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ROBUST DYNAMIC INPUT RECONSTRUCTION FOR DELAY SYSTEMS

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A problem of reconstruction of a non-observable control input for a system with a time delay is analyzed within the framework of the dynamical input reconstruction approach (see Kryazhimskii and Osipov, 1987; Osipov and Kryazhimskii, 1995; Osipov *et al.*, 1991). In (Maksimov, 1987; 1988) methods of dynamical input reconstruction were described for delay systems with fully observable states. The present paper provides an input reconstruction algorithm for partially observable systems. The algorithm is robust to the observation perturbations.

Keywords: delay system, input reconstruction, observation, robust algorithm

1. Introduction and Problem Statement

Consider a dynamical system described by a pair of evolutionary equations in Hilbert spaces $(X_1, |\cdot|_{X_1})$ and $(X_2, |\cdot|_{X_2})$:

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + CDx_2(t) + f_1(t), \\ \dot{x}_2(t) = A_2 x_2(t) + f\big(t, x_1(t), Dx_2(t)\big) + \mathcal{B}\big(t, x_1(t)\big)u(t), \\ t \in T = [0, \vartheta], \quad x_1(0) = x_0^{(1)} \in X_1, \quad x_2(0) = x_0^{(2)} \in X_2. \end{cases} \tag{1}$$

Here A_i's are the infinitesimal generators of strongly continuous semigroups of bounded linear operators $\mathcal{X}_i(t) : X_i \to X_i$ $(t \in T)$, $D : X_2 \to X_\pi$ signifies the projection onto the subspace $X_\pi \subset X_2$, $(|x|_\pi = |x|_{X_2}\ \forall x \in X_\pi)$, $C \subset L(X_\pi; X_1)$ is a continuous linear operator, $f_1(\cdot) \in L_2(T; X_1)$ stands for a given disturbance, $f(t, x_1, x_2)$ denotes a Lipschitz function, $f(0,0,0) = 0$, $(U, |\cdot|_U)$ is a uniformly convex Banach space of controls, $\mathcal{B}(t, x_1)$ stands for a family of operators satisfying the following conditions:

(i) $\mathcal{D}(\mathcal{B}(t,x)) = U$, $\mathcal{B}(t,x) \in L(U, X_2)$ for all $(t,x) \in T \times X_1$ ($\mathcal{D}(\mathcal{B})$ is the domain of the definition of the operator $\mathcal{B}$),

(ii) $|\mathcal{B}(t,x) - \mathcal{B}(t,y)|_{L(U,X_2)} \le L|x-y|_{X_1}$ for all $x, y \in X_1$,

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(iii) $|\mathcal{B}(t,x)|_{L(U,X_2)} \le d + K|x|_{X_1}$ for all $x \in X_1$, and

(iv) the mapping $(t,x) \to \mathcal{B}(t,x) \in L(U,X_2)$ is continuous.

A weak solution to the system (1) corresponding to a control $u(\cdot) \in U_T$ and an initial state $x_0 = \{x_0^{(1)}, x_0^{(2)}\}$ is defined to be a continuous function $x(t) = \{x_1(t), x_2(t)\} \in X_1 \times X_2$, $t \in T$, satisfying the equalities

$$x_1(t) = \mathcal{X}_1(t)x_0^{(1)} + \int_0^t \mathcal{X}_1(t-\tau)\Big\{CDx_2(\tau) + f_1(\tau)\Big\}\,\mathrm{d}\tau,$$

and

$$x_2(t) = \mathcal{X}_2(t)x_0^{(2)} + \int_0^t \mathcal{X}_2(t-\tau)\Big\{f\big(\tau, x_1(\tau), Dx_2(\tau)\big) + \mathcal{B}\big(\tau, x_1(\tau)\big)u(\tau)\Big\}\,\mathrm{d}\tau.$$

Here U_T is the set of admissible controls,

$$U_T = \Big\{u(\cdot) \in L_2(T;U) : \ u(t) \in P \ \text{ a.a. on } t \in T\Big\},$$

where $P \subset U$ is a convex, bounded and closed set. In what follows, we assume for simplicity that the initial state x_0 is known precisely. Using Conditions (i)–(iv) and the principle of contraction mappings in an ordinary way (Varga, 1975), it is not difficult to show that for every $x_0 \in X_1 \times X_2$ and every $u(\cdot) \in U_T$ there exists a unique weak solution to (1). We denote this weak solutions by $x(\cdot\,; x_0, u(\cdot)) = \{x_1(\cdot\,; x_0, u(\cdot)), x_2(\cdot\,; x_0, u(\cdot))\}$. We shall be concerned with the following problem of robust input reconstruction: Let the motion of the system (1), i.e. the evolution of the state $x(t) = x(t; x_0, u(\cdot))$, start from x_0 under the action of the input control $u(\cdot) \in U_T$. At discrete, sufficiently frequent time instants $\tau_i \in T$, $\tau_i = \tau_{i-1} + \delta$, $i \in [1:m]$, $\tau_0 = 0$, $\tau_m = \vartheta$, the component $x_1(\tau_i)$ of the state vector $x(\tau_i)$ is observed. The observation results $\xi_i \in X_1$ are in general inaccurate and satisfy the inequalities

$$\big|x_1(\tau_i) - \xi_i\big|_{X_1} \le h. \tag{2}$$

It is required to provide an algorithm of real-time reconstruction of an input $u(\cdot)$ generating the observed output $x_1(\cdot)$.

Now we give an exact statement of the problem: Fix a family (Δ_h), $h \in (0,1]$,

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}, \quad \tau_{h,0} = t_0, \quad \tau_{h,m_h} = \vartheta \tag{3}$$

of uniform partitions of the segment T with the diameters

$$\delta(h) = \delta(\Delta_h) = \tau_{h,i+1} - \tau_{h,i} \to 0 \ \text{ as } \ h \to 0. \tag{4}$$

Choose some $u_*(\cdot\,; x_1(\cdot))$ in the set $U(x_1(\cdot)) \in U_T$ of all controls compatible with the output $x_1(\cdot)$.

Definition 1. A family D_h, $h \in (0,1)$, of operators acting from Ξ_T into U_T is called regularizing if for any output $x_1(\cdot)$

$$\lim_{h\to 0} \sup \Big\{ \big|D_h\xi(\cdot) - u_*(\cdot\,; x_1(\cdot))\big|_{L_2(T;U)} : \ \xi(\cdot) \in \Xi\big(x_1(\cdot), h\big) \Big\} = 0.$$

Here $\Xi(x_1(\cdot), h)$ is the set of all admissible measurements, i.e. all piecewise constant functions $\xi(t) = \xi_i$, $t \in [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i} \in \Delta_h$ satisfying (2), $\Xi_T = \{\Xi(x_1(\cdot), h) : \ h \in (0,1),\ x(\cdot) \in X_T^{(1)}\}$, $X_T = \{x(\cdot\,; t_0, x_0, u(\cdot)) : \ u(\cdot) \in U_T\}$ is the set of all trajectories of the system (1), and $X_T^{(1)}$ ($X_T^{(2)}$) denotes the projection of X_T onto the space $C(T; X_1)$ ($C(T; X_2)$).

The problem of real-time robust input reconstruction consists in constructing a family of algorithms

$$D_h\colon \ \{\tau_i, \xi_{t_0,\tau_i}(\cdot)\} \longmapsto v^h_{\tau_i,\tau_{i+1}}(\cdot) \in U_{\tau_i,\tau_{i+1}}$$

such that

$$\big|v^h(\cdot) - u_*(\cdot\,; x_1(\cdot))\big|_{L_2(T;U)} \to 0 \quad \text{as} \quad h \to 0$$

under a suitable relation between h and $\delta = \delta(h)$. We denote by $U_{\tau_i,\tau_{i+1}}$ (respectively, by $v^h_{\tau_i,\tau_{i+1}}(\cdot)$) the restriction of the functional set U_T (resp. the function $v^h(\cdot)$) to the interval $[\tau_i, \tau_{i+1})$.

Our approach is based on the use of auxiliary control models (Krasovskii and Subbotin, 1988; Osipov and Kryazhimskii, 1995). A solution procedure is organized as follows: We select an auxiliary dynamical system M (we call it the model) whose motion originates from some initial state $w_0 \in X$ and is identified with a (weak) solution of an appropriate system of differential equations. We shall denote the model's motions by

$$\begin{aligned} w^h(t) &= M\big(h, \xi(\cdot), v(\cdot); w_0\big)(t) \\ &= w^h\big(t; w_0, \xi_{0,t}(\cdot), v_{0,t}(\cdot)\big) \in X, \quad t \in T. \end{aligned} \tag{5}$$

Here $X = X_1 \times X_2$ with the norm

$$\big|\{x_1, x_2\}\big|_X = \Big(|x_1|^2_{X_1} + |x_2|^2_{X_2}\Big)^{1/2}, \quad V_T = L_2(T; X_\pi \times U),$$

$v(\cdot) = \{v^{1,h}(\cdot), v^h(\cdot)\} \in V_T$ is the model's control and $\xi(\cdot) \in \Xi_T$ denotes an observation result. In the present paper, we assume that

$$w_0 = x_0. \tag{6}$$

The rules of forming controls $v(\cdot)$ will be called the strategies (we use the terminology of the theory of feedback control, see Krasovskii and Subbotin, 1988). Each strategy is identified with a pair $\mathcal{S}_h = (\Delta_h, \mathcal{U}_h)$. Here $\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}$ is a partition of the segment T and $\mathcal{U}_h$ is a function which assigns to every triple

$(\tau_i, \xi_{0,\tau_i}(\cdot), w_{0,\tau_i}(\cdot))$, $i \in [0 : m_h - 1]$, $(\tau_i = \tau_{h,i}$, $\xi_{0,\tau_i}(\cdot) \in \Xi(x_1(\cdot), h)_{0,\mathfrak{B}}$, $w_{0,\tau_i}(\cdot) \in C([0,\tau_i]; X)$, $x_1(\cdot) \in X_T^{(1)})$, an element

$$\left\{v^{1,h}_{\tau_i,\tau_{i+1}}(\cdot), v^h_{\tau_i,\tau_{i+1}}(\cdot)\right\} = \mathcal{U}_h(\tau_i, \xi_{0,\tau_i}(\cdot), w_{0,\tau_i}(\cdot)) \in V_{\tau_i,\tau_{i+1}}. \tag{7}$$

Thus, for every $h \in (0,1)$, the triple $(M, \Delta_h, \mathcal{U}_h)$ determines a real-time algorithm $D_h : \Xi_T \to V_T$ which transforms each observation result $\xi(\cdot) \in \Xi(x_1(\cdot), h)$ into an output $\{v^{1,h}(\cdot), v^h(\cdot)\} = D_h\xi(\cdot) \in V_T$ through the feedback model controller (5)–(7). Later, we shall group algorithms of this kind into a desired regularizing family D_h, $h \in (0,1)$. Each algorithm D_h will be defined by the triple $(M, \Delta_h, \mathcal{U}_h)$.

