297	1.343	1.359	1.366	1.370	1.373	1.375	1.377	1.378	1.379

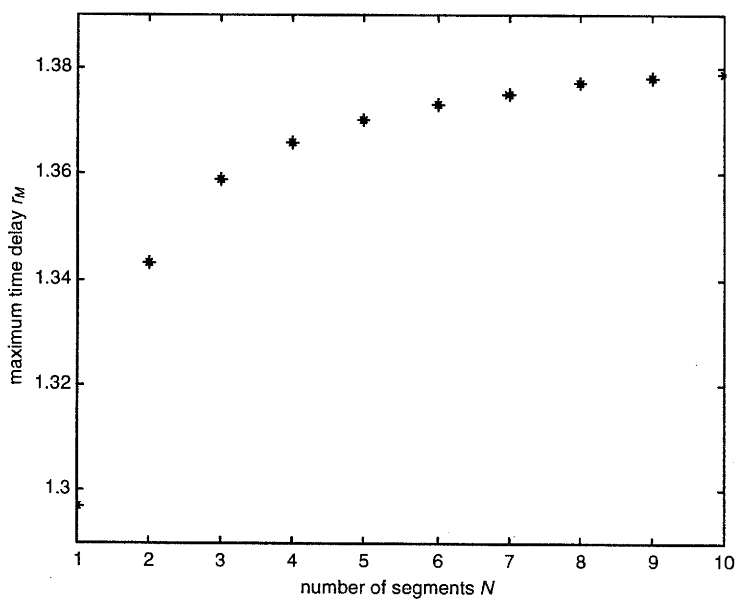

Fig. 4. Variability of r_M for $r_m = 0.9r_M$, $\beta = 0.1$.

Tab. 5. Bound r_M for $r_m = \delta r_M$, $\beta = 0.1$, $N = 1$.

δ	0.90	0.91	0.92	0.93	0.94	0.95
r_M	1.297	1.340	1.388	1.442	1.504	1.576
δ	0.96	0.97	0.98	0.99	0.999	0.9999
r_M	1.663	1.769	1.905	2.089	2.331	2.362

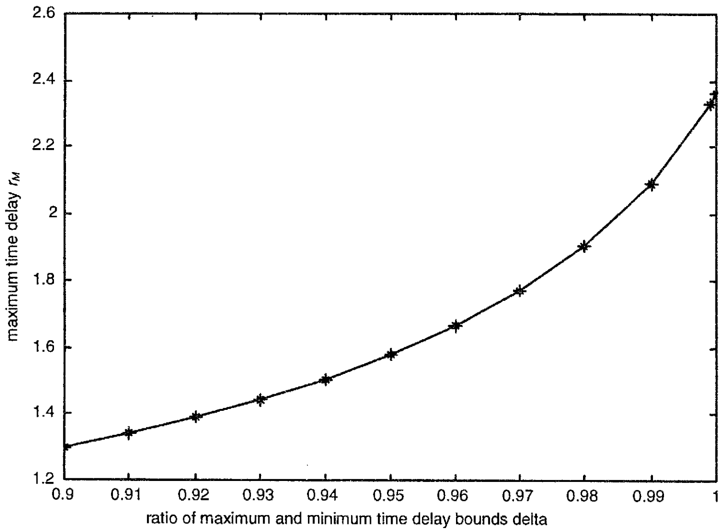

Fig. 5. Variability of r_M for $r_m = \delta r_M$, $\beta = 0.1$, $N = 1$.

7. Conclusion

The stability problem for a class of uncertain linear system with a single time-varying delay has been investigated. Delay-dependent criteria have been proposed by employing the *discretized Lyapunov functional approach.* Numerical examples show significant improvements over the results existing in the literature. As the lower bound approaches the upper one, a fixed delay limit is recovered even though the rate of change of delay remains non-zero.

References

Boyd S., Ghaoni L.El., Feron E. and Balakrishnan V. (1994): *Linear Matrix Inequalities in Systems and Control Theory.* — Philadelphia: SIAM.

Chen J. and Latchman H.A. (1995): *Frequency sweeping tests for stability independent of delay.* — IEEE Trans. Automat. Contr., AC-40, No.9, pp.1640–1645.

Gahinet P., Nemirovski A., Laub A.J. and Chilali M. (1995): *LMI Control Toolbox: For Use with MATLAB.* — Natick, MA: MathWorks.

Gu K. (1997): *Discretized LMI set in the stability problem of linear uncertain time-delay systems.* — Int. J. Contr., Vol.68, No.4, pp.923–934.

Gu K. (1999a): *A generalized discretization scheme of Lyapunov functional in the stability problem of linear uncertain time-delay systems.* — Int. J. Robust Nonlin. Contr., Vol.9, No.1, pp.1–14.

Gu K. (1999b): *Discretized Lyapunov functional for uncertain systems with multiple time-delay.* — Int. J. Contr., Vol.72, No.16, pp.1436–1445.

Gu K., Luo A.C.J. and Niculescu S.-I. (1999): *Discretized Lyapunov functional for systems with distributed delays.* — Proc. *European Control Conference*, Karlsruhe, Germany.

Hale J.K. and Verduyn Lunel S.M. (1993): *Introduction to Functional Differential Equations.* — New York: Springer-Verlag.

Han Q.-L. and Mehdi D. (1998): *Comments on 'Robust control for parameter uncertain delay systems in state and control input'.* — Automatica, Vol.34, No.12, pp.1665–1666.

Han Q.-L. and Mehdi D. (1999a): *Robust stabilization for uncertain time-varying delay constrained systems with delay-dependence.* — Int. J. Appl. Math. Comp. Sci., Vol.9, No.2, pp.293–311.

Han Q.-L. and Mehdi D (1999b): *Robust H_∞ controller synthesis for uncertain systems with multiple time-varying delays: An LMI approach.* — Proc. 14th IFAC *World Congress*, Beijing, P.R. China, Vol.C, pp.271–276.

Li X. and de Souza C.E. (1997): *Criteria for robust stability and stabilization of uncertain linear systems with state delay.* — Automatica, Vol.33, No.9, pp.1657–1662.

Malek-Zavarei M. and Jamshidi M. (1987): *Time Delay Systems: Analysis, Optimization and Applications.* — Amsterdam: North-Holland.

Mori T. (1985): *Criteria for asymptotic stability of linear time-delay systems.* — IEEE Trans. Automat. Contr., Vol.AC-30, No.2, pp.158–161.

Niculescu S.-I., Dion J.-M. and Dugard L. (1996): *Robust stabilization for uncertain time-delay systems containing saturating actuators.* — IEEE Trans. Automat. Contr., Vol.AC-41, No.5, pp.742–747.

Niculescu S.-I., Neto A.T., Dugard L. and Dion J.M. (1995): *Delay-dependent stability of linear systems with delayed state: An LMI approach.* – Proc. 34th IEEE Conf. *Decision and Control*, New Orleans, USA, Vol.2, pp.1495–1496.

Phoojaruenchanachai S. and Furuta K. (1992): *Memoryless stabilization of uncertain linear systems including time-varying state delays.* — IEEE Trans. Automat. Contr., Vol.AC-37, No.7, pp.1022–1026.

Received: 18 May 1999
Revised: 26 January 2000