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DYNAMIC ALGORITHM FOR LINEAR QUADRATIC GAUSSIAN PREDICTIVE CONTROL

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In this paper, the optimal control law is derived for a multi-variable state-space Linear Quadratic Gaussian Predictive Controller (LQGPC). A dynamic performance index is utilized resulting in an optimal steady-state controller. Knowledge of future reference values is incorporated into the controller design and the solution is derived using the method of Lagrange multipliers. It is shown how the well-known GPC controller can be obtained as a special case of the LQGPC controller design. The important advantage of using the LQGPC framework for designing predictive controllers is that, based on stabilizing properties of LQG control, it enables a systematic approach to selection of the design parameters to yield a stable closed-loop system. The system model considered in this paper can be further extended to also include direct feed-through and knowledge about future external inputs.

Keywords: state-space design, multi-variable control, linear quadratic Gaussian predictive control, generalized predictive control

1. Introduction

An increasing popularity of Model Based Predictive Control algorithms may be noted over the recent years. Among predictive control schemes the Generalized Predictive Controller (GPC) is perhaps the best known and one of the most successful representatives. Several papers analyze the properties of GPC, see e.g. (Clarke *et al.*, 1987; Clarke and Mohtadi, 1989; Ordys and Clarke, 1993). The GPC is a *static* variance minimization algorithm, i.e. it separates the dynamic problem into individual steps and a solution is obtained for each such step. This does not necessarily give the optimal steady-state solution. Furthermore GPC control formulae do not lend themselves easily to an analysis of the closed-loop stability and performance properties (Bitmead *et al.*, 1990). This is a problem one faces when tuning the GPC controller parameters.

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As an alternative to static optimization in predictive controllers, dynamic optimization can be used. A range of methods are available to deal with dynamic variance minimization (Grimble, 1984; 1986; 1988; 1990). These methods are derived using a frequency domain approach. A state-space dynamic optimization algorithm is proposed in (Ordys and Grimble, 1996). The problem is formulated in state space and a so-called Dynamic Predictive Controller (DPC) is derived using a state feedback LQ formulation. The stochastic case, involving use of a Kalman filter is also considered.

In this paper, the multivariable LQGPC controller is derived using Lagrange multipliers. The integral action is introduced to the state-space equations. The stability of the controller is discussed. It is shown explicitly how to incorporate the output and control horizons into the system equations. The relations between the proposed (LQGPC) controller and the state-space version of the GPC are discussed.

In LQGPC the optimal predictive control law is derived using an LQG approach. Working with a predictive control scheme like e.g. GPC in the LQGPC framework offers a systematic way of selecting the adjustable parameters to yield a stable closed loop system.

2. Problem Formulation

2.1. Preliminaries

Consider the linear system model in the ordinary discrete-time state-space form

$$\begin{aligned} x_{t+1}^{(0)} &= A^{(0)}x_t^{(0)} + B^{(0)}u_t + \xi_{v,t} \\ y_t^{(0)} &= C^{(0)}x_t^{(0)} + \xi_{w,t}. \end{aligned} \tag{1}$$

Here $x_t^{(0)}$ is the system state vector with dimensions $n_{x^{(0)}} \times 1$. The vector of control signals u_t has dimensions $n_u \times 1$. The vector of output signals $y_t^{(0)}$ has dimensions $n_{y^{(0)}} \times 1$. The process noise $\xi_{v,t}$ and the measurement noise $\xi_{w,t}$ have dimensions $n_{x^{(0)}} \times 1$ and $n_{y^{(0)}} \times 1$, respectively. The system matrices $A^{(0)}$, $B^{(0)}$ and $C^{(0)}$ are constant and can be obtained using any appropriate system identification method. Introducing an integral action, we define

$$\Delta u_t = u_t - u_{t-1}. \tag{2}$$

Equations (1) and (2) can be combined defining an extended state vector x:

$$\underbrace{\begin{bmatrix} x_{t+1}^{(0)} \\ u_t \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} A^{(0)} & B^{(0)} \\ 0 & I \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_t^{(0)} \\ u_{t-1} \end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix} B^{(0)} \\ I \end{bmatrix}}_{B} \Delta u_t + \underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix} \xi_{v,t}}_{v_t}, \tag{3}$$

$$y_t = \underbrace{\begin{bmatrix} C^{(0)} & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_t^{(0)} \\ u_{t-1} \end{bmatrix}}_{x_t} + \underbrace{\xi_{w,t}}_{w_t}. \tag{4}$$

Hence, the state-space equations for the system can be written in the form

$$\begin{aligned} x_{t+1} &= Ax_t + B\Delta u_t + v_t, \\ y_t &= Cx_t + w_t. \end{aligned} \tag{5}$$

The system of interest is subject to stochastic process noise (v) and measurement noise (w). However, from this point on it will be useful to consider the deterministic optimal-control problem, assuming that states are available for feedback. Notice that to obtain a solution to the stochastic optimal control problem, by invoking the *Separation Principle*, the deterministic control problem and state estimation problem can be solved independently. From the state equation (5), by dropping the noise terms, the future values of the system outputs can be calculated as

$$\begin{aligned} x_{t+k} &= Ax_{t+k-1} + B\Delta u_{t+k-1} = A^k x_t + \sum_{j=1}^{k} A^{k-j} B\Delta u_{t+j-1}, \\ y_{t+k} &= Cx_{t+k} = CA^k x_t + \sum_{j=1}^{k} CA^{k-j} B\Delta u_{t+j-1}. \end{aligned} \tag{6}$$

