Int. J. Appl. Math. and Comp. Sci., 2000, Vol.10, No.2, 381–404

SHAPE OPTIMIZATION OF LABYRINTH SEALS[†]

RAJAE ABOULAÏCH*, KARIM AZELMAT*,**
JACQUES BARANGER***, ALI SOUISSI*,**

In this work, we determine the optimal shape of a labyrinth seal in a hydraulic Francis turbine. The numerical approximation of the optimal shape is obtained using shape optimization techniques. The flow is governed by Navier-Stokes equations. We first prove the existence of an optimal domain, and later we present the computation of the shape gradient which allows us to approximate numerically the optimal domain.

Keywords: shape optimization, hydraulic turbine, Navier-Stokes equations, finite element method

1. Introduction

Optimization of the shape of a hydraulic turbine in order to minimize the leakage at the input is an important industrial topic. A mathematical model neglecting inertia for this problem and some numerical experiments have been described in (Samouh, 1992).

In this work, we give a proof of the existence of the optimal shape for a model including inertial terms using a technique described in (Haslinger and Neittaanmäki, 1988). We also calculate the shape gradient by domain parametrization techniques (Zolésio, 1981). The problem is then approximated by the FEM (Finite Element Method) and a convergence result for a subsequence is proved. Finally, we report on some numerical experiments.

2. Problem Statement

We consider the flow governed by steady Navier-Stokes equations in labyrinth seals. Since the flow is supposed to be the same in each seal of the labyrinth, we reduce the study to one seal represented by a domain Ω whose boundary is $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where Γ_0, Γ_1 and Γ_2 are assumed to be fixed. The geometrical situation is described in Fig. 1.

† This work was partly supported by C.N.C.P.R.S.T. of Morocco and A.I. 845/95.

* Université Mohamed V, Ecole Mohammadia d'Ingénieurs, L.E.R.M.A, Av. Ibn Sina, B.P. 765 AGDAL, Rabat, Morocco, e-mail: aboul@emi.ac.ma

** Université Mohamed V, Faculté des Sciences, Département de Mathématiques et Informatique, Av. Ibn Battouta, B.P. 1014, Rabat, Morocco, e-mail: {azalmat,souissi}@fsr.ac.ma

*** Université Claude Bernard Lyon I, Laboratoire d'analyse numérique CNRS UMR 5585, 43, Boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France, e-mail: baranger@cdcsp.univ-lyon1.fr

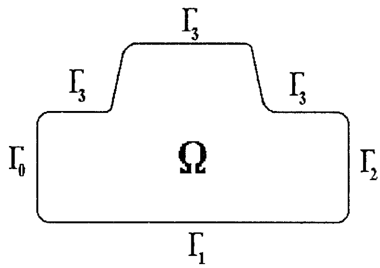

Fig. 1. The domain representing a seal.

The velocity u and the pressure p of the fluid satisfy the following system:

$$P(\Omega):\begin{cases} -\nu\Delta u+(u\cdot\nabla)u+\nabla p=0 & \text{in } \Omega,\\ \nabla\cdot u=0 & \text{in } \Omega,\\ u=0 & \text{on } \Gamma\backslash\Gamma_0\cup\Gamma_2,\\ -p+\nu\dfrac{\partial u\cdot n}{\partial n}=g\chi_{\Gamma_2} & \text{on } \Gamma_0\cup\Gamma_2,\\ u\cdot\tau=0 & \text{on } \Gamma_0\cup\Gamma_2, \end{cases}$$

where ν is the viscosity of the fluid.

We consider the spaces

$$\mathbb{H}^1(\Omega)=H^1(\Omega)\times H^1(\Omega),$$

$$\mathcal{V}=\left\{v\in\mathbb{H}^1(\Omega): v=0 \text{ on } \Gamma_1\cup\Gamma_3,\ \ v\cdot\tau=0 \text{ on } \Gamma_0\cup\Gamma_2\right\},$$

$$Q=L^2(\Omega),$$

$$H^1_{\Gamma_{13}}(\Omega)=\left\{w\in H^1(\Omega): w=0 \text{ on } \Gamma_1\cup\Gamma_3\right\},$$

and the following forms:

$$a(u,v)=\nu\int_\Omega\nabla u\cdot\nabla v\,\mathrm{d}x, \tag{1}$$

$$b(v,q)=\int_\Omega q\,\operatorname{div}v\,\mathrm{d}x, \tag{2}$$

$$c(u,v,w)=\int_\Omega(u\cdot\nabla)v\cdot w\,\mathrm{d}x, \tag{3}$$

$$L(v)=\int_{\Gamma_2}g\,v\cdot n\,\mathrm{d}\sigma \quad\text{with}\quad g\in\left(H^{\frac{1}{2}}_{00}(\Gamma_0\cup\Gamma_2)\right)'. \tag{4}$$

where

$$H^{\frac{1}{2}}_{00}(\Gamma_0\cup\Gamma_2)=\left\{v\in L^2(\Gamma_0\cup\Gamma_2):\exists\, w\in H^1_{\Gamma_{13}}(\Omega) \text{ such that } w=v \text{ on } \Gamma_0\cup\Gamma_2\right\}$$

is the space of traces of $H^1_{\Gamma_{13}}(\Omega)$ on $\Gamma_0 \cup \Gamma_2$ (Lions and Magenes, 1972). This space is introduced to insure the compatibility with the homogeneous limit conditions on other parts of the boundary. $H^{\frac{1}{2}}_{00}(\Gamma_0 \cup \Gamma_2)$ is equipped with the norm

$$||v||_{00} = \inf_{w \in H^1_{\Gamma_{13}}(\Omega)} \left\{ ||w||_{H^1(\Omega)} : w = v \text{ on } \Gamma_0 \cup \Gamma_2 \right\}$$

which defines a lifting operator from $H^{\frac{1}{2}}_{00}(\Gamma_0 \cup \Gamma_2)$ to $\mathcal{V}$.

Note that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are respectively continuous bilinear forms on $\mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega)$ and $\mathbb{H}^1(\Omega) \times L^2(\Omega)$, $c(\cdot,\cdot,\cdot)$ is a continuous trilinear form on $\mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega)$ and L is a continuous linear form on $\mathbb{H}^1(\Omega)$. Moreover, we have the coercivity property

$$a(v,v) \geq \beta \, ||v||_1^2 \quad \forall\, v \in \mathcal{V} \tag{5}$$

and the inf-sup condition

$$\inf_{q \in Q} \sup_{\mathcal{V}} \frac{b(v,q)}{||v||_1 \, ||q||_0} \geq \mu,$$

where β and μ are two nonnegative constants.

The two possible variational formulations associated with Problem $P(\Omega)$ are the following:

$$P_1(\Omega) : \begin{cases} a(u,v) + c(u,u,v) - b(p,v) = L(v) & \forall v \in \mathcal{V}, \\ b(q,u) = 0 \quad \forall q \in Q \end{cases}$$

and

$$P_2(\Omega) : \quad a(u,v) + c(u,u,v) = L(v) \quad \forall v \in \mathcal{V}_1,$$

where

$$\mathcal{V}_1 = \{v \in \mathcal{V} : \operatorname{div} v = 0 \text{ in } \Omega\}.$$

We will use respectively $P_1(\Omega)$ and $P_2(\Omega)$ to approximate the solution to $\mathcal{P}_{\text{opt}}$ of Section 3 and to prove the existence result.

Theorem 1. *If β and L are such that*

$$\frac{\beta^2}{16\gamma} > ||L||_{(\mathbb{H}^1(\Omega))'}, \tag{6}$$

where γ is the constant in the inequality $c(u,v,w) \leq \gamma \, ||u||_{1,\Omega} \, ||v||_{1,\Omega} \, ||w||_{1,\Omega}$, then $P_2(\Omega)$ has a unique solution in the ball $M = \{v \in \mathcal{V} : ||v||_{\mathcal{V}} < \beta/4\gamma\}$.

For the proof, we consider the problem

$$P_{u_0}(\Omega) : \begin{cases} \text{Find } u \in \mathcal{V}_1 \text{ such that} \\ a(u,v) = L(v) - c(u_0,u_0,v) \quad \forall v \in \mathcal{V}_1, \end{cases}$$

and show that the operator defined by

$$K : \begin{cases} \mathcal{V}_1 & \longrightarrow & \mathcal{V}_1 \\ u_0 & \longmapsto & u \end{cases}$$

is a contraction in M. Then the assertion follows from the Banach fixed-point theorem. For more details, refer to (Azelmat, 1997; Gunzburger and Hou, 1992; Orlt and Sändig, 1994; Temam, 1983).