For a fixed h, an algorithm D_h runs in real time step by step. At a preliminary step, before the starting time t_0, a partition $\Delta = \Delta_h = \{\tau_i\}_{i=0}^m$, $(\tau_i = \tau_{h,i}$, $m = m_h)$ of the segment T is fixed. The i-th step is performed during the time interval $[\tau_i, \tau_{i+1})$ and comprises the following operations. The state $x_1(\tau_i)$ is observed with accuracy h, and an observation result ξ_i satisfying (2) is accepted. A model's control is designed using the rule (7). Finally, the history of the model's trajectory, $w^h(t)$, $t \in (\tau_i, \tau_{i+1}]$, is memorized. The procedure stops at time ϑ. We call a triple

$$D_h = (M, \Delta_h, \mathcal{U}_h)$$

defined by the relationships (5)–(7) a *positional modelling algorithm*. Below, we shall describe a regularizing family of positional modelling algorithms.

2. Solution Algorithm

Let us turn to the description of the algorithm for solving the problem. In accordance with the approach stated above, we should indicate a rule of choosing the model M as well as the form of the strategy $S_h = (\Delta_h, \mathcal{U}_h)$, $h \in (0,1)$.

We define the model M to be a mapping which puts into correspondence to every triple $(w_0, \xi(\cdot), v(\cdot)) \in X \times \Xi_T \times V_T$, where $v(\cdot) = \{v^{1,h}(\cdot), v^h(\cdot)\}$, $w_0 = x_0$, a weak solution to the system

$$\begin{aligned} \dot{w}_1^h(t) &= A_1 w_1^h(t) + Cv^{1,h}(t) + f_1(t), \\ \dot{w}_2^h(t) &= A_2 w_2^h(t) + f\big(t, \xi(t), v^{1,h}(t)\big) + \mathcal{B}\big(t, \xi(t)\big)v^h(t), \end{aligned}$$

i.e. the continuous function $w^h(t) = w^h(t; w_0, \xi_{0,t}(\cdot), v_{0,t}(\cdot)) = \{w_1^h(t), w_2^h(t)\} \in C(T; X)$ of the form

$$\begin{aligned} w_1^h(t) &= \mathcal{X}_1(t)w_0^{(1)} + \int_0^t \mathcal{X}_1(t-\tau)\Big\{Cv^{1,h}(\tau) + f_1(\tau)\Big\}\,\mathrm{d}\tau, \\ w_2^h(t) &= \mathcal{X}_2(t)w_0^{(2)} \\ &\quad + \int_0^t \mathcal{X}_2(t-\tau)\Big\{f_h\big(\tau, \xi(\tau), v^{1,h}(\tau)\big) + \mathcal{B}\big(\tau, \xi(\tau)\big)v^h(t)\Big\}\,\mathrm{d}\tau. \end{aligned}$$

Here

$$f_h(\tau, \xi(\tau), v^{1,h}(\tau)) = f(\tau_i, \xi_i, v_i^{1,h}), \quad \mathcal{B}(\tau, \xi(\tau)) = \mathcal{B}(\tau_i, \xi_i)$$

for almost all $t \in [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i} \in \Delta_h$, $i \in [0 : m_h - 1]$.

In what follows, for brevity we write τ_i instead of $\tau_{h,i}$. Let the family Δ_h of partitions (3), (4) be fixed. We define a strategy $\mathcal{S}_h = (\Delta_h, \mathcal{U}_h)$, $h \in (0, 1)$, by

$$\mathcal{U}_h = \mathcal{U}_h(\tau_i, \xi_i, w^h(\tau_i)) = \left\{ v^{1,h}_{\tau_i,\tau_{i+1}}(\cdot), v^h_{\tau_i,\tau_{i+1}}(\cdot) \right\}, \tag{8}$$

$$v^{1,h}(t) = v_i^{1,h} = \arg\min \{ l_1(\alpha, v, s_i^0) : v \in S_\pi(d_*) \},$$

$$v^h(t) = v_i^h = \arg\min \{ l_2(\beta, v, s_i^*) : \ v \in P \}, \quad t \in [\tau_i, \tau_{i+1}),$$

where $\alpha(h), \beta(h) \colon \mathbb{R}^+ \to \mathbb{R}^+ = \{ r \in \mathbb{R} \colon \ r > 0 \}$ are some functions which play the role of Tikhonov's regularizators,

$$l_1(\alpha, v, s_i^0) = \alpha(h)|v|^2_{X_\pi} + 2(s_i^0, Cv)_1, \quad S_\pi(d_*) = X_\pi \cap S(d_*), \tag{9}$$

$$s_i^0 = \exp(-2\omega_1 \tau_{i+1}) \{ w_1^h(\tau_i) - \xi_i \}, \quad S(d_*) = \{ v \in X_2 : \ |v|_{X_2} \le d_* \},$$

$$l_2(\beta, v, s_i^h) = \beta(h)|v|^2_U + 2\left(s_i^h, D\mathcal{B}(\tau_i, \xi_i) v(\tau) \right)_2,$$

$$s_i^h = \exp(-2\omega_2 \tau_{i+1}) \left\{ D w_2^h(\tau_i) - v_i^{1,h} \right\}, \quad i \in [0 : m_h - 1].$$

A number $d_* < +\infty$ is such that $\sup\{ |x(\cdot)|_{C(T;X)} \colon \ x(\cdot) \in X_T \} \le d_*$. It is easily seen that the set X_T is bounded in $C(T; X)$.

Let the following conditions be fulfilled:

C1. In the space X_j, the norm $|\cdot|_j$ generated by some scalar product $(\cdot, \cdot)_j$ is equivalent to the norm $|\cdot|_{X_j}$, i.e.

$$c_1 |\cdot|_j \le |\cdot|_{X_j} \le c_2 |\cdot|_j, \quad c_1, c_2 = \text{const} \in (0, +\infty),$$

and the semigroup $\mathcal{X}_j(t)$ is ω_j-dissipative on X_j, with respect to $|\cdot|_j$, i.e.

$$\left| \mathcal{X}_j(t) x \right|_j \le \exp(\omega_j t) |x|_j$$

holds for every $x \in X_j$, where $j = 1, 2$.

C2. There exists a family $Y(t) \in L(X_\pi, X_\pi)$, $t \in T$ of one-to-one operators such that

$$c_0 \left| \int_0^t Y(\tau) D y(\tau) \, d\tau \right|_{X_\pi} \le \left| \int_0^t \mathcal{X}_1(t - \tau) C D y(\tau) \, d\tau \right|_1,$$

$$\forall\, y(\cdot) \in C^+(T; X_2), \quad t \in T, \quad c_0 > 0,$$

$$t \to Y(t) y(t) \in C^+(T; X_\pi) \quad \forall\, y(\cdot) \in C^+(T; X_\pi).$$

C3. The semigroup $\mathcal{X}_1(t)$ possesses the following property: for every bounded set $X_* \subset X_1$ there exist $\delta_* \in (0,1)$, $k_0 = k_0(X_*) \in (0,+\infty)$ and $\gamma(\cdot): [0,\delta_*) \to \mathbb{R}^+$ continuous at zero and satisfying $\gamma(0) = 0$ such that the inequality

$$\left|\left(\mathcal{X}_1(\delta)x, \mathcal{X}_1(\delta_1)CDv(\delta_1)\right)_1 - \left(x, CDv(\delta_1)\right)_1\right| \le k_0\gamma(\delta)$$

holds uniformly with respect to $x \in X_*$, $\delta \in (0,\delta_*)$, $\delta_1 \in (0,\delta]$, $|v(\delta_1)|_{X_2} \le d_*$.

C4. $DY_T = \{y(\cdot): y(t) = (Y^{-1}(t))^* Dx_2(t) \ \ \forall t \in T, \ \ x_2(\cdot) \in X_T^{(2)}\} \subset V(T;X_\pi)$.

C5. The semigroup $\mathcal{X}_2(t)$ possesses the following property: for every bounded set $Y_* \subset X_2$ there exist $\delta_* \in (0,1)$, $k_* = k_*(Y_*) \in (0,+\infty)$ and $\gamma_1(\cdot): [0,\delta_*) \to \mathbb{R}^+$ continuous at zero and satisfying $\gamma_1(0) = 0$ such that the inequality

$$\left|\left(\mathcal{X}_2(\delta)x, \mathcal{X}_2(\delta_1)\mathcal{B}(t,y)v(\delta_1)\right)_2 - \left(Dx, D\mathcal{B}(t,y)v(\delta_1)\right)_2\right| \le k_*\gamma_1(\delta)$$

holds uniformly with respect to all $x, y \in Y_*$, $\delta \in (0,\delta_*)$, $\delta_1 \in (0,\delta]$, $|v(\delta_1)|_U \le d_*$ and a.e. on T.

Here $C^+(T;X_1)$ is the space of all piecewise continuous functions with a finite number of discontinuity points, $V(T;X_\pi)$ denotes the space of all functions $t \to y(t) \in X_\pi$ with the bounded variation $\mathrm{var}_{X_\pi}(T;y(\cdot))$, and symbol $(Y^{-1}(t))^*$ denotes the adjoint operator of operator $Y^{-1}(t)$.