The equations for the predicted future outputs, (6), can be rewritten in a more compact block matrix form

$$\underbrace{\begin{bmatrix} y_{t+1} \\ y_{t+2} \\ \vdots \\ y_{t+N+1} \end{bmatrix}}_{Y_{t,N}} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^N \end{bmatrix}}_{\Phi_N} Ax_t$$

$$+ \underbrace{\begin{bmatrix} CB & 0 & \cdots & \cdots & 0 \\ CAB & CB & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ CA^N B & CA^{N-1}B & \cdots & \cdots & CB \end{bmatrix}}_{S_N} \underbrace{\begin{bmatrix} \Delta u_t \\ \Delta u_{t+1} \\ \vdots \\ \Delta u_{t+N} \end{bmatrix}}_{U_{t,N}}$$

$$= \Phi_N A x_t + S_N U_{t,N}. \tag{7}$$

If the prediction horizon is limited to $N+1$, it is reasonable to assume that $N+1$ future reference signals are known, and the references further into the future are unknown. However, if optimisation over a horizon longer than $N+1$ is to be performed (e.g. infinite horizon optimization), then some assumptions must be made about the values of the reference signal for the whole duration of the optimisation horizon (Tomizuka and Whitney, 1975). In this paper it is assumed that the future values of the reference may be evaluated from the equation

$$R_{t+1,N} = \Theta_{R,N} R_{t,N}, \tag{8}$$

where

$$R_{t,N} = \begin{bmatrix} r_{t+1}^T & r_{t+2}^T & \cdots & r_{t+N+1}^T \end{bmatrix}^T. \tag{9}$$

Here $\Theta_{R,N}$ represents the transition matrix for the reference signal. In order to incorporate the knowledge of future reference values, the extended state vector is defined and the output equation is rewritten:

$$\begin{aligned}\begin{bmatrix} x_{t+1} \\ R_{t+1,N} \end{bmatrix} = \chi_{t+1} &= \underbrace{\begin{bmatrix} A & 0 \\ 0 & \Theta_{R,N} \end{bmatrix}}_{\Lambda} \chi_t + \underbrace{\begin{bmatrix} \beta \\ 0 \end{bmatrix}}_{\Psi} U_{t,N} \\ &= \Lambda \chi_t + \Psi U_{t,N},\end{aligned} \tag{10}$$

$$Y_{t,N} = \begin{bmatrix} \Phi_N A & 0 \end{bmatrix} \chi_t + S_N U_{t,N}, \tag{11}$$

$$\beta = \begin{bmatrix} B & 0 & \cdots & 0 \end{bmatrix}. \tag{12}$$

Now the error vector $e_{t,N}$ can be defined for the predicted tracking error signals as

$$\begin{aligned} e_{t,N} &= Y_{t,N} - R_{t,N} \\ &= \underbrace{\begin{bmatrix} \Phi_N A & -I \end{bmatrix}}_{L_N} \underbrace{\begin{bmatrix} x_t \\ R_{t,N} \end{bmatrix}}_{\chi_t} + S_N U_{t,N}. \end{aligned} \tag{13}$$

First, consider the so-called static performance index of the type which is usually associated with predictive control problems:

$$\begin{aligned} J_t &= \sum_{j=0}^{N} \Big[(y_{t+j+1} - r_{t+j+1})^T \Lambda_e^0 (y_{t+j+1} - r_{t+j+1}) + \Delta u_{t+j}^T \Lambda_u^0 \Delta u_{t+j} \Big] \\ &= \sum_{j=0}^{N} \Big[e_{t+j+1}^T \Lambda_e^0 e_{t+j+1} + \Delta u_{t+j}^T \Lambda_u^0 \Delta u_{t+j} \Big]. \end{aligned} \tag{14}$$

Using (13) this can be written down as

$$J_t = e_{t,N}^T \Lambda_e e_{t,N} + U_{t,N}^T \Lambda_u U_{t,N}, \tag{15}$$

where

$$\Lambda_e = \mathrm{diag}\Big(\underbrace{\Lambda_e^0 \quad \cdots \quad \Lambda_e^0}_{N+1}\Big), \quad \Lambda_u = \mathrm{diag}\Big(\underbrace{\Lambda_u^0 \quad \cdots \quad \Lambda_u^0}_{N+1}\Big).$$

Here it is assumed that the same horizon $j = 0, 1, \ldots, N$ is used for the control signal (GPC: $k = 1, \ldots, N_u$) and for the output error signal (GPC: $l = N_1, N_1 + 1, \ldots, N_2$). The GPC design parameters N_1, N_2 and N_u can be incorporated in the LQGPC framework by a proper adjustment of various matrices. This will be discussed later.

When using (15) as the performance index, the vector $U_{t,N}$ of optimal control actions within the horizon N_u is calculated. However, only the first element Δu_t is applied and the procedure is repeated in the next step. Therefore, the algorithm solves a static optimization problem in each step.

2.2. Dynamic Optimization

The dynamic performance index will be defined as an infinite average over time of the indices of the form as in (15):

$$\begin{aligned} J &= \lim_{T_h \to \infty} \frac{1}{T_h + 1} \sum_{t=t_0}^{t_0+T_h} J_t, \\ &= \lim_{T_h \to \infty} \frac{1}{T_h + 1} \sum_{t=t_0}^{t_0+T_h} \left(e_{t,N}^T \Lambda_e e_{t,N} + U_{t,N}^T \Lambda_u U_{t,N}\right), \end{aligned} \tag{16}$$

where $T_h + 1$ is the summation horizon. There are some special cases where the performance indices (15) and (16) yield the same optimal control solution, e.g. if the problem can be transformed to the Åström minimum variance controller (Åström, 1970). However, apart from these special cases, the solutions obtained when minimizing (15) and (16) are different.