The existence of p in Q is obtained under the inf-sup condition (Brezzi and Fortin, 1991; Girault and Raviart, 1979; Temam, 1983).

3. Existence of the Optimal Domain

The boundaries Γ_0, Γ_1 and Γ_2 of the admissible domains are fixed. Therefore we look for a shape of the part Γ_3 of the boundary minimizing the flow of the leakage losses. The set of the admissible controls will be the set of the admissible functions which parametrize Γ_3. It is defined by

$$\mathcal{U}_{\text{ad}} = \Big\{ \alpha \in C^{0,1}(0,a) \ : |\alpha(x) - \alpha(y)| \leq k\,|x-y|\,, \\ 0 < b \leq \alpha(x) \leq c,\ \alpha(0) = \alpha(a) = d \Big\},$$

where a, b, c, d and k are fixed real constants, and $C^{0,1}(0,a)$ is the set of Lipschitz continuous functions on $(0,a)$.

In order to minimize the flow of the leakage losses at the input, we consider the cost functional given by

$$\mathcal{J}(\Omega) = \int_{\Gamma_0} u \cdot n \, \mathrm{d}\sigma.$$

To prove the existence of the optimal domain, we will use an approach similar to the one introduced by Haslinger and Neittaanmäki (1988). For that purpose, we consider the domains $\Omega(\alpha)$ parametrized by functions $\alpha \in \mathcal{U}_{\text{ad}}$ such that

$$\Omega \equiv \Omega(\alpha) = \big\{ (x_1, x_2) : 0 < x_1 < a,\ 0 < x_2 < \alpha(x_1) \big\}.$$

and the shape optimization problem considered is

$$\mathcal{P}_{\text{opt}} : \begin{cases} \min \big\{ J(\alpha) = \mathcal{J}\big(\Omega(\alpha)\big),\ \alpha \in \mathcal{U}_{\text{ad}} \big\} \\ \text{such that } (u,p) \text{ is a solution to } P(\Omega) \end{cases}$$

We define the convergence of domains by the uniform convergence of the functions which parametrize Γ_3, i.e.

$$\text{`}\Omega_n \longrightarrow \Omega\text{'} \iff \alpha_n \rightrightarrows \alpha \ \text{ in } (0,a),$$

where $\Omega_n \equiv \Omega(\alpha_n)$ and $\Omega \equiv \Omega(\alpha)$.

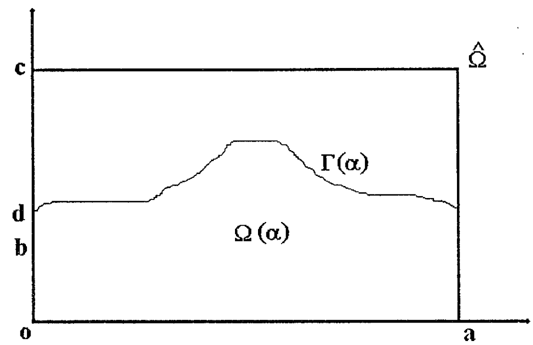

Fig. 2. Parametrization of the domains.

Let

$$\mathcal{V}(\Omega_n) = \{v \in \mathbb{H}^1(\Omega_n) : v = 0 \text{ on } \Gamma_1 \cup \Gamma(\alpha_n),\ v \cdot \tau = 0 \text{ on } \Gamma_0 \cup \Gamma_2\}.$$

We establish below the continuity of the state with respect to the domain and the compactness of $\mathcal{U}_{\mathrm{ad}}$.

Denote by $\hat{\Omega}$ the open rectangle $(0,a) \times (0,c)$ which contains all domains $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{\mathrm{ad}}$. To prove the continuity result, we use the next proposition whose proof is given in (Haslinger and Neittaanmäki, 1988).