Let $\varphi_X(\cdot)$ be the modulo of continuity of the set X_T, i.e.

$$\varphi_X(\delta) = \sup\Big\{\left|x(\tau_1) - x(\tau_2)\right|_X : \ \tau_1,\tau_2 \in T, \\ |\tau_1 - \tau_2| \le \delta, \quad x(\cdot) \in X_T\Big\} \to 0 \ \text{ as } \ \delta \to 0.$$

Moreover, let

$$\varphi_{\mathcal{B}}(\delta) = \sup\Big\{\left|\mathcal{B}(\tau,x_1(\tau)) - \mathcal{B}(t,x_1(t))\right|_{L(U;X_2)} : \ t,\tau \in T, \\ |t-\tau| \le \delta, \quad x(\cdot) = \{x_1(\cdot),x_2(\cdot)\} \in X_T\Big\} \to 0 \ \text{ as } \ \delta \to 0.$$

Assume that the following relationships between the parameters take place:

$$\alpha(h) \to 0, \quad h\alpha^{-1}(h) \to 0, \\ \Big(\delta(h) + \gamma(\delta(h))\Big)\alpha^{-1}(h) \to 0 \ \text{ as } \ h \to 0, \tag{10}$$

$$\beta(h) \to 0, \quad \varrho(h) = \Big\{\Big(h + \delta(h) + \gamma(\delta(h)) + \alpha(h)\Big)^{1/2} \\ +\Big(h + \delta(h) + \gamma(\delta(h))\Big)\alpha^{-1}(h)\Big\}^{1/2}\beta^{-1}(h) \to 0, \tag{11}$$

$$\Big(\gamma_1(\delta(h)) + \varphi_X(\delta(h)) + \varphi_{\mathcal{B}}(\delta(h))\Big)\beta^{-1}(h) \to 0 \ \text{ as } \ h \to 0. \tag{12}$$

Fix a sufficiently large $K \in (0, +\infty)$ and set

$$U_K(X_2) = \Big\{x_2(\,\cdot\,) \in V(T; X_2): \quad \mathrm{var}_{X_2}\big(T; Dx_2(\,\cdot\,)\big) \le K\Big\}.$$

Before passing to the proof of the main result of the section (Theorem 1), we formulate auxiliary statements which are necessary in what follows.

Proposition 1. *(a) Every motion of the system (1) possesses the semigroup property. (b) Every motion of the model M which corresponds to a disturbance $\xi(\,\cdot\,) \in \Xi(x_1(\,\cdot\,), 1)$, $x_1(\,\cdot\,) \in X_T^{(1)}$, possesses the semigroup property.*

Proposition 2. *Let $u_i(\,\cdot\,)$, $u_*(\,\cdot\,) \in U_T$, $u^*(\,\cdot\,) \in U(x_1(\,\cdot\,))$, $u_i(\,\cdot\,) \to u_*(\,\cdot\,)$ weakly in $L_2(T; U)$, and*

$$\sup_{t\in T}\Big|\int_0^t \mathcal{X}_2(t-\tau)\mathcal{B}\big(\tau, x_1(\tau)\big)\big\{u_i(\tau) - u^*(\tau)\big\}\,\mathrm{d}\tau\Big|_2 \to 0.$$

Then $u_*(\,\cdot\,) \in U(x_1(\,\cdot\,))$, where $U(x_1(\,\cdot\,))$ is the set of all the controls compatible with the output $x_1(\,\cdot\,)$.

Proposition 1 can be easily checked with the help of the semigroup property of the operators $\mathcal{X}_j(t)$, $j = 1, 2$ and Conditions (i)–(iv). Proposition 2 is proved by contradiction.

Define the set

$$\begin{aligned} &W_T(x_0) \\ &\quad = \Big\{w(\,\cdot\,): \;\; w(t) = w\big(t; x_0, \xi_{0,t}(\,\cdot\,), v_{0,t}(\,\cdot\,)\big), \quad t \in T, \quad v(\,\cdot\,) \in S(d_*), \\ &\qquad\qquad \xi(\,\cdot\,) \in \Xi_T\big(x_1(\,\cdot\,), h\big), \quad x_1(\,\cdot\,) \in X_T^{(1)}, \quad h \in (0,1)\Big\} \subset C(T; X). \end{aligned}$$

Using Conditions (i)–(iv), one can easily prove the following.

Proposition 3. *The set $W_T(x_0)$ is bounded in $C(T; X)$.*

Proposition 4. *(Maksimov, 1994; Osipov and Kryazhimskii, 1995) Let X be a Banach space, $u(\,\cdot\,) \in L_\infty(T; X^*)$, $v(\,\cdot\,) \in V(T; X)$, $|\int_0^t u(\tau)\,\mathrm{d}\tau|_{X^*} \le \varepsilon$, $|v(t)|_X \le K$ $\forall t \in T$. Then*

$$\Big|\int_0^\vartheta \big\langle v(\tau), u(\tau)\big\rangle_{X\times X^*}\,\mathrm{d}\tau\Big| \le \varepsilon\Big(K + \mathrm{var}_X(T; v(\,\cdot\,))\Big).$$

Lemma 1. *Let Conditions C1–C4 be fulfilled and* $x_2(\cdot) = x_2(\cdot\,; x_0, u_*(\cdot\,; x_1(\cdot))) \in Y_T \cap U_K(X_2)$. *Then*

$$\big|v^{1,h}(\cdot) - Dx_2(\cdot)\big|^2_{L_2(T;X_\pi)}$$

$$\le \nu(h) = d_0\Big\{\big(h+\delta+\gamma(\delta)+\alpha\big)^{1/2} + \big(h+\delta+\gamma(\delta)\big)\alpha^{-1}\Big\}, \quad (13)$$

where $\delta = \delta(h)$, *and* $\alpha = \alpha(h)$.

Proof. Introduce the Lyapunov functional

$$\lambda_h\big(t, x(\cdot), v^{1,h}(\cdot), w_1^h(\cdot)\big) = \alpha^{-1}(h)\exp(-2\omega_1 t)\big|w_1^h(t) - x_1(t)\big|_1^2 + \int_0^t \Big\{\big|v^{1,h}(\tau)\big|^2_{X_\pi} - \big|Dx_2(\tau)\big|^2_{X_\pi}\Big\}\,\mathrm{d}\tau$$

and estimate the increment in

$$\varepsilon_h(t) = \alpha(h)\lambda_h\big(t, x(\cdot), v^{1,h}(\cdot), w_1^h(\cdot)\big).$$

Due to Proposition 1, for $t \in [\tau_i, \tau_{i+1})$ the following relations hold:

$$w_1^h(t) = \mathcal{X}_1(t-\tau_i)w_1^h(\tau_i) + \int_{\tau_i}^t \mathcal{X}_1(t-\tau)\big\{Cv^{1,h}(\tau) + f_1(\tau)\big\}\,\mathrm{d}\tau,$$

$$x_1(t) = \mathcal{X}_1(t-\tau_i)x_1(\tau_i) + \int_{\tau_i}^t \mathcal{X}_1(t-\tau)\big\{CDx_2(\tau) + f_1(\tau)\big\}\,\mathrm{d}\tau.$$

Furthermore, we have

$$\exp(-2\omega_1\tau_{i+1})\,\big|w_1^h(\tau_{i+1}) - x_1(\tau_{i+1})\big|_1^2$$

$$\le \exp(-2\omega_1\tau_{i+1})\Bigg\{|s_i^*|_1^2 + 2\Bigg(s_i^*, \int_{\tau_i}^{\tau_{i+1}} \mathcal{X}_1(\tau_{i+1}-\tau)C\Big\{v_i^{1,h} - Dx_2(\tau)\Big\}\,\mathrm{d}\tau\Bigg)_1 + \delta\int_{\tau_i}^{\tau_{i+1}} \Big|\mathcal{X}_1(\tau_{i+1}-\tau)C\Big\{v_i^{1,h} - Dx_2(\tau)\Big\}\Big|_1^2\,\mathrm{d}\tau\Bigg\},$$

where

$$s_i^* = \mathcal{X}_1(\tau_{i+1}-\tau_i)\big\{w_1^h(\tau_i) - x_1(\tau_i)\big\}.$$

Therefore, for $i \in [0 : m-1]$ owing to (2) and Condition C1, the following estimates take place:

$$\varepsilon_h(\tau_{i+1}) \le \exp(-2\omega_1\tau_i)\ \left|w_1^h(\tau_i) - x_1(\tau_i)\right|_1^2 + \lambda_i + \mu_i$$

$$+ \alpha(h) \int\limits_{t_0}^{\tau_{i+1}} \left\{ \left|v^{1,h}(\tau)\right|_{X_\pi}^2 - |Dx_2(\tau)|_{X_\pi}^2 \right\} \mathrm{d}\tau, \quad \left|s_i^* - \tilde{s}_i\right|_1 \le k_0 h, \quad (14)$$

where

$$\tilde{s}_i = \mathcal{X}_1(\tau_{i+1} - \tau_i)\left(w_1^h(\tau_i) - \xi_i\right),$$

$$\lambda_i = 2\exp(-2\omega_1\tau_{i+1})\left(\tilde{s}_i,\ \int\limits_{\tau_i}^{\tau_{i+1}} \mathcal{X}_1(\tau_{i+1} - \tau)C\left\{v_i^{1,h} - Dx_2(\tau)\right\}\mathrm{d}\tau\right)_1 + k_1\delta h,$$

$$\mu_i = \delta\exp(-2\omega_1\tau_i)\int\limits_{\tau_i}^{\tau_{i+1}} \left|C\left\{Dx_2(\tau) - v_i^{1,h}\right\}\right|_1^2 \mathrm{d}\tau.$$

On account of Condition C3 and Proposition 4, we have

$$\left|\left(\tilde{s}_i,\ \int\limits_{\tau_i}^{\tau_{i+1}} \mathcal{X}_1(\tau_{i+1} - \tau)C\left\{v_i^{1,h} - Dx_2(\tau)\right\}\mathrm{d}\tau\right)_1 - \left(s_i,\ \int\limits_{\tau_i}^{\tau_{i+1}} C\left\{v_i^{1,h} - Dx_2(\tau)\right\}\mathrm{d}\tau\right)_1\right| \le k_2\delta\gamma(\delta), \quad (15)$$

$$s_i = w_1^h(\tau_i) - \xi_i.$$

Using the boundedness of X_T in $C(T;X)$, we find that there exists a number k_3 such that

$$\mu_i \le k_3\delta^2, \quad \delta = \delta(h). \quad (16)$$

Combining (14)–(16) and the definition of strategy $\mathcal{S}_h$, we get

$$\varepsilon_h(\tau_{i+1}) \le \varepsilon_h(\tau_i) + k_3\delta\{h + \delta + \gamma(\delta)\}, \quad i \in [0 : m_h - 1]. \quad (17)$$