It is easy to observe that (14) corresponds to the situation when $\lim_{T_h \to \infty}$ is dropped in (16), and then T_h is assumed to be zero. From the point of view of the system equations (10), (4) and the performance index (16) it means a one-step ahead, static optimization.

Therefore, starting from the same initial conditions the two performance indices may lead to two different control strategies which may settle on different steady state values. This is demonstrated by an example in Section 6. In steady state, the algorithm resulting from (16) will provide the minimum value of both the performance indices (16) and (15). Moreover, the value obtained from the algorithm resulting from (15) may be higher, e.g. the steady-state value of the performance index will not be minimized (Ordys and Grimble, 1996).

Using (7), (13) and (16), for any value of T_h the LQGPC performance index can be written in the form

$$J = \frac{1}{T_h+1}\sum_{t=t_0}^{t_0+T_h}\Big[e_{t,N}^T\Lambda_e e_{t,N} + U_{t,N}^T\Lambda_u U_{t,N}\Big]$$

$$= \frac{1}{T_h+1}\sum_{t=t_0}^{t_0+T_h}\Big[\chi_t^T \underbrace{L_N^T\Lambda_e L_N}_{Q}\chi_t + \chi_t^T \underbrace{L_N^T\Lambda_e S_N}_{M} U_{t,N} + U_{t,N}^T \underbrace{S_N^T\Lambda_e L_N}_{M^T}\chi_t$$

$$+ U_{t,N}^T\underbrace{\left(S_N^T\Lambda_e S_N + \Lambda_u\right)}_{R}U_{t,N}\Big]$$

$$= \frac{1}{T_h+1}\sum_{t=t_0}^{t_0+T_h}\Big[\chi_t^T Q\chi_t + \chi_t^T M U_{t,N} + U_{t,N}^T M^T\chi_t + U_{t,N}^T R U_{t,N}\Big]. \tag{17}$$

Without loss of generality it may be assumed that Λ_e and Λ_u are symmetric and therefore so are Q and R.

The performance index J in (17) is to be minimized under the constraints (10):

$$\chi_{t+1} = \Lambda\chi_t + \Psi U_{t,N}, \quad t = t_0, t_0+1, \ldots, t_0+T_h. \tag{18}$$

This is equivalent (Ogata, 1987) to minimizing the performance index

$$J = \frac{1}{T_h+1}\sum_{t=t_0}^{t_0+T_h}\Big[\chi_t^T\hat{Q}\chi_t + V_t^T R V_t\Big] \tag{19}$$

subject to

$$\chi_{t+1} = \hat{G}\chi_t + \Psi V_t \tag{20}$$

with the initial condition $\chi_{t_0} = c$ and for $t = t_0, t_0+1, \ldots, t_0+T_h$, where

$$\hat{Q} = Q - MR^{-1}M^T, \tag{21}$$

$$V_t = R^{-1}M^T\chi_t + U_{t,N}, \tag{22}$$

$$\hat{G} = \Lambda - \Psi R^{-1}M^T. \tag{23}$$

This is an equality constrained problem in two dimensions since the function to be minimized and the constraints are functions of two variables, namely χ and U. Such a problem can be solved using various methods. One possibility, applied in this paper, is the method of Lagrange multipliers.

3. Solution

Minimizing J in (19) under the constraints in (20) corresponds (Ogata, 1987) to minimizing the performance index

$$L_0 = \frac{1}{T_h+1}\sum_{t=t_0}^{t_0+T_h}\Big[\big(\chi_t^T\hat{Q}\chi_t + V_t^TRV_t\big) + \lambda_{t+1}^T\big(\hat{G}\chi_t + \Psi V_t - \chi_{t+1}\big) + \big(\hat{G}\chi_t + \Psi V_t - \chi_{t+1}\big)^T\lambda_{t+1}\Big]$$

subject to

$$\begin{cases} \chi_{t+1} = \hat{G}\chi_t + \Psi V_t, \\ \chi_{t_0} = c, \end{cases} \tag{24}$$

where c is a constant vector. The vectors $\lambda_{t_0+1}, \lambda_{t_0+2}, \dots, \lambda_{t_0+T_h}$ are Lagrange multipliers. The Lagrange multipliers are eliminated from the equations by assuming that they can be written in the form

$$\lambda_t = P_t\chi_t = \begin{bmatrix} P_t^1 & P_t^2 \\ P_t^{2^T} & P_t^3 \end{bmatrix}\chi_t. \tag{25}$$

In the partition of the matrix P_t the square parts P_t^1 and P_t^3 correspond respectively to the components x_t and R_t of the vector χ_t. Now, solving the minimization problem with respect to P_t results in the block matrix Riccati equation

$$P_t = \hat{Q} + \hat{G}^TP_{t+1}\hat{G} - \hat{G}^TP_{t+1}\Psi\big(R + \Psi^TP_{t+1}\Psi\big)^{-1}\Psi^TP_{t+1}\hat{G} \tag{26}$$

with

$$P_{t+T_h+1} = 0.$$

After some algebra this can be simplified to the following block matrix Riccati equation:

$$P_t = Q + \Lambda^TP_{t+1}\Lambda - \big(M + \Lambda^TP_{t+1}\Psi\big)\big(R + \Psi^TP_{t+1}\Psi\big)^{-1}\big(M^T + \Psi^TP_{t+1}\Lambda\big). \tag{27}$$