Proposition 1. *Given* $A_\alpha(y,\xi) = \int_0^a y(x_1,\alpha(x_1))\xi(x_1,\alpha(x_1))\,\mathrm{d}x_1$, $y_n, y \in \mathbb{H}^1(\hat{\Omega})$ *and* $\alpha_n, \alpha \in \mathcal{U}_{\mathrm{ad}}$ *such that* $y_n \rightharpoonup y$ *in* $\mathbb{H}^1(\hat{\Omega})$ *and* $\alpha_n \rightrightarrows \alpha$ *in* $(0,a)$, *we have* $A_{\alpha_n}(y_n,\xi) \xrightarrow[n\to\infty]{} A_\alpha(y,\xi) \quad \forall \xi \in (C^\infty(\bar{\hat{\Omega}}))^2$.

Remark 1. In order to prove the continuity property, we assume that the constants β and γ of formulae (5) and (6) do not depend on Ω_n when considering the multilinear forms defined on products of $\mathbb{H}^1(\hat{\Omega})$, where $\Omega(\alpha_n) \subset \hat{\Omega}$ for all $\alpha_n \in \mathcal{U}_{\mathrm{ad}}$ and $\beta^2/16\gamma \geq \|L\|_{(\mathbb{H}^1(\hat{\Omega}))'}$.

Proposition 2. *Assume that* $\beta^2/16\gamma \geq \|L\|_{(\mathbb{H}^1(\Omega_n))'}$ *and let* $u_n = u(\alpha_n) \in \mathcal{V}(\alpha_n) = \mathcal{V}(\Omega_n)$ *be the solution to* $P_2(\Omega((\alpha_n))$, $\alpha_n \in U_{\mathrm{ad}}$. *Then there exists a subsequence of* $\{(\alpha_n, u_n)\}$, *denoted again by* $\{(\alpha_n, u_n)\}$, *and elements* $\alpha \in U_{\mathrm{ad}}, u \in \mathcal{V}(\alpha)$ *such that* '$\Omega(\alpha_n) \to \Omega(\alpha)$' *and* '$u_n \to u$' *in the sense that* $\tilde{u}_n \rightharpoonup \tilde{u}$ *in* $\mathbb{H}^1(\hat{\Omega})$, *where* $\tilde{u}_n$ *and* $\tilde{u}$ *denote respectively the uniform extensions of* u_n *and* u *to* $\hat{\Omega}$. *Moreover,* u *is a solution to* $P_2(\Omega(\alpha))$.

Proof. Let $\{\alpha_n\}$ be a sequence of functions in $\mathcal{U}_{\mathrm{ad}}$. This sequence is uniformly bounded and equicontinuous, and therefore by Ascoli-Arzela's theorem there exists a subsequence, denoted again by $\{\alpha_n\}$, which converges uniformly towards α in $(0,a)$. We consider a sequence of solutions of the following variational equations:

$$\int_{\Omega_n} \big(\nu \nabla u_n \cdot \nabla v + (u_n \cdot \nabla) u_n \cdot v\big)\,\mathrm{d}x = L(v) \quad \forall v \in \mathcal{V}_1(\alpha_n), \tag{7}$$

where

$$\mathcal{V}_1(\alpha_n) = \{v \in \mathcal{V}(\alpha_n) : \operatorname{div} v = 0 \ \text{ in } \ \Omega(\alpha_n)\}.$$

For all n, u_n is uniquely determined in the ball M. To get back to a fixed space, we extend u_n by 0 in $\hat{\Omega}\backslash\Omega$, which is feasible through the homogeneous Dirichlet condition on $\Gamma(\alpha_n)$. Denoting by $\tilde{u}_n$ that extension, we get

$$\|\tilde{u}_n\|_{\mathbb{H}^1(\hat{\Omega})} = \|u_n\|_{\mathbb{H}^1(\Omega_n)} < C. \tag{8}$$

We can then extract a subsequence of $\{\tilde{u}_n\}$, denoted again by $\{\tilde{u}_n\}$, which converges weakly to $\tilde{u}$ in $\mathbb{H}^1(\hat{\Omega})$.

It remains to show that $\tilde{u}|_{\Omega(\alpha)}$ is a solution to $P_2(\Omega(\alpha))$.

Step 1: $\tilde{u}|_{\Omega(\alpha)} \in \mathcal{V}_1(\Omega(\alpha))$.