Then from (17) we obtain

$$\lambda_h\left(\tau_{i+1}, x(\,\cdot\,), v^{1,h}(\,\cdot\,), w_1^h(\,\cdot\,)\right) \le \lambda_h\left(\tau_i, x(\,\cdot\,), v^{1,h}(\,\cdot\,), w_1^h(\,\cdot\,)\right) + \varphi_i(\delta, h), \quad (18)$$

where

$$\varphi_i(\delta, h) = k_4\delta\{h + \delta + \gamma(\delta)\}\alpha^{-1}(h),$$

$$\sum_{i=1}^{m_{h_k}-1} \varphi_i(\delta, h) \le k_5\{h + \delta + \gamma(\delta)\}\alpha^{-1}(h). \quad (19)$$

Due to the choice of the initial state of the model, c.f. (6), along with the relations (18) and (19), we conclude that

$$\lambda_h\big(\tau_{i+1}, x(\cdot), v^{1,h}(\cdot), w_1^h(\cdot)\big) \le k_5\{h+\delta+\gamma(\delta)\}\alpha^{-1}, \quad i \in [0:m_h-1]. \tag{20}$$

Hence, we see that

$$\big|w_1^h(\tau_i) - x_1(\tau_i)\big|_1^2 \le k_6\{h+\delta+\gamma(\delta)+\alpha\}, \quad i \in [1:m_h]. \tag{21}$$

From Condition C2 and estimate (21) we derive

$$\sup_{t\in T}\left|\int_0^t Y(\tau)\left\{v^{1,h}(\tau) - Dx_2(\tau)\right\}\,\mathrm{d}\tau\right|_{X_\pi} \le k_7\{\delta+\gamma(\delta)+h+\alpha\}^{1/2}. \tag{22}$$

From (20) for $i = m_h - 1$ it follows that

$$\big|v^{1,h}(\cdot)\big|^2_{L_2(T;X_\pi)} \le |Dx_2(\cdot)|^2_{L_2(T;X_\pi)} + k_5\big(h+\delta+\gamma(\delta)\big)\alpha^{-1}. \tag{23}$$

Applying (23) we have

$$\begin{aligned}
&\big|v^{1,h}(\cdot) - Dx_2(\cdot)\big|^2_{L_2(T;X_\pi)} \\
&= \big|v^{1,h}(\cdot)\big|^2_{L_2(T;X_\pi)} - 2\int_0^\vartheta \big(v^{1,h}(\tau), Dx_2(\tau)\big)_{X_\pi}\,\mathrm{d}\tau + \big|Dx_2(\cdot)\big|^2_{L_2(T;X_\pi)} \\
&\le 2\big|Dx_2(\cdot)\big|^2_{L_2(T;X_\pi)} - 2\int_0^\vartheta \big(v^{1,h}(\tau), Dx_2(\tau)\big)_{X_\pi}\,\mathrm{d}\tau \\
&\qquad + k_5\{h+\delta+\gamma(\delta)\}\alpha^{-1}.
\end{aligned}$$

This fact, (22), Condition C4 and Proposition 5 imply the estimate

$$\begin{aligned}
&\big|v^{1,h}(\cdot) - Dx_2(\cdot)\big|^2_{L_2(T;X_\pi)} \\
&\le 2\int_0^\vartheta \Big(Y(\tau)\left(Dx_2(\tau) - v^{1,h}(\tau)\right), \left(Y^{-1}(\tau)\right)^* Dx_2(\tau)\Big)_{X_\pi}\,\mathrm{d}\tau \\
&\qquad + k_5\{h+\delta+\gamma(\delta)\}\alpha^{-1} \\
&\le k_8\{\delta+\gamma(\delta)+h+\alpha\}^{1/2} + k_5\{h+\delta+\gamma(\delta)\}\alpha^{-1}.
\end{aligned}$$

Consequently, the lemma is proved. ∎

Introduce the notation

$$U_0(x_1(\cdot)) = \Big\{ u(\cdot) \in U_T : \quad Dx_2(t) = D\Big\{ \mathcal{X}_2(t)x_0^{(2)} + \int_0^t \mathcal{X}_2(t-\tau) \times \Big\{ f\big(\tau, x_1(\tau), Dx_2(\tau)\big) + \mathcal{B}\big(\tau, x_1(\tau)\big)u(\tau) \Big\} \,\mathrm{d}\tau \Big\} \ \text{for} \ t \in T \Big\}.$$

In the sequel, we assume that $U_0(x_1(\cdot)) = U(x_1(\cdot))$. The proposition below follows from this equality and Lemma 1.

Proposition 5. *For every output $x_1(\cdot) \in X_T^{(1)}$ of system (1), the set $U(x_1(\cdot))$ is convex, bounded and closed.*

From this result, it follows that the set

$$U_*\big(x_1(\cdot)\big) = \arg\min \Big\{ |u(\cdot)|_{L_2(T;U)} : \ u(\cdot) \in U\big(x_1(\cdot)\big) \Big\}$$

is a singleton, i.e. $U_*(x_1(\cdot)) = \{u_*(\cdot\,; x_1(\cdot))\}$.

Theorem 1. *Let $x_2(\cdot) = x_2(\cdot\,; x_0, u_*(\cdot\,; x_1(\cdot))) \in Y_T \cap U_K(X_2)$. If conditions (10)–(12) are fulfilled, then*

$$v^h(\cdot) \to u_*\big(\cdot\,; x_1(\cdot)\big) \ \ \text{in} \ \ L_2(T;U) \ \ \text{as} \ \ h \to 0.$$

In other words, the family of positional modelling algorithms $D_h = (M, \Delta_h, \mathcal{U}_h)$ constructed above is regularizing provided that (10)–(12) are satisfied.

Proof of Theorem 1. Let us now estimate the increment in

$$\nu_h(t) = \beta(h)\mu_h\left(t, x_2(\cdot), v^h(\cdot), w_2^h(\cdot)\right),$$

where

$$\mu_h\left(t, x_2(\cdot), v^h(\cdot), w_2^h(\cdot)\right) = \beta^{-1}(h)\exp(-2\omega_2 t)\ \left|w_2^h(t) - x_2(t)\right|_2^2 + \int_0^t \Big\{ \left|v^h(\tau)\right|_U^2 - |u_*(\tau)|_U^2 \Big\} \,\mathrm{d}\tau, \tag{24}$$

$u_*(\cdot) = u_*(\cdot\,; x_1(\cdot))$. Conditions (i), (ii), (iv) and inequalities (2), for $t \in [\tau_i, \tau_{i+1})$ imply the following relations:

$$\left|\mathcal{B}\big(\tau, x_1(\tau)\big)v - \mathcal{B}(\tau_i, \xi_i)v\right|_2 \le b_1\big\{h + \varphi_{\mathcal{B}}(\delta)\big\}, \quad \forall\, v \in P, \quad \tau \in [\tau_i, \tau_{i+1}]. \tag{25}$$

Then we see that

$$g_{i+1} \equiv \exp(-2\omega_2\tau_{i+1})\ \left|w_2^h(\tau_{i+1}) - x_2(\tau_{i+1})\right|_2^2 = \exp(-2\omega_2\tau_{i+1})\Big\{ \left|s_i^e\right|_2^2 + \sum_{j=1}^4 \rho_i^{(j)} \Big\},$$

where

$$s_i^e = \mathcal{X}_2(\tau_{i+1} - \tau_i)\left\{w_2^h(\tau_i) - x_2(\tau_i)\right\},$$

$$\rho_i^{(1)} = 2\Big(s_i^e, \int_{\tau_i}^{\tau_{i+1}} \mathcal{X}_2(\tau_{i+1} - \tau)\left\{\mathcal{B}(\tau_i, \xi_i)v^h(t) - \mathcal{B}\big(\tau, x_1(\tau)\big)u_*(\tau)\right\}\,\mathrm{d}\tau\Big)_2,$$

$$\rho_i^{(2)} = 2\Big(s_i^e, \int_{\tau_i}^{\tau_{i+1}} \mathcal{X}_2(\tau_{i+1} - \tau)\Big\{f\big(\tau_i, \xi_i, v_i^{1,h}\big) - f\big(\tau, x_1(\tau), Dx_2(\tau)\big)\Big\}\,\mathrm{d}\tau\Big)_2,$$

$$\rho_i^{(3)} = 2\Big|\int_{\tau_i}^{\tau_{i+1}} \mathcal{X}_2(\tau_{i+1} - \tau)\left\{\mathcal{B}(\tau_i, \xi_i)v^h(t) - \mathcal{B}\big(\tau, x_1(\tau)\big)u_*(\tau)\right\}\,\mathrm{d}\tau\Big|_2^2,$$

$$\rho_i^{(4)} = 2\Big|\int_{\tau_i}^{\tau_{i+1}} \mathcal{X}_2(\tau_{i+1} - \tau)\Big\{f\big(\tau_i, \xi_i, v_i^{1,h}\big) - f\big(\tau, x_1(\tau), Dx_2(\tau)\big)\Big\}\,\mathrm{d}\tau\Big|_2^2.$$

Owing to Condition C1, Proposition 3, and the Lipschitz condition for the function f, we have

$$\exp(-2\omega_2\tau_{i+1})\,|s_i^e|_2^2 \le g_i \le b_2 < +\infty, \tag{26}$$

$$\begin{aligned} &\Big|f\big(\tau_i, \xi_i, v_i^{1,h}\big) - f\big(\tau, x_1(\tau), Dx_2(\tau)\big)\Big|_{X_2} \\ &\quad \le L\Big\{|\tau - \tau_i| + |x_1(\tau) - \xi_i|_{X_1} + \big|v_i^{1,h} - Dx_2(\tau)\big|_{X_\pi}\Big\} \\ &\quad \le L\Big\{\delta + h + \omega_X(\delta) + \big|v_i^{1,h} - Dx_2(\tau)\big|_{X_\pi}\Big\}, \quad \tau \in [\tau_i, \tau_{i+1}]. \end{aligned} \tag{27}$$