Substituting definitions from (21)–(23) and (3), (4), (12) this equation is further decomposed into two matrices P_t^1 and P_t^2 (cf. (25)), and the optimal-control law can then (Hangstrup, 1997; Ordys and Grimble, 1996) be written as

$$
\begin{aligned}
U_{t,N} &= -\big(R+\Psi^T P_{t+1}\Psi\big)^{-1}\big(M^T+\Psi^T P_{t+1}\Lambda\big)\chi_t \\
&= -\big(R+\Psi^T P_{t+1}\Psi\big)^{-1}\big(M^T+\Psi^T P_{t+1}\Lambda\big)\begin{bmatrix} x_t \\ R_{t,N}\end{bmatrix} \\
&= -\Gamma_{x,t}x_t - \Gamma_{R,t}R_{t,N} \\
&= -\big(\Lambda_u + S_N^T\Lambda_e S_N + \beta^T P_{t+1}^1\beta\big)^{-1}\big(S_N^T\Lambda_e\Phi_N A + \beta^T P_{t+1}^1 A\big)x_t \\
&\quad - \big(\Lambda_u + S_N^T\Lambda_e S_N + \beta^T P_{t+1}^1\beta\big)^{-1}\big(\beta^T P_{t+1}^2\Theta_{R,N} - S_N^T\Lambda_e\big)R_{t,N},
\end{aligned}
$$

where

$$
\begin{aligned}
P_t^1 &= A^T\big(\Phi_N^T\Lambda_e\Phi_N + P_{t+1}^1\big)A - A^T\big(\Phi_N^T\Lambda_e S_N + P_{t+1}^1\beta\big) \\
&\quad \times \big(S_N^T\Lambda_e S_N + \Lambda_u + \beta^T P_{t+1}^1\beta\big)^{-1}\big(S_N^T\Lambda_e\Phi_N + \beta^T P_{t+1}^1\big)A, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
P_t^2 &= A^T\big(P_{t+1}^2\Theta_{R,N} - \Theta_{R,N}^T\Lambda_e\big) - A^T\big(\Phi_N^T\Lambda_e S_N + P_{t+1}^1\beta\big) \\
&\quad \times \big(S_N^T\Lambda_e S_N + \Lambda_u + \beta^T P_{t+1}^1\beta\big)^{-1}\big(\beta^T P_{t+1}^2\Theta_{R,N} - S_N^T\Lambda_e\big). \qquad (29)
\end{aligned}
$$

Only the first (vector) element of $U_{t,N}$ is actually applied and can be written in the form

$$
\Delta u_t = \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}}_{\beta_u} U_{t,N} = -\underbrace{\beta_u\Gamma_{x,t}}_{L_{x,t}}x_t - \underbrace{\beta_u\Gamma_{R,t}}_{L_{R,t}}R_{t,N}. \qquad (30)
$$

Hence, the resulting controller is a 2 degree-of-freedom controller consisting of a feedback controller matrix $L_{x,t}$ and a tracking controller matrix $L_{R,t}$. In most practical situations, the state vector x is not measurable. However, it will be shown (see also (Hangstrup, 1997)) that the Separation Theorem holds for the problem of computing an optimal controller for the system defined by (5). Hence, in the optimal control law the state x_t can be substituted with $\hat{x}_t$ resulting in

$$
\Delta u_t = -L_{x,t}\hat{x}_t - L_{R,t}R_{t,N}. \qquad (31)
$$

4. Stability Using the Steady-State Solution Including an Observer

In this section, the steady-state stability of the overall closed-loop system including an observer is evaluated. The approach taken here follows (Ordys and Pike, 1998).

Since the reference model is not part of a closed loop and is itself stable by construction, $R_{t,N}$ can and will be set equal to zero in this section yielding the control law $\Delta u_t = -L_{x,t}\hat{x}_t$.

Remembering that

$$\hat{x}_t = \begin{bmatrix} \hat{x}_t^{(0)} \\ \hat{u}_{t-1} \end{bmatrix} = \begin{bmatrix} \hat{x}_t^{(0)} \\ u_{t-1} \end{bmatrix}, \tag{32}$$

the observer is given by

$$\hat{x}_t^{(0)} = A^{(0)}\hat{x}_t^{(0)} + B^{(0)}u_t + K_{x^{(0)},t}\left(y_t^{(0)} - C^{(0)}\hat{x}_t^{(0)}\right), \tag{33}$$

where $K_{x^{(0)},t}$ results from evaluation of a standard Kalman filter Riccati equations (Åström, 1970). The observer error is given by

$$\begin{aligned} \tilde{x}_{t+1}^{(0)} &= x_{t+1}^{(0)} - \hat{x}_{t+1}^{(0)} \\ &= \left(A^{(0)} - K_{x^{(0)},t}C^{(0)}\right)\tilde{x}_t^{(0)} + v_t - K_{x^{(0)},t}w_t. \end{aligned} \tag{34}$$

Collecting the equation for the state x and the equation for the observer error $\tilde{x}^{(0)}$ into a block matrix form yields

$$\begin{aligned} \begin{bmatrix} x_{t+1} \\ \tilde{x}_{t+1}^{(0)} \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} A^{(0)} & B^{(0)} \\ 0 & I \end{bmatrix} & 0 \\ 0 & \left(A^{(0)} - K_{x^{(0)},t}C^{(0)}\right) \end{bmatrix} \begin{bmatrix} x_t \\ \tilde{x}_t^{(0)} \end{bmatrix} \\ &+ \begin{bmatrix} \begin{bmatrix} B^{(0)} \\ I \end{bmatrix} \\ 0 \end{bmatrix} \underbrace{\left(-L_{x,t}\hat{x}_t\right)}_{\Delta u_t} + \begin{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \\ I \end{bmatrix} v_t + \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ -K_{x^{(0)},t} \end{bmatrix} w_t. \end{aligned} \tag{35}$$