We have $\operatorname{div} \tilde{u}_n = 0$ in $\hat{\Omega}$, but $\operatorname{div} \tilde{u}_n \rightharpoonup \operatorname{div} \tilde{u}$ in $L^2(\hat{\Omega})$, so $\operatorname{div} \tilde{u} = 0$ in $\hat{\Omega}$. Since Γ_1 is fixed, we have $\tilde{u}|_{\Gamma_1} = 0$.

By Proposition 1, we have

$$A_\alpha(\tilde{u}_n, \xi) = 0 \quad \forall n \in IN, \quad \forall \xi \in \left(C^\infty(\overline{\hat{\Omega}})\right)^2$$

on $\Gamma(\alpha)$. Hence, in the limit, we have

$$\int_0^a \tilde{u}_n(x_1, \alpha(x_1))\xi(x_1, \alpha(x_1))\, dx_1 = 0 \quad \forall \xi \in \left(C^\infty(\overline{\hat{\Omega}})\right)^2,$$

which implies

$$\tilde{u}|_{\Gamma(\alpha)} = 0.$$

Step 2: $\tilde{u}|_{\Omega(\alpha)}$ is a solution to $P_2(\Omega(\alpha))$.

We need to show that the restriction to $\Omega(\alpha)$ of the weak limit $\tilde{u}$ of $\tilde{u}_n$, denoted by $\tilde{u}|_{\Omega(\alpha)} = u$, is a solution to $P_2(\Omega(\alpha))$. To achieve this goal, we introduce the following spaces:

$$\mathcal{W}_n = \Big\{\Phi \in \left(C^\infty(\bar{\Omega}_n)\right)^2 : \operatorname{div} \Phi = 0 \ \text{ in } \ \Omega_n, \Phi = 0 \ \text{ in a neighbourhood of } \ \Gamma_1 \cup \Gamma(\alpha_n)\Big\},$$

$$\mathcal{W} = \Big\{\Phi \in \left(C^\infty(\bar{\Omega}(\alpha))\right)^2 : \operatorname{div} \Phi = 0 \ \text{ in } \ \Omega, \Phi = 0 \ \text{ in a neighbourhood of } \ \Gamma_1 \cup \Gamma(\alpha)\Big\}.$$

We have (Tartar, 1974)

$$\overline{\mathcal{W}_n \cap \{v \in \mathbb{H}^1(\Omega_n) : v \cdot \tau = 0 \ \text{ on } \ \Gamma_0 \cup \Gamma_2\}}^{\mathbb{H}^1(\Omega_n)} = \mathcal{V}_1(\alpha_n)$$

and

$$\overline{\mathcal{W} \cap \{v \in \mathbb{H}^1(\Omega(\alpha)) : v \cdot \tau = 0 \text{ on } \Gamma_0 \cup \Gamma_2\}}^{\mathbb{H}^1(\Omega(\alpha))} = \mathcal{V}_1(\alpha).$$

We must have $\Phi \in \mathcal{W} \cap \{v \in \mathbb{H}^1(\Omega(\alpha)) : v \cdot \tau = 0 \text{ on } \Gamma_0 \cup \Gamma_2\} = \mathcal{W}_1$, since $\alpha_n \rightrightarrows \alpha$ in $(0, a)$, $\Phi \in \mathcal{W}_n \cap \{v \in \mathbb{H}^1(\Omega_n) : v \cdot \tau = 0 \text{ on } \Gamma_0 \cup \Gamma_2\}$ for large values of n, $n \geq n_0$, and

$$\int_{\Omega_n} \big(\nu \nabla u_n \cdot \nabla \Phi + (u_n \cdot \nabla) u_n \cdot \Phi\big)\, \mathrm{d}x = L(\Phi). \tag{9}$$