Taking into account (13), (26) and (27), we get

$$\sum_{i=0}^{m_h-1} \Big\{\rho_i^{(2)} + \rho_i^{(4)}\Big\} \le \mu_1(h) = b_3\Big\{h + \delta + \omega_X(\delta) + \nu^{1/2}(h)\Big\}, \tag{28}$$

$$\sum_{i=0}^{m_h-1} \rho_i^{(3)} \le b_4\delta. \tag{29}$$

Now consider the value $\rho_i^{(1)}$. We have (cf. (25), (26) and Condition C5)

$$\rho_i^{(1)} \le 2\Big(s_i^e, \int\limits_{\tau_i}^{\tau_{i+1}} \mathcal{X}_2(\tau_{i+1}-\tau)\mathcal{B}(\tau_i,\xi_i)\{v^h(\tau)-u_*(\tau)\}\,\mathrm{d}\tau\Big)_2 + b_5\delta\{h+\varphi_{\mathcal{B}}(\delta)\}$$

$$\le 2\int\limits_{\tau_i}^{\tau_{i+1}} \Big(Dw_2^h(\tau_i)-v_i^{1,h}, D\mathcal{B}(\tau_i,\xi_i)\{v^h(\tau)-u_*(\tau)\}\Big)_2\,\mathrm{d}\tau$$

$$+ b_5\delta\{\varphi_{\mathcal{B}}(\delta)+\gamma_1(\delta)+h\} + b_6\delta\left|v_i^{1,h}-Dx_2(\tau_i)\right|_{X_2}. \tag{30}$$

Here we also used the equality $Dv_i^{1,h} = v_i^{1,h}$. Combining (28)–(30) and using the definition of the control $v^h(\cdot)$, we get

$$\nu_h(\tau_{i+1}) \le \nu_h(\tau_i) + \varphi_i(h) + b_7\delta\left|v_i^{1,h}-Dx_2(\tau_i)\right|_{X_2}, \tag{31}$$

$$\sum_{i=0}^{m_h-1}\varphi_i(h) \le b_9\mu^{(0)}(h),$$

$$\mu^{(0)}(h) = h+\delta+\omega_X(\delta)+\nu^{1/2}(h)+\varphi_{\mathcal{B}}(\delta)+\gamma_1(\delta), \quad \delta=\delta(h).$$

From (13) it may be concluded that

$$\sum_{i=0}^{m_h-1} b_6\delta\left|v_i^{1,h}-Dx_2(\tau_i)\right|_{X_2} \le b_8\left\{\omega_X(\delta)+\nu^{1/2}(h)\right\}. \tag{32}$$

In turn, from (31) and (32) we obtain

$$\nu_h(\tau_{i+1}) \le \nu_h(\tau_i) + b_9\mu^{(0)}(h).$$

Thus, taking into account the equality $w_0 = x_0$, we have

$$\mu_h\left(\tau_{i+1}, x_2(\cdot), v^h(\cdot), w_2^h(\cdot)\right) \le b_{10}\mu^{(0)}(h)\beta^{-1}(h).$$

Therefore

$$\left|w_2^h(\tau_i)-x_2(\tau_i)\right|_2^2 \le b_{11}\mu^{(0)}(h), \quad i\in[0:m_h], \tag{33}$$

$$\left|v^h(\cdot)\right|_{L_2(T;U)}^2 \le |u_*(\cdot)|_{L_2(T;U)}^2 + b_{11}\mu^{(0)}(h)\beta^{-1}(h). \tag{34}$$

From (13), (25) and (31) we deduce that

$$\sup_{t\in T}\left|\int\limits_{t_0}^{t}\mathcal{X}_2(t-\tau)\mathcal{B}(\tau,x_1(\tau))\{u_*(\tau)-v^h(\tau)\}\,\mathrm{d}\tau\right|_2 \le \mu(\delta,h),$$

where

$$\mu(\delta,h)\to 0 \ \text{ as } \quad \delta\to 0,\ h\to 0.$$

Now the use of Proposition 2, inequality (34) and a standard argument in the spirit of (Kryazhimskii and Osipov, 1987; Osipov *et al.*, 1991; Osipov and Kryazhimskii, 1995) complete the proof. ■

Further, along with Conditions C1–C5, we assume the following ones:

C6. There exists a family $Z(t) \in L(U,U)$, $t \in T$, of one-to-one operators, such that $\forall\, x(\cdot) \in X_T$, $t \in T$, $u_1(\cdot)$, $u_2(\cdot) \in U_T$

$$C^0 \left| \int_0^t Z(\tau)\{u_1(\tau) - u_2(\tau)\}\, \mathrm{d}\tau \right|_U \le \left| \int_0^t \mathcal{X}_2(t-\tau)\mathcal{B}(\tau, x(\tau))\{u_1(\tau) - u_2(\tau)\}\, \mathrm{d}\tau \right|_2, \quad C^0 > 0,$$

$$t \to Z(t)u(t) \in C^+(T;U) \quad \forall\, u(\cdot) \in U_T.$$

C7. $x(\cdot) \in X_V = \{x(\cdot) \in X_T \colon Z^{-1}u_*(\cdot\,; x_1(\cdot)) \in V(T;U), \mathrm{var}_U(T; Z^{-1}u_*(\cdot\,; x_1(\cdot))) \le K\}$, where $(Z^{-1}u_*(\cdot\,; x_1(\cdot)))(t) = (Z^{-1}(t))^* u_*(t; x_1(\cdot))$ for almost all $t \in T$.

Theorem 2. *Let Conditions C1–C7 be fulfilled. Then we have the following estimate of the convergence rate for our algorithm:*

$$|v^h(\cdot) - u_*(\cdot\,; x_1(\cdot))|^2_{L_2(T;U)} \le C\left\{\mu^{(0)}(h)^{1/2} + \mu^{(0)}(h)\beta^{-1}(h)\right\}. \tag{35}$$

Here the constant C does not depend on $u_*(\cdot)$ and $\mu^{(0)}(h)$ is defined in (31).

Inequality (35) is established similarly to (13). For this purpose, we use the estimates (33), (34), and the inequalities

$$|v^h(\cdot) - u_*(\cdot\,; x_1(\cdot))|^2_{L_2(T;U)} \le 2\int_0^{\vartheta} \Big(Z(\tau)\left(u_*(\tau; x_1(\cdot)) - v^h(\tau)\right), \left(Z^{-1}(\tau)\right)^* u_*(\tau; x_1(\cdot))\Big)_U \,\mathrm{d}\tau + b_{11}\mu^{(0)}(h)\beta^{-1}(h),$$

$$C^0 \left| \int_0^t Z(\tau)\left\{v^h(\tau) - u_*(\tau; x_1(\cdot))\right\} \mathrm{d}\tau \right|_U \le b_{12}\left(\mu^{(0)}(h)\right)^{1/2},$$

$$\left| \int_0^t X_2(t-\tau)\left\{\mathcal{B}(\tau, x_1(\tau)) - \mathcal{B}(\tau, \xi(\tau))\right\} v^h(\tau)\, \mathrm{d}\tau \right|_2 \le b_{13}\left(\mu^{(0)}(h)\right)^{1/2}.$$

3. Delay System

3.1. Bilinear System with Scalar Control

Consider the system of equations with time delay

$$\begin{cases} \dot{x}_1(t) = L_1\big(x_{1t}(s)\big) + C_1 x_2(t) + f_0(t), \\ \dot{x}_2(t) = L_2\big(x_{2t}(s)\big) + E\big(x_1(t)\big) + u(t)Bx_1(t), \quad t \in T, \\ L_j\big(y_t(s)\big) = \sum\limits_{i=0}^{l_j} A_i^{(j)} y\big(t - \tau_i^{(j)}\big) + \int\limits_{\tau_{l_j}^{(j)}}^{0} A_*^{(j)}(s) y(t+s)\,\mathrm{d}s, \quad j = 1,2 \end{cases} \tag{36}$$

with the initial conditions

$$\begin{cases} x_1(0) = x_1^0, \quad x_1(s) = x_1^1(s) \;\text{ for }\; s \in \big[-\tau_{l_1}^{(1)}, 0\big], \\ x_2(0) = x_2^0, \quad x_2(s) = x_2^1(s) \;\text{ for }\; s \in \big[-\tau_{l_2}^{(2)}, 0\big]. \end{cases} \tag{37}$$

Here $x_1(t) \in \mathbb{R}^N$, $x_2(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $x_1^0 \in \mathbb{R}^N$, $x_2^0 \in \mathbb{R}^n$, $x_1^1(s) \in L_2([-\tau_{l_1}^{(1)}, 0]; \mathbb{R}^N)$, $x_2^1(s) \in L_2([-\tau_{l_2}^{(2)}, 0]; \mathbb{R}^n)$, $0 = \tau_0^{(j)} < \tau_1^{(j)} < \cdots < \tau_{l_j}^{(j)}$, $x_{1t}(s):\; s \to x_1(t+s)$, $s \in [-\tau_{l_1}^{(1)}, 0]$, $x_{2t}(s):\; s \to x_2(t+s)$, $s \in [-\tau_{l_2}^{(2)}, 0]$. Moreover, $A_i^{(j)}$, B, and C_1 are constant matrices of the dimensions $N \times N$ (for $j = 1$), $n \times n$ (for $j = 2$), $n \times N$ and $N \times n$, respectively. The elements of the matrix functions $s \to A_*^{(j)}(s)$, $s \in [-\tau_{l_j}^{(j)}, 0]$, $j = 1, 2$, are square integrable and $E(\,\cdot\,)\colon \mathbb{R}^N \to \mathbb{R}^n$ is a matrix function satisfying the global Lipschitz condition.

Following (Banks and Kappel, 1979; Kappel, 1986), we denote by $X_1 = \mathbb{R}^N \times L_2([-\tau_{l_1}^{(1)}, 0]; \mathbb{R}^N)$ the Hilbert space of all pairs $x = (x^0, x^1(s))$, with the scalar product

$$(x, y)_{X_1} = \big(x^0, y^0\big)_{\mathbb{R}^N} + \int\limits_{\tau_{l_1}^{(1)}}^{0} \big(x^1(s), y^1(s)\big)_{\mathbb{R}^N}\,\mathrm{d}s$$

and the norm $|\cdot|_{X_1}$. In a similar manner, we define the space $X_2 = \mathbb{R}^n \times L_2([-\tau_{l_2}^{(2)}; 0]; \mathbb{R}^n)$.