Now, since

$$\hat{x}_t = \begin{bmatrix} \hat{x}_t^{(0)} \\ u_{t-1} \end{bmatrix} = \begin{bmatrix} x_t^{(0)} - \tilde{x}_t^{(0)} \\ u_{t-1} \end{bmatrix}, \tag{36}$$

the matrix $L_{x,t}$ can be divided into two parts, i.e.

$$L_{x,t} = \begin{bmatrix} L_{x^{(0)},t} & L_{u,t} \end{bmatrix}, \tag{37}$$

and (35) can then be rewritten as

$$\begin{bmatrix} x_{t+1} \\ \tilde{x}^{(0)}_{t+1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A^{(0)} & B^{(0)} \\ 0 & I \end{bmatrix} & 0 \\ 0 & \left(A^{(0)} - K_{x^{(0)},t}C^{(0)}\right) \end{bmatrix} \begin{bmatrix} x_t \\ \tilde{x}^{(0)}_t \end{bmatrix}$$

$$+ \begin{bmatrix} \begin{bmatrix} B^{(0)} \\ I \end{bmatrix} \\ 0 \end{bmatrix} \left(- \begin{bmatrix} L_{x^{(0)},t} & L_{u,t} \end{bmatrix} \begin{bmatrix} x^{(0)}_t - \tilde{x}^{(0)}_t \\ u_{t-1} \end{bmatrix} \right)$$

$$+ \begin{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \\ I \end{bmatrix} v_t + \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ -K_{x^{(0)},t} \end{bmatrix} w_t$$

$$= \underbrace{\begin{bmatrix} \Omega_{L,t} & \begin{bmatrix} B^{(0)} \\ I \end{bmatrix} L_{x^{(0)},t} \\ 0 & \Omega_{K,t} \end{bmatrix}}_{\Omega_t} \begin{bmatrix} x_t \\ \tilde{x}^{(0)}_t \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \\ I \end{bmatrix} v_t + \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ -K_{x^{(0)},t} \end{bmatrix} w_t, \tag{38}$$

where

$$\Omega_{L,t} = \begin{bmatrix} A^{(0)} - B^{(0)}L_{x^{(0)},t} & B^{(0)}\left(I - L_{u,t}\right) \\ -L_{x^{(0)},t} & I - L_{u,t} \end{bmatrix}, \tag{39}$$

$$\Omega_{K,t} = \left(A^{(0)} - K_{x^{(0)},t}C^{(0)}\right). \tag{40}$$

As is well-known, the stability of the closed-loop steady-state solution including an observer can be evaluated by checking the location of the eigenvalues of the transition matrix Ω_t. If in (26) the upper horizon T_h is infinite, then the controller Riccati equations (27) and consequently (28) and (29) become algebraic (steady-state) Riccati equations. Denote by $L_{x^{(0)},\infty}$ and $L_{u,\infty}$ the matrices obtained by using the steady-state solutions to the controller Riccati equations, and by $K_{x^{(0)},\infty}$ the steady-state solution to the observer Riccati equation. The value of Ω_t obtained by inserting these steady-state matrices will be denoted by Ω_∞. The eigenvalues of Ω_∞ are given as the solutions with respect to z of the characteristic equation:

$$\det\left(zI - \Omega_\infty\right) = 0 \iff \det\left(zI - \Omega_{L,\infty}\right)\det\left(zI - \Omega_{K,\infty}\right) = 0, \tag{41}$$

where

$$\Omega_{L,\infty} = \begin{bmatrix} A^{(0)} - B^{(0)}L_{x^{(0)},\infty} & B^{(0)}\left(I - L_{u,\infty}\right) \\ -L_{x^{(0)},\infty} & I - L_{u,\infty} \end{bmatrix}, \tag{42}$$

$$\Omega_{K,\infty} = \left(A^{(0)} - K_{x^{(0)},\infty}C^{(0)}\right). \tag{43}$$

From the above equations, it is seen that the controller eigenvalues and the observer eigenvalues can be chosen independently as they do not influence one another. The controller eigenvalues depend upon $L_{x^{(0)},\infty}$ and $L_{u,\infty}$; whereas the observer eigenvalues depend upon $K_{x^{(0)},\infty}$. Therefore, the Separation Principle holds for the above system.

5. Output and Control Horizons in the LQGPC Framework

In the GPC minimization problem, assuming a model in the form (5), the output and control horizons (N_1, N_2 and N_u) are commonly introduced:

$$\underbrace{\min}_{\Delta u_t, \Delta u_{t+1}, \ldots, \Delta u_{t+N_u-1}} \quad E\left[J_t\right], \tag{44}$$

where

$$\begin{aligned} J_t &= \sum_{l=N_1}^{N_2} \left[\left(y_{t+l} - r_{t+l}\right)^T \Lambda_e \left(y_{t+l} - r_{t+l}\right) \right] \\ &\quad + \sum_{k=1}^{N_u} \left[\Delta u_{t+k-1} \Lambda_u \Delta u_{t+k-1} \right], \end{aligned} \tag{45}$$

$$\begin{aligned} x_{t+1} &= A x_t + B \Delta u_t + v_t, \\ y_t &= C x_t + w_t. \end{aligned} \tag{46}$$