Let us examine each term separately:

$$\int_{\Omega_n} \nu \nabla u_n \cdot \nabla \Phi \, \mathrm{d}x = \int_{\hat{\Omega}} \nu \nabla \tilde{u}_n \cdot \nabla \tilde{\Phi}\, \mathrm{d}x \xrightarrow[n\to\infty]{} \int_{\hat{\Omega}} \nu \nabla \tilde{u} \cdot \nabla \tilde{\Phi}\, \mathrm{d}x = \int_{\Omega(\alpha)} \nu \nabla u \cdot \nabla \Phi \, \mathrm{d}x,$$

where $\tilde{\Phi}$ is the extension of Φ by 0 in $\hat{\Omega}$. To prove the convergence of the non-linear term, we use the fact that the embedding of $\mathbb{H}^1(\hat{\Omega})$ in $(L^2(\hat{\Omega}))^2$ and the trace mapping of $\mathbb{H}^1(\hat{\Omega})$ in $(L^2(\partial\hat{\Omega}))^2$ are compact. We can then extract a subsequence, denoted again by $\tilde{u}_n$, which satisfies

$$\tilde{u}_n \longrightarrow \tilde{u} \text{ in } \big(L^2(\hat{\Omega})\big)^2$$

and

$$\tilde{u}_n|_{\partial\hat{\Omega}} \longrightarrow \tilde{u}|_{\partial\hat{\Omega}} \text{ in } \big(L^2(\partial\hat{\Omega})\big)^2.$$

We then get

$$\begin{aligned}\int_{\Omega_n} (u_n \cdot \nabla) u_n \cdot \Phi \, \mathrm{d}x &= \int_{\hat{\Omega}} (\tilde{u}_n \cdot \nabla)\tilde{u}_n \cdot \tilde{\Phi}\, \mathrm{d}x \xrightarrow[n\to\infty]{} \int_{\hat{\Omega}} (\tilde{u} \cdot \nabla)\tilde{u} \cdot \tilde{\Phi}\, \mathrm{d}x \\ &= \int_{\Omega(\alpha)} (u \cdot \nabla) u \cdot \Phi \, \mathrm{d}x,\end{aligned}$$

and therefore u satisfies

$$\int_{\Omega(\alpha)} \big(\nu \nabla u \cdot \nabla \Phi + (u \cdot \nabla) u \cdot \Phi\big)\, \mathrm{d}x = L(\Phi) \quad \forall \Phi \in \mathcal{W}_1. \tag{10}$$

Since $\mathcal{W}_1$ is a dense subspace of $\mathcal{V}_1(\alpha)$, we can conclude that u satisfies

$$\int_{\Omega(\alpha)} \big(\nu \nabla u \cdot \nabla v + (u \cdot \nabla) u \cdot v\big)\, \mathrm{d}x = L(v) \text{ for all } v \in \mathcal{V}_1(\alpha). \tag{11}$$

This completes the proof. ■

Theorem 2. *Problem $\mathcal{P}_{\text{opt}}$ has at least one solution.*

Indeed, we have constructed a sequence $\{\Omega_n\}$ which converges to Ω and a sequence $\{u_n\}$ which converges to the solution to the state problem in the domain Ω, which proves the continuity of the state with respect to the domain. Moreover, the sequence $\{\tilde{u}_n|_{\partial\hat{\Omega}(\alpha)}\}_{n\geq 0}$ converges in $(L^2(\partial\hat{\Omega}))^2$ towards $\tilde{u}|_{\partial\hat{\Omega}(\alpha)}$, which implies the continuity of the cost function $J(\alpha) = \int_{\Gamma_0} u \cdot n \, \mathrm{d}\sigma$. Since the set $\mathcal{U}_{\mathrm{ad}}$ is compact and the function $J(\alpha)$ is continuous on $\mathcal{U}_{\mathrm{ad}}$, we deduce that an optimal domain exists.

Remark 2. In the case where the sequence u_n is a solution to Stokes equations, the inequality (8) is checked through the coercivity of the bilinear form $a(\cdot,\cdot)$ and the continuity of the linear form L. Recall that $a(\cdot,\cdot)$ and L are respectively defined by (1) and (4).