Substantially, the input reconstruction problem we are concerned with consists in the following. Let the system (36) be affected by a non-observable control input $u = u(t) \in P = [\alpha, \beta]$, $-\infty < \alpha < \beta < +\infty$. Let the first state component, $x_1(\tau_i)$, be observed at every time τ_i. The observation results are represented by vectors $\xi_i \in \mathbb{R}^N$ such that

$$|x_1(\tau_i) - \xi_i|_{\mathbb{R}^N} \le h.$$

Our task is to reconstruct the input $u(\,\cdot\,)$ in real time.

The equation

$$\dot{x}_j(t) = L_j\big(x_{jt}(s)\big), \quad j = 1,2$$

is known to generate a C_0-semigroup of bounded linear operators $\mathcal{X}_j(t),\ t \geq 0$ which are defined as follows (see Bernier and Manitius, 1978; Banks and Kappel, 1979). Let $s_j(\cdot)$ be a unique solution of the functional-differential matrix equation

$$\frac{\mathrm{d}s_j(t)}{\mathrm{d}t} = A_0^{(j)} s_j(t) + \sum_{i=1}^{l_j} A_i^{(j)} s_j\big(t + \tau_i^{(j)}\big) + \int_{-\tau_{l_j}^{(j)}}^{0} A_*^{(j)}(s) s_j(t+s)\,\mathrm{d}s \ \text{ a.a. on } T$$

with the initial state $s_j(t) = E^*,\ t \leq 0$. Here E^* is the $q \times q$ identity matrix. The operator $B_*^{(j)}:\ L_2([-\tau_{l_j}^{(j)}, 0]; \mathbb{R}^q) \to L_2([-\tau_{l_j}^{(j)}, 0]; \mathbb{R}^q)$ has the form

$$\Big(B_*^{(j)}\varphi\Big)(\tau) = \sum_{i=1}^{l_j} A_i^{(j)} \chi_{[\tau_i^{(j)}, 0]}(\tau)\varphi\big(-\tau_i^{(j)} - \tau\big) + \int_{-\tau_{l_j}^{(j)}}^{0} A_*^{(j)}(\xi)\varphi(\xi - \tau)\,\mathrm{d}\xi$$

for almost all $\tau \in [-\tau_{l_j}^{(j)}, 0]$. Moreover, $q = N$ if $j = 1$ and $q = n$ if $j = 2$, $\chi_{[a,b]}(\cdot)$ is the characteristic function of the interval $[a,b]$ and $F_j:\ X_j \to X_j$ is given by

$$(F_j\varphi)^0 = \varphi^0, \quad (F_j\varphi)^1 = B_*^{(j)}\varphi^1 \quad \big(\varphi = \big(\varphi^0, \varphi^1(s)\big) \in X_j\big).$$

The following equality holds (Bernier and Manitius, 1978, p. 903):

$$\mathcal{X}_j(t)\varphi = G_j^t F_j \varphi + S_j(t)\varphi, \tag{38}$$

where $G_j^t:\ X_j \to X_j$,

$$\big(G_j^t\varphi\big)^1(\tau) = s_j(t+\tau)\varphi^0 + \int_{-\tau_{l_j}^{(j)}}^{0} s_j(t+\tau+\xi)\varphi^1(\xi)\,\mathrm{d}\xi, \quad \tau \in \big[-\tau_{l_j}^{(j)}, 0\big],$$

$$\big(G_j^t\varphi\big)^0 = \big(G_j^t\varphi\big)^1(0), \quad \big(S_j(t)\varphi\big)^0 = 0,$$

$$\big(S_j(t)\varphi\big)^1(\tau) = \varphi(t+\tau)\chi_{[-\tau_{l_j}^{(j)}, -t]}(\tau).$$

In what follows, we assume that $U = \mathbb{R}$. Recall that P and U_T are defined in Section 1. For each $u(\cdot) \in U_T$ let $\tilde{x}(t; x_0, u(\cdot)) = \{\tilde{x}_1(t; x_0, u(\cdot)), \tilde{x}_2(t; x_0, u(\cdot))\}$, $(x_0 = \{x_0^{(1)}, x_0^{(2)}\},\ x_0^{(1)} = (x_1^0, x_1^1(s)) \in X_1,\ x_0^{(2)} = (x_2^0, x_2^1(s)) \in X_2)$ be a unique Carathéodory solution to eqn. (36) with the initial conditions (37), $\tilde{x}_t(s) =$

$\{\tilde{x}_{1t}(s), \tilde{x}_{2t}(s)\} \in X = X_1 \times X_2$. Let $x(t; x_0, u(\cdot))$ be a weak solution to the differential equation (1), where we set

$$\begin{aligned}
&Dy = (y^0, 0) \subset X_\pi, \quad CDy = (C_1 y^0, 0) \in X_1 \quad \forall\, y = (y^0, y^1(s)) \in X_2,\\
&X_\pi = \mathbb{R}^n \times \{0\} \quad \Big(0 \in L_2\big([-\tau_{l_2}^{(2)}, 0]; \mathbb{R}^n\big)\Big),\\
&f(t, x, Dy) = f(x) = \big(E\,(x^0)\,, 0\big),\\
&\mathcal{B}(t, x) = \mathcal{B}(x) \in L(U, X_2): \quad \mathcal{B}(x)u = \big(uBx^0, 0\big), \qquad (39)\\
&\Big(u \in U, \quad x = (x^0, x^1(s)) \in X_1, \quad 0 \in L_2\big([-\tau_{l_2}^{(2)}; 0]; \mathbb{R}^n\big)\Big).
\end{aligned}$$

The operator D is defined in Section 2. In turn the operator A_j is given by (Bernier and Manitius, 1978, Proposition 2.1):

$$\begin{aligned}
D(A_j) = \Big\{\varphi = (\varphi^0, \varphi^1(s)) \in X_j :\\
\varphi^1(s) \in W^{1,2}\big([-\tau_{l_j}^{(j)}, 0]; \mathbb{R}^q\big), \quad \varphi^1(0) = \varphi^0\Big\}, \qquad (40)\\
A_j\varphi = \big(L_j\,(\varphi^1)\,, \dot{\varphi}^1(s)\big)\,, \quad \varphi = (\varphi^0, \varphi^1(s)) \in D(A_j).
\end{aligned}$$

Then $A_j : D(A_j) \subset X_j \to X_j$ is the infinitesimal generator of the C_0-semigroup $\mathcal{X}_j(t)$, $t \geq 0$, of the form (38).

The following theorem establishes a one-to-one correspondence between the Carathéodory solutions to the system (36), (37) and the weak solutions to the system (1) with the operators A_j, $j = 1, 2$, of the form (40) and the operator $\mathcal{B}$ of the form (39).

Theorem 3. *For every $x_0^{(1)} = (x_1^0, x_1^1(s)) \in X_1$, $x_0^{(2)} = (x_2^0, x_2^1(s)) \in X_2$, $x_0 = \{x_0^{(1)}, x_0^{(2)}\}$, $u(\cdot) \in U_T$, and $t \in T$, we have*

$$x(t; x_0, u(\cdot)) = \Big(\big(\tilde{x}(t; x_0, u(\cdot)), \tilde{x}_t(s; x_0, u(\cdot))\big)\Big).$$

The proof of this theorem is based on Lemma 2.4 from (Bernier and Manitius, 1978).

In the space X_j $(j = 1, 2)$ define the norm $|\cdot|_j$ equivalent to the norm $|\cdot|_{X_j}$:

$$\big|(\varphi^0, \varphi^1(s))\big|_j = \Bigg(|\varphi^0|^2_{\mathbb{R}^q} + \int_{-\tau_{l_j}^{(j)}}^{0} |\varphi^1(\tau)|^2_{\mathbb{R}^q}\, g(\tau)\, \mathrm{d}\tau\Bigg)^{1/2}, \quad (\varphi^0, \varphi^1(s)) \in X_j,$$

where $g(\tau) = k$ for $\tau \in (-\tau^{(j)}_{l_j-k-1}, -\tau^{(j)}_{l_j-k})$, $k \in [1 : l_j]$. The scalar product corresponding to the norm $|\cdot|_j$ has the form

$$\Big(\big(\varphi^0, \varphi^1(s)\big)\,, \big(\bar{\varphi}^0, \bar{\varphi}^1(s)\big)\Big)_j = \big(\varphi^0, \bar{\varphi}^0\big)_{\mathbb{R}^q} + \int_{-\tau_{l_j}^{(j)}}^{0} \big(\varphi^1(\tau), \bar{\varphi}^1(\tau)\big)_{\mathbb{R}^q}\, g(\tau)\, \mathrm{d}\tau.$$

As follows from (Banks and Kappel, 1979, Lemma 2.3) for the norm $|\cdot|_j$ in X_j, the semigroup $\mathcal{X}_j(t)$, $t \in T$ is ω_j-dissipative:

$$|\mathcal{X}_j(t)z|_j \le \exp(\omega_j t)|z|_j \quad (z \in X_j, \quad j = 1, 2),$$

where

$$\omega_j = \frac{1+l_j}{2} + |A_0^{(j)}| + \frac{1}{2}\sum_{i=1}^{l_j} |A_i^{(j)}|^2 + \frac{1}{2}\int_{-\tau_{l_j}^{(i)}}^{0} |A_*^{(j)}(\tau)|^2 \, d\tau$$

and $|A|$ stands for the matrix norm.

Proposition 6. *Let $X_0 \subset X$ be bounded. Then the set of all solutions to the system (36),*

$$X(X_0) = \left\{\tilde{x}(\,\cdot\,; x_0, u(\,\cdot\,)) : \quad x_0 \in X_0,\ u(\,\cdot\,) \in U_T\right\}$$

is bounded in $W^{1,2}(T; \mathbb{R}^{N+n}) = \{x(\,\cdot\,) \in L_2(T; \mathbb{R}^{N+n}) : x_t(\,\cdot\,) \in L_2(T; \mathbb{R}^{N+n})\}$.