If v_t and w_t are independent Gaussian white noise sequences, this minimization problem is equivalent to the deterministic minimization

$$\underbrace{\min}_{\Delta u_t, \Delta u_{t+1}, \ldots, \Delta ut+N_u-1} \quad J_t,$$

where

$$J_t = \sum_{l=N_1}^{N_2} \left[\left(y_{t+l} - r_{t+l}\right)^T \Lambda_e \left(y_{t+l} - r_{t+l}\right) \right] \tag{47}$$

$$+ \sum_{k=1}^{N_u} \left[\Delta u_{t+k-1} \Lambda_u \Delta u_{t+k-1} \right],$$

$$x_{t+1} = A x_t + B \Delta u_t, \tag{48}$$

$$y_t = C x_t. \tag{49}$$

In GPC and LQGPC control the output and control intervals may be treated as tuning parameters. In order to incorporate the output interval $[N_1; N_2]$ and the control interval $[1; N_u]$ in the design, it is easy to cut out the corresponding parts from the matrices in the output prediction equation (7), adjust various matrices accordingly and make certain substitutions. This will be shown in the following. First recall the output prediction equation (6):

$$y_{t+k} = CA^k x_t + \sum_{i=1}^{k} CA^{k-i} B \Delta u_{t+i-1}.$$

If the output interval is $[N_1; N_2]$ and the control interval is $[1; N_u]$ like in (45) and (47), then the block matrix form in (7) is modified to

$$\underbrace{\begin{bmatrix} y_{t+N_1} \\ y_{t+N_1+1} \\ \vdots \\ y_{t+N_2} \end{bmatrix}}_{Y_{t,N_1,N_2}} = \underbrace{\begin{bmatrix} CA^{N_1-1} \\ CA^{N_1} \\ \vdots \\ CA^{N_2-1} \end{bmatrix}}_{\Phi_{N_1,N_2}} Ax_t$$

$$+ \underbrace{\begin{bmatrix} CA^{N_1-1}B & CA^{N_1-2}B & \cdots & CA^{N_1-N_u}B \\ CA^{N_1}B & CA^{N_1-1}B & \cdots & CA^{N_1-N_u+1}B \\ \vdots & \vdots & \vdots & \vdots \\ CA^{N_2-1}B & CA^{N_2-2}B & \cdots & CA^{N_2-N_u} \end{bmatrix}}_{S_{N_1,N_2}} \underbrace{\begin{bmatrix} \Delta u_t \\ \Delta u_{t+1} \\ \vdots \\ \Delta u_{t+N_u-1} \end{bmatrix}}_{U_{t,N_u}}. \quad (50)$$

Hence the output equation in (10) takes the form

$$Y_{t,N_1,N_2} = \Phi_{N_1,N_2} A x_t + S_{N_1,N_2} U_{t,N_u}. \quad (51)$$

The reference model is modified to

$$R_{t+1,N_1,N_2} = \Theta_{R,N_1,N_2} R_{t,N_1,N_2}, \quad (52)$$

where

$$R_{t,N_1,N_2} = \begin{bmatrix} r_{t+N_1} & r_{t+N_1+1} & \cdots & r_{t+N_2} \end{bmatrix}. \quad (53)$$

Now the error vector in (13) reduces to

$$\begin{aligned} e_{t,N_1,N_2} &= Y_{t,N_1,N_2} - R_{t,N_1,N_2} \\ &= \underbrace{\begin{bmatrix} \Phi_{N_1,N_2}A & -I \end{bmatrix}}_{L_{N_1,N_2}} \underbrace{\begin{bmatrix} x_t \\ R_{t,N_1,N_2} \end{bmatrix}}_{\chi_{t,N_1,N_2}} + S_{N_1,N_2} U_{t,N_u}. \end{aligned} \quad (54)$$

Define the weighting matrices Λ_{e,N_1,N_2} and Λ_{u,N_u} with appropriate dimensions corresponding to e_{t,N_1,N_2} and U_{t,N_u}, respectively. With these definitions the GPC performance index in (47) can be written down as follows:

$$\begin{aligned} J_t &= e^T_{t,N_1,N_2}\Lambda_{e,N_1,N_2}e_{t,N_1,N_2} + U^T_{t,N_u}\Lambda_{u,N_u}U_{t,N_u} \\ &= \left(L_{N_1,N_2}\chi_{t,N_1,N_2} - S_{N_1,N_2}U_{t,N_u}\right)^T\Lambda_{e,N_1,N_2} \\ &\quad \times \left(L_{N_1,N_2}\chi_{t,N_1,N_2} - S_{N_1,N_2}U_{t,N_u}\right) + U^T_{t,N_u}\Lambda_{u,N_u}U_{t,N_u}. \end{aligned} \tag{55}$$

Summing up over t yields the LQGPC performance index

$$\begin{aligned} J = \frac{1}{T_h+1}\sum_{t=t_0}^{t_0+T_h} &\Big[(L_{N_1,N_2}\chi_{t,N_1,N_2} - S_{N_1,N_2}U_{t,N_u})^T\Lambda_{e,N_1,N_2} \\ &\times \left(L_{N_1,N_2}\chi_{t,N_1,N_2} - S_{N_1,N_2}U_{t,N_u}\right) + U^T_{t,u}\Lambda_{u,N_u}U_{t,N_u}\Big]. \end{aligned} \tag{56}$$

The structures of the performance indices (17) and (56) are equal and with the substitutions

$$\begin{cases} L_N = L_{N_1,N_2}, & \chi_t = \chi_{t,N_1,N_2}, \quad S_N = S_{N_1,N_2}, \\ U_{t,N} = U_{t,N_u}, & \Lambda_e = \Lambda_{e,N_1,N_2}, \quad \Lambda_u = \Lambda_{u,N_u}, \end{cases} \tag{57}$$

exactly the same formulae as used in the common horizon case from (17) and forward can be used for deriving the optimal control law.

6. Comparison of GPC and LQGPC

In both GPC and LQGPC, the vector U_{t,N_u} is to be found. For the GPC controller (performance index (55)) this can be obtained through straightforward static minimization. For the LQGPC controller, especially as $T_h \to \infty$ in performance index (56), a solution to the control Riccati equation is needed. Therefore, for this case, the LQGPC controller retains the stability features characteristic for infinite horizon LQG design (Kwakernaak and Sivan, 1972). In particular, if the control algebraic Riccati equation has multiple solutions, it is possible to select a stable solution, leading to a stable control system. In a standard GPC formulation (Clarke *et al.*, 1987) the infinite horizon is not available. A traditional approach to deal with instability of the system would be to increase the output horizon or to decrease the control horizon. The resulting controller may feature a slower, more sluggish response. In some cases, it may be beneficial to be able to operate with relatively short output horizons and relatively long control horizons and the LQGPC control synthesis may prove easier.

In the example presented below, we make a comparision between the GPC and LQGPC controllers. The purpose is not to analyse the dynamic features and tuning of the two algorithms (this would be beyond the scope of this paper) but to illustrate possible differences. The system which is subject to the test is given by the state-space equation

$$x_{t+1} = \underbrace{\begin{bmatrix} A^{(0)} & B^{(0)} \\ 0 & I \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_t^{(0)} \\ u_{t-1} \end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix} B^{(0)} \\ I \end{bmatrix}}_{B} \Delta u_t + \underbrace{\begin{bmatrix} G \\ 0 \end{bmatrix}} \xi_{v,t}$$

$$= \begin{bmatrix} 1.8850 & 1 & 0 & 0 & 0 & 0 & 0 & -0.0320 \\ -0.1637 & 0 & 1 & 0 & 0 & 0 & 0 & 0.4240 \\ -1.1060 & 0 & 0 & 1 & 0 & 0 & 0 & 0.9740 \\ 0.0159 & 0 & 0 & 0 & 1 & 0 & 0 & 1.7129 \\ 0.3966 & 0 & 0 & 0 & 0 & 1 & 0 & 0.5162 \\ 0.0554 & 0 & 0 & 0 & 0 & 0 & 1 & -0.9110 \\ -0.0839 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2324 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x_t$$

$$+ \begin{bmatrix} -0.032 \\ 0.42 \\ 0.974 \\ 1.7129 \\ 0.5162 \\ -0.911 \\ -0.2324 \\ 1 \end{bmatrix} \Delta u_t + \begin{bmatrix} 2.6850 \\ 0.4763 \\ -0.5940 \\ 0.4255 \\ 0.7243 \\ 0.3175 \\ 0.1258 \\ 0 \end{bmatrix} \xi_{v,t}$$

and the output equation

$$y_t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_t + \xi_{v,t}.$$

Notice that the above state-space formulation corresponds to a stable but non-minimum phase system. The roots of the polynomial accompanying the noise signal are all located on the circle of radius 0.8. For such a system, the parameters of the GPC controller (see Section 5) were selected as $N_1 = 1$, $N_2 = 50$, $N_u = 1$, $\lambda_u = 0$. The set-point signal is a step at the time instant $t_s = 25$. Figure 1 shows the mean values of the output signal for both the GPC and LQGPC algorithms. Notice that, due to the output horizon N_2 greater than 25, both the algorithms start reacting to the set point change right from the beginning of the simulation. The LQGPC algorithm is slightly faster in this case but otherwise the responses are very similar and they both settle at the value of one, due to the integral action. Figure 2 shows the value of the variance of the output error for both the algorithms. Now it is visible that the two algorithms have indeed different stochastic properties and, in particular, they settle at different steady-state values. The steady-state value of the variance which is achieved with the LQGPC algorithm is noticeably smaller than that achieved with the GPC algorithm. This smaller value of the output error is achieved through a more active control signal, as can be seen in Fig. 3.

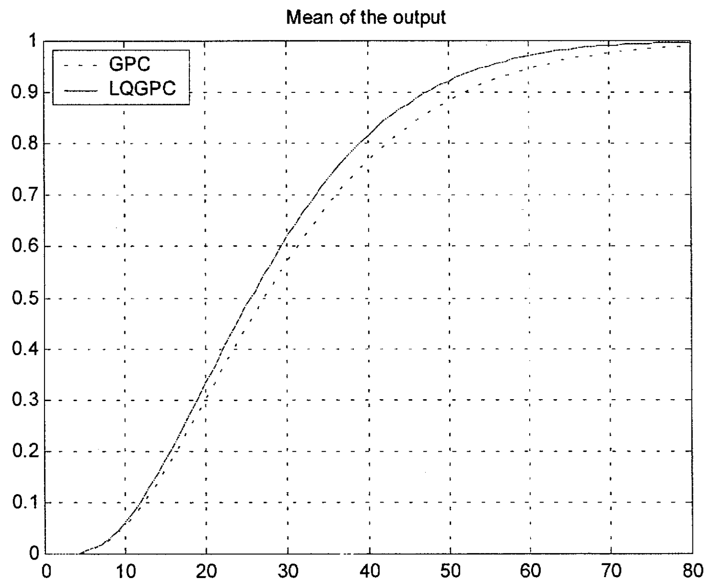

Fig. 1. Mean value of the output signal.