Proposition 7. *Set*

$$x_1^1(s) \in C_1\big([-\tau_{l_1}^{(1)}, 0]; \mathbb{R}^N\big), \quad x_2^1(s) \in C_1\big([-\tau_{l_2}^{(2)}, 0]; \mathbb{R}^n\big), \tag{41}$$
$$x_1^0 = x_1^1(0), \quad x_2^0 = x_2^1(0)$$

in (37). Then the semigroup $\mathcal{X}_1(t)$, $t \in T$ ($\mathcal{X}_2(t)$, $t \in T$) satisfies Conditions C3 and C5 for $\gamma(\delta) = \delta^{1/2}$ ($\gamma_1(\delta) = \gamma(\delta)$).

We omit the proofs of Propositions 6 and 7, since they utilize standard algebraic transformations and Gronwall's inequality. In what follows, we assume that condition (41) is fulfilled.

Thus, for system (1), which is equivalent to the system of functional-differential equations (36) and (37), all the assumptions of Section 2 are satisfied. Therefore, for solving the input reconstruction problem we use the method presented earlier. Let us describe it in detail. Referring to the definition of the operator $\mathcal{B}$, cf. (39), and using Theorem 3, we conclude that the model M is described by the delay system

$$\dot{w}_1^h(t) = L_1\left(w_{1t}^h(s)\right) + C_1 v^{1,h}(t) + f_0(t),$$
$$\dot{w}_2^h(t) = L_2\left(w_{2t}^h(s)\right) + E(\xi(t)) + v^h(t)B\xi(t), \quad t \in T,$$
$$w_j^h(s) = x_j^1(s) \text{ for } s \in \big[-\tau_{l_j}^{(j)}, 0\big], \quad j = 1, 2$$

with controls $v^{1,h}(t) \in \mathbb{R}^n$, $v^h(t) \in \mathbb{R}$. We denote by $w^h(t; x_0, \xi(\,\cdot\,), v(\,\cdot\,))$ the Carathéodory solution of this system on the segment T. Define a positional strategy $S_h = (\Delta_h, \mathcal{U}_h)$, $h \in (0, 1)$ by (3), (4) and

$$\mathcal{U}_h^{(1)} = \mathcal{U}_h^{(1)}\left(\tau_i, \xi_i, w^h(\tau_i)\right) = \left\{\tilde{v}_{\tau_i,\tau_{i+1}}^{1,h}(\,\cdot\,), \tilde{v}_{\tau_i,\tau_{i+1}}^{h}(\,\cdot\,)\right\},$$

$$\tilde{v}^{1,h}(t) = \tilde{v}_i^{1,h} = \arg\min\Big\{L_1\big(\alpha, v, \tilde{s}_i^0\big): \quad v \in S_0(d_1)\Big\}, \tag{42}$$

$$\tilde{v}^h(t) = \tilde{v}_i^h = \arg\min\Big\{L_2\big(\beta, v, \tilde{s}_i^*\big): \quad v \in P\Big\}.$$

Here

$$w^h(\tau_i) = \Big\{w_1^h(\tau_i), w_2^h(\tau_i)\Big\} \in \mathbb{R}^{N+n},$$

$$L_1\left(\alpha, v, \tilde{s}_i^0\right) = \alpha(h)|v|^2_{\mathbb{R}^N} + 2\big(\tilde{s}_i^0, C_1 v\big)_{\mathbb{R}^N},$$

$$L_2\big(\beta, v, \tilde{s}_i^*\big) = \beta(h)|v|^2 + 2\big(\tilde{s}_i^*, B\xi_i\big)_{\mathbb{R}^n} v,$$

$$\tilde{s}_i^0 = \big(w_1^h(\tau_i) - \xi_i\big)\exp(-2\omega_1\tau_{i+1}),$$

$$\tilde{s}_i^* = \big(w_2^h(\tau_i) - \tilde{v}_i^{1,h}\big)\exp(-2\omega_2\tau_{i+1}),$$

$$d_1 = \sup\Big\{\, \big|x_2\big(t; x_0, u(\,\cdot\,)\big)\big|_{\mathbb{R}^n}: \quad u(\,\cdot\,) \in U_T, \quad t \in T\Big\},$$

$$S_0(d_1) = \Big\{v \in \mathbb{R}^n: \quad |v|_{\mathbb{R}^n} \le d_1\Big\}.$$

Assume that the following relationships between the parameters are valid:

$$\alpha(h) \to 0, \quad \beta(h) \to 0,$$
$$\Big\{\Big(h + \delta^{1/2}(h) + \alpha(h)\Big)^{1/2} + \Big(h + \delta^{1/2}(h)\Big)\,\alpha^{-1}(h)\Big\}\beta^{-1}(h) \to 0 \tag{43}$$

as $h \to 0$. For example, we can set $\delta = h^2$, $\alpha = h^{1/2}$, $\beta = h^\nu$, $\nu \in (0, 1/4)$.

Theorem 4. *Let a control $\tilde{v}^h(\,\cdot\,)$ be determined by (42). Let $n \le N$ and the following conditions be fulfilled:*

(a) $$\inf_{t\in T}\big|s_1^{-1}(t)x\big|_{\mathbb{R}^N} \ge d_1|x|_{\mathbb{R}^N} \quad \forall\, x \in \mathbb{R}^N \quad (d_1 > 0), \tag{44}$$

(b) there exist a number $d_2 > 0$ and an n-th order minor of the matrix $s_1(t)C_1$, such that the $n \times n$-matrix $\overline{s_1(t)C_1}$ corresponding to this minor satisfies the inequality

$$\inf_{t\in T}|\overline{s_1(t)C_1}v|_{\mathbb{R}^n} \ge d_2|v|_{\mathbb{R}^n} \tag{45}$$

for all $v \in \mathbb{R}^n$,

(c) for any solution $x_2(\,\cdot\,)$ to the system (36), (37) we have $(\overline{s_1(\vartheta - t)C_1})^{-1}x_2(t) \in V(T; \mathbb{R}^n)$.

Then

$$\tilde{v}^h(\,\cdot\,) \to u_*\big(\,\cdot\,; x_1(\,\cdot\,)\big) \quad \textit{in} \quad L_2(T; \mathbb{R}).$$

Let the following conditions be also fulfilled:

(d) $$\inf_{t\in T} \left|s_2^{-1}(t)x\right|_{\mathbb{R}^n} \geq d^{(1)}|x|_{\mathbb{R}^n} \quad \forall\, x \in \mathbb{R}^n \quad (d^{(1)} > 0)$$

(e) there exists a coordinate of vector $s_2(\vartheta - t)Bx_1(t)$ *(denote it by* $\{s_2(\vartheta - t)Bx_1(t)\}_*$*) such that*

$$\inf_{t\in T} \left|\{s_2(\vartheta - t)Bx_1(t)\}_*\right| > 0.$$

If $\{s_2(\vartheta - t)Bx_1(t)\}_*^{-1}u_*(t, x_1(\cdot)) \in V(T;\mathbb{R})$*, then the following estimate of the rate of convergence takes place:*

$$\left|\tilde{v}(\cdot) - u_*(\cdot\,; x_1(\cdot))\right|_{L_2(T;\mathbb{R})} \leq C\left\{\mu^{(1)}(h)^{1/2} + \mu^{(1)}(h)\beta^{-1}(h)\right\},$$

where

$$\mu^{(1)}(h) = \left(h + \delta^{1/2}(h) + \alpha(h)\right)^{1/2} + \left(\left(h + \delta^{1/2}(h)\right)\alpha^{-1}(h)\right)^{1/2}.$$

Proof. To prove the first statement, we refer to Theorem 1. One can easily see that conditions (44) and (45) imply Condition C2. Indeed, in this case, due to (44), for all $y(\cdot)$, $t \to y(t) = \{y^0(t), y_t^1(s)\} \in C^+(T; X_2)$, we have

$$\begin{aligned}
\left|\int_0^t \mathcal{X}_1(t-\tau)CDy(\tau)\,\mathrm{d}\tau\right|_1 &\geq \left|\int_0^t s_1(t-\tau)C_1y^0(t-\tau)\,\mathrm{d}\tau\right|_{\mathbb{R}^N} \\
&= \left|s_1^{-1}(\vartheta - t)\int_0^t s_1(\vartheta-\tau)C_1y^0(t-\tau)\,\mathrm{d}\tau\right|_{\mathbb{R}^N} \\
&\geq d_1\left|\int_0^t s_1(\vartheta-\tau)C_1y^0(t-\tau)\,\mathrm{d}\tau\right|_{\mathbb{R}^N} \\
&\geq d_1\left|\int_0^t \overline{s_1(\vartheta-\tau)C_1}y^0(t-\tau)\,\mathrm{d}\tau\right|_{\mathbb{R}^n} \\
&= d_1\left|\int_0^t Y(\tau)Dy(\tau)\,\mathrm{d}\tau\right|_{X_\pi},
\end{aligned}$$

where

$$Y(t)x = \left(\overline{s_1(\vartheta - t)C_1}x^0, 0\right): \quad X_\pi \to X_\pi \subset X_2, \quad x = \left(x^0, 0\right) \in X_\pi.$$

(Thus, in Condition C2 one can take $c_0 = d_1$.) The injectivity of mapping $Y(t)$, $t \in T$ follows from (45). The validity of Conditions C3 and C5 may easily be derived from the form of the operators C and $\mathcal{B}(x)$ (see (39)) and Proposition 7. The conditions (9)–(11) follow from (43) and Proposition 4. The correctness of the equality $U_0(x_1(\cdot)) = U(x_1(\cdot))$ is easy to check. Finally, Condition C4 is a consequence of condition (c). The validity of the second statement of our Theorem follows from Theorem 2. ■