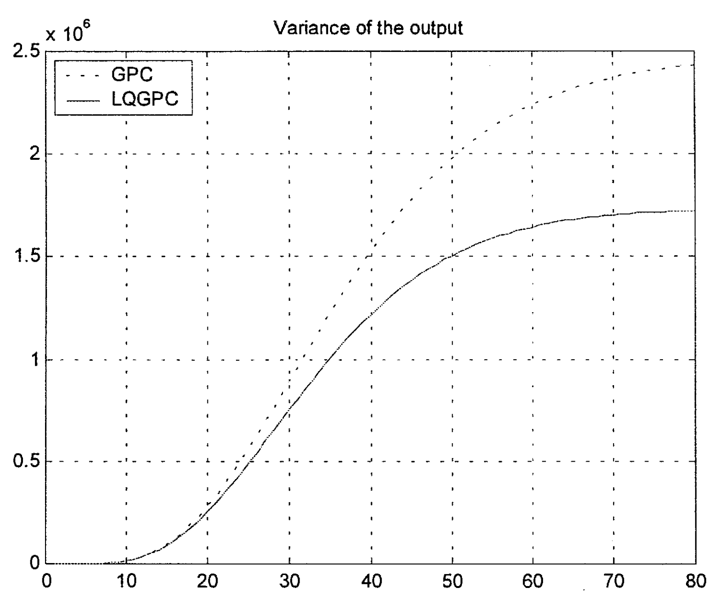

Fig. 2. Variance of the output signal.

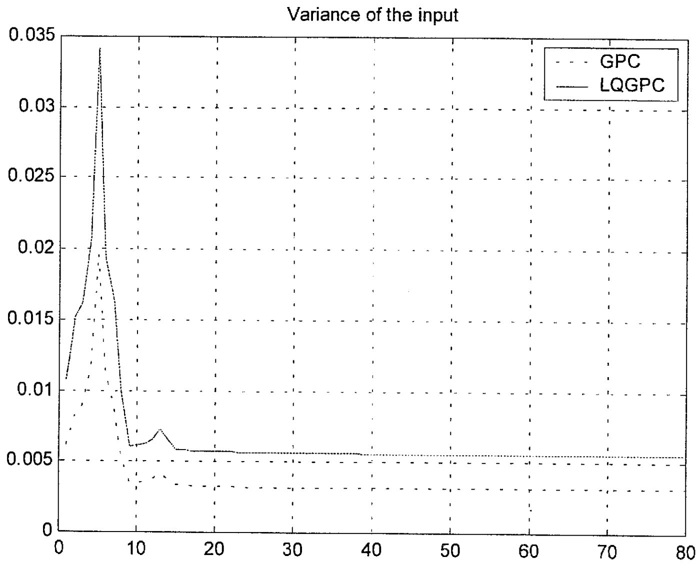

Fig. 3. Variance of the input signal.

7. Conclusion

In this paper, the multivariable state space LQGPC predictive control problem utilizing a dynamic performance index was formulated. The algorithm resulting from using the dynamic performance index yields the optimal steady-state values of both the static and dynamic performance indices. The values of these indices will be higher when using the algorithm resulting from the controller design based on the static performance index. In the stochastic case, if stochastic distributions of the disturbances are known, the LQGPC algorithm enables minimisation of the influence of future disturbances over an infinite horizon. The GPC algorithm is able to take into account only the disturbances within its output horizon which is finite and usually short.

The solution to the optimal control problem was derived using the method of Lagrange multipliers. The resulting controller was shown to have two degrees of freedom. The optimal control signal is therefore a linear combination of the state vector and the future reference values. The formulae necessary for checking the steady-state stability of the closed loop system including an observer were given.

The GPC controller can be computed as a special case of the LQGPC. Making adjustments of certain matrices, the GPC performance index can be rewritten into an LQGPC form. When doing so it is possible to use the LQGPC derivations/formulae to systematically obtain GPC design parameters yielding a stable closed-loop system.

The control problem described in this paper can also be extended to include knowledge of future external input values, i.e. known signals from other parts of the plant, and direct feed through in the system model (Hangstrup, 1997; Ordys and

Grimble, 1996; Ordys and Pike, 1998). For state-space LQGPC these extensions can be made by extending the state vector and then the state transition, input and output vectors accordingly. In (Ordys and Pike, 1998) such direct feed through terms and external inputs arose from generating a linearized model for a part of a larger system.

Simulation studies (Taube and Lampe, 1992) have shown that GPC is more sensitive to the choice of tuning parameters such as the output and control horizons, whereas the LQGPC with infinite T_h in the performance index and the same tuning parameters is almost invariant. Furthermore, the LQGPC controller gives smoother responses with less overshoot than the GPC controller (Ordys and Grimble, 1996).

The LQGPC algorithm is computationally more involved than GPC. The GPC requires one inverse of a matrix to calculate the control and, in the stochastic case, a solution to the filtering Riccati equation. The LQGPC algorithm, in addition to the filtering Riccati equation, requires a solution to two coupled Riccati equations for control. However, it is worth noticing that if the system parameters are constant (no adaptation), those calculations of control law and of filtering equation are performed off-line and the coefficients calculated once can then be used through the process with simple multiplication and addition operations. Those operations are of the same order of complexity for both GPC and LQGPC. Moreover, the resuling controllers are of the same order.

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