3.2. Linear System with Vector Control

Substitute the term $u(t)Bx(t)$ on the right-hand side of the second equation of (36) by $Bu(t)$, where $u(t) \in P \subset \mathbb{R}^r$, P is a convex bounded closed set, and B is an $n \times r$-matrix, i.e. the system to be solved is of the form

$$\begin{cases} \dot{x}_1(t) = L_1\big(x_{1t}(s)\big) + C_1 x_2(t) + f_0(t), \\ \dot{x}_2(t) = L_2\big(x_{2t}(s)\big) + E\big(x_1(t)\big) + Bu(t), \quad t \in T, \\ L_j\big(y_t(s)\big) = \sum\limits_{i=0}^{l_j} A_i^{(j)} y\Big(t - \tau_i^{(j)}\Big) + \int\limits_{\tau_{l_j}^{(j)}}^{0} A_*^{(j)}(s) y(t+s)\,\mathrm{d}s, \quad j = 1,2 \end{cases} \tag{46}$$

with the initial conditions (37). For reconstruction of a vector control $u_*(\cdot\,;x_1(\cdot))$ generating the solution $x_1(\cdot)$ of (46) we use the scheme described above. As the model we take the system

$$\dot{w}_1^h(t) = L_1\left(w_{1t}^h(s)\right) + C_1 v^{1,h}(t) + f_0(t),$$

$$\dot{w}_2^h(t) = L_2\left(w_{2t}^h(s)\right) + E\big(\xi(t)\big) + Bv^h(t), \quad t \in T,$$

$$v^{1,h}(t) \in \mathbb{R}^r, \quad v^h(t) \in \mathbb{R}^r$$

with the initial state coinciding with the one of the system (46). Define the strategy S_h according to (3), (4), (42), where we assume, though, that

$$L_2\big(\beta, v, \tilde{s}_i^*\big) = \beta(h)|v|^2_{\mathbb{R}^r} + 2\big(\tilde{s}_i^*, Bv\big)_{\mathbb{R}^r}.$$

Theorem 5. *Let conditions (a)–(c) of Theorem 4 be fulfilled. Then*

$$\tilde{v}^h(\cdot) \to u_*(\cdot\,;x(\cdot)) \quad in \quad L_2(T;\mathbb{R}^r).$$

Let condition (d) of Theorem 4 and the following conditions be also fulfilled:

(a) there exist a number $d_2 > 0$ and an r-th order minor of the matrix $s_2(t)B$, such that the $r \times r$-matrix $\overline{s_2(t)B}$ corresponding to this minor satisfies the inequality

$$\inf_{t\in T} |\overline{s_2(t)B}v|_{\mathbb{R}^r} \geq d_2|v|_{\mathbb{R}^r}$$

for all $v \in \mathbb{R}^r$,

(b) for any solution $x_2(\cdot)$ *of (36), (37), we have* $\{(\overline{s_2(\vartheta - t)B})^{-1}\}' u_*(t, x_1(\cdot)) \in V(T; \mathbb{R}^r)$.

Then the following estimate of the rate of convergence takes place:

$$\left|\tilde{v}(\cdot) - u_*(\cdot\,; x_1(\cdot))\right|_{L_2(T;\mathbb{R}^r)} \le C_* \left\{\mu^{(1)}(h)^{1/2} + \mu^{(1)}(h)\beta^{-1}(h)\right\}.$$

Let us recall that the prime above indicates transposes.

4. Example

The algorithm described was tested by a model example. The following system was considered (Kappel, 1986):

$$\dot{x}_1(t) = -r^{-1}x_1(t) + kr^{-1}x_2(t-\tau),$$

$$\dot{x}_2(t) = x_3(t),$$

$$\dot{x}_3(t) = -\omega^2 x_2(t) - 2q\omega x_3(t) + \omega^2 u(t),$$

on time interval $T = [0, 2]$. It was assumed that $\omega = 1{,}1$, $q = 0{,}1$, $x_1(t) = 1$ for $t \in [-\tau, 0]$, $x_2(t) = a/b \sin bt$, $x_3(t) = a \cos bt$, $u(t) = -a(\omega^2 b)^{-1} \sin bt + ab^{-1} \sin bt + 2qa\omega^{-1} \cos bt$. At moments τ_i the value

$$\xi_i = x_2(\tau_i) + h \sin M\tau_i$$

was measured. As a model we took the system

$$\dot{w}_1(t) = -r^{-1}w_1(t) + kr^{-1}w_1^{(1)}(t-\tau),$$

$$\dot{w}_1^{(1)}(t) = \tilde{v}^{1,h}(t),$$

$$\dot{w}_2(t) = -\omega^2 \xi(t) - 2q\omega w_2(t) + \omega^2 \tilde{v}^h(t)$$

with the initial state $w_1(0) = x_1(0)$, $w_1^{(1)}(0) = x_2(0)$, $w_2(t) = x_3(t)$ for $t \in [-\tau, 0]$. Controls $\tilde{v}_i^{1,h}$ and $\tilde{v}_i^h$ at moments τ_i were calculated from condition (42) which, in this case, was of the form

$$\tilde{v}_i^{1,h} = \arg\min\left\{2\bar{s}_i^0 v^{1,h} + \alpha(h)\left(v^{1,h}\right)^2 : \quad \left|v^{1,h}\right| \le K\right\},$$

$$\tilde{v}_i^h = \arg\min\left\{s_i^0 v^h + \beta(h)\left(v^h\right)^2 : \quad \left|v^h\right| \le L\right\},$$

$$s_i^0 = \left(w_2(\tau_i) - v_i^{1,h}\right)\exp(-2\omega_2\tau_{i+1}),$$

$$\bar{s}_i^0 = \left(w_1^{(1)}(\tau_i) - \xi_i\right)\exp(-2\omega_1\tau_{i+1}),$$

$$\omega_1 = 1 + r^{-1} + 0.5(kr^{-1})^2, \qquad \omega_2 = 0.5 + 2\xi\omega.$$

In Figs. 1–3 the results of calculations are presented for the case when $\delta(h) = k_1 h^2$, $\alpha(h) = k_2 h^{1/2}$, $\beta(h) = k_3 h^\nu$, $\tau = 1$, $r = -0.9$, $k = 0.01$, $k_1 = 1$, $k_2 = 0.1$, $k_3 = 0.08$, $\nu = 0.22$, $M = 10$, $a = 5$, $b = 5$, $K = 5$, $L = 5$. Solid (dashed) lines represent the real trajectory $x_2(t)$ and the control $u(t)$ (model controls $\tilde{v}^{1,h}(t)$ and $\tilde{v}^h(t)$).

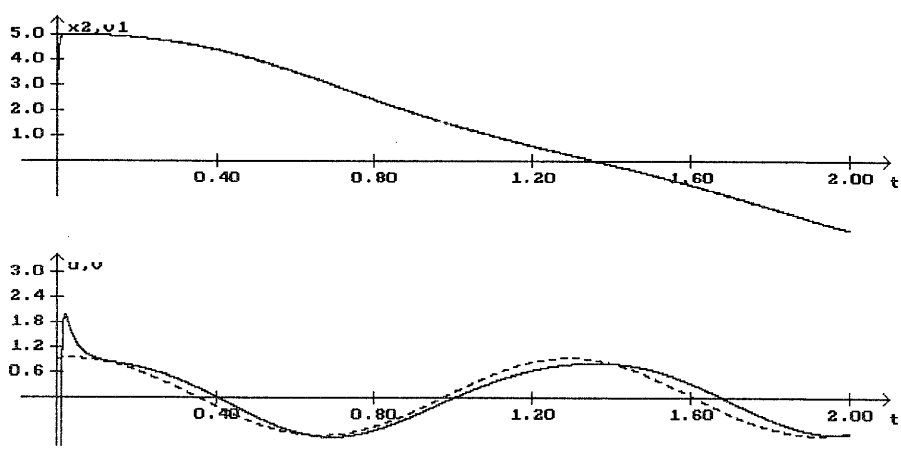

Fig. 1. Numerical results for $h = 10^{-3}$.

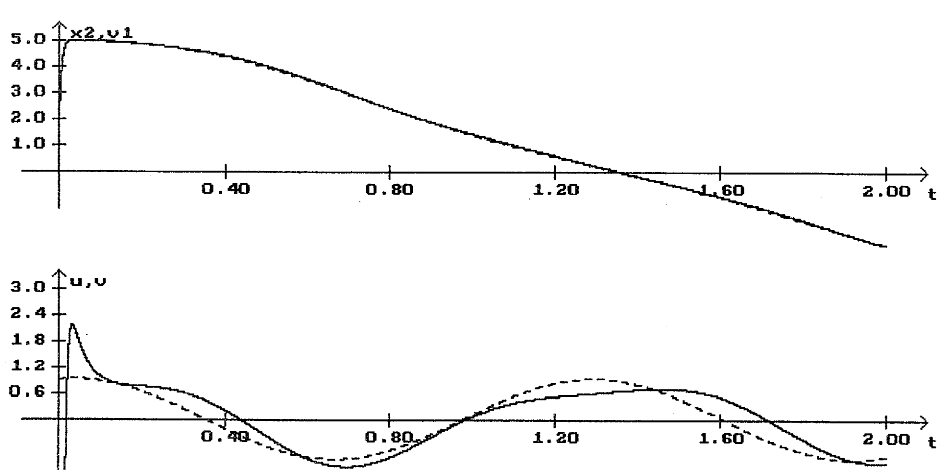

Fig. 2. Numerical results for $h = 10^{-5/2}$.

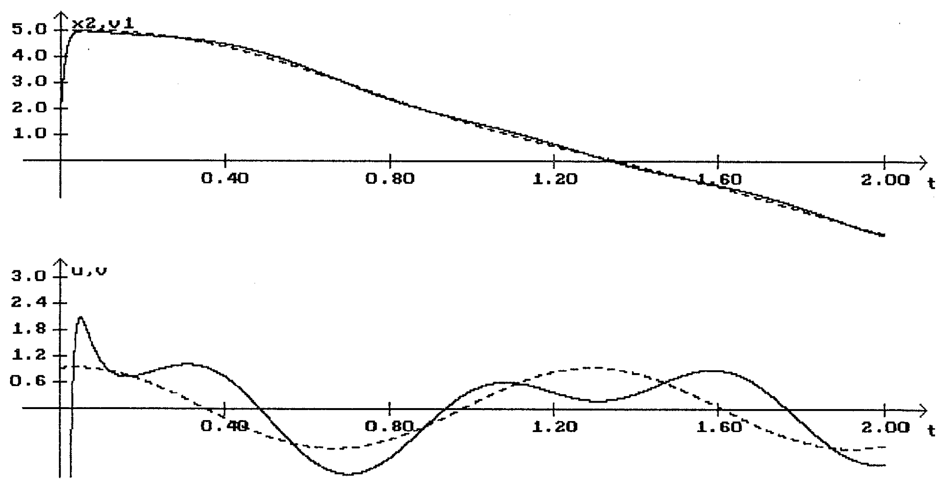

Fig. 3. Numerical results for $h = 10^{-2}$.

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