4. Parametrization of the Domain and Calculation of the Shape Gradient

To compute the gradient of $\mathcal{J}(\Omega)$ with respect to Ω, we first parametrize the domain Ω as follows: Given an open Ω with a generic point X, we transform Ω into Ω_t through the function T_t defined by

$$x_t = T_t(X), \quad x_t \in \Omega_t, \quad t \geq 0,$$

where x_t is a solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x_t = V(t, x_t), \quad x_0 = X.$$

Here $V(t, x_t)$, being a regular vector field, represents the speed of displacement of points of Ω. For t close to zero, we get

$$x_t = X + t \frac{\mathrm{d}}{\mathrm{d}t} T_t(X)\Big|_{t=0} + \cdots$$

and

$$V(x) = \frac{\mathrm{d}}{\mathrm{d}t} T_t(X)\Big|_{t=0}.$$

Hence

$$x_t \simeq X + t\,V(X).$$

Notice that the transformations $T_t(X)$ and $X + t\,V(X)$ are tangent at the origin. It is well-known (Zolésio, 1981) that two tangent transformations generate the same results of derivation at the origin. We assume that under the effect of the deformation field V, we can transform Ω into Ω_t, Γ into Γ_t and $H^1(\Omega)$ into $H^1(\Omega_t)$ for all $t \geq 0$. The functional spaces on Ω_t are defined by

$$\mathcal{V}_t = \{u \circ T_t^{-1} = u_t : u \in \mathcal{V}\}$$

and

$$Q_t = \{q \circ T_t^{-1} = q_t : q \in Q\}.$$

The cost functional $\mathcal{J}$ has the form

$$\mathcal{J}(\Omega_t) = \int_{\Gamma_0} u_t \cdot n \, \mathrm{d}\sigma. \tag{12}$$

In order to minimize $\mathcal{J}$ with respect to Ω, we need to compute the Eulerian semi-derivative at Ω in the direction of the velocity field V associated with the deformation T_t.

The variational formulation associated with the Navier-Stokes problem on Ω_t is given by

$$P(\Omega_t) : \begin{cases} \text{Find } (u_t, p_t) \in \mathcal{V}_t \times Q_t \text{ such that} \\ \nu \int_{\Omega_t} \nabla u_t \cdot \nabla v_t \, \mathrm{d}x_t + \int_{\Omega_t} (u_t \cdot \nabla) u_t \cdot v_t \, \mathrm{d}x_t - \int_{\Omega_t} p_t \operatorname{div} v_t \, \mathrm{d}x_t \\ \qquad = \int_{\Gamma_2} g \, v_t \cdot n \, \mathrm{d}\sigma \quad \forall v_t \in \mathcal{V}_t, \\ \int_{\Omega_t} q_t \operatorname{div} u_t \, \mathrm{d}x_t = 0 \quad \forall q_t \in Q_t. \end{cases}$$

We change the variables to get back to integrals on Ω

$$P(\Omega_t) : \begin{cases} \nu \int_\Omega {}^t\mathrm{D}T_t^{-1} \nabla(u^t) \cdot {}^t\mathrm{D}T_t^{-1} \nabla v \, J_t \, \mathrm{d}x + \int_\Omega (u^t \cdot {}^t\mathrm{D}T_t^{-1} \nabla) u^t \cdot v \, J_t \, \mathrm{d}x \\ \qquad - \int_\Omega p_t \circ T_t \, \nabla v \cdot \mathrm{D}T_t^{-1} \, J_t \, \mathrm{d}x - \int_\Omega p_t \circ T_t \, \nabla v \cdot \mathrm{D}T_t^{-1} \, J_t \, \mathrm{d}x \\ \qquad = \int_{\Gamma_2} g \, v \cdot n \, \mathrm{d}\sigma \quad \forall v \in \mathcal{V}, \\ \int_\Omega q \, \nabla(u^t) \cdot \mathrm{D}T^{-1} \, J_t \, \mathrm{d}x = 0 \quad \forall q \in Q, \end{cases}$$

$$\nu \int_\Omega {}^t\mathrm{D}T_t^{-1} \nabla(u^t) \cdot {}^t\mathrm{D}T_t^{-1} \nabla(v) \, J_t \, \mathrm{d}x = \nu \int_\Omega A(t) \nabla(u^t) \cdot \nabla v \, \mathrm{d}x$$

with $u^t = u_t \circ T_t$, $v = v_t \circ T_t$, $J_t = \det(\mathrm{D}T_t)$ and $A(t) = J_t \mathrm{D}T_t^{-1} \, {}^t\mathrm{D}T_t$.

Differentiating $P(\Omega_t)$ with respect to t for $t = 0$, we obtain the variational equation satisfied by $(\dot{u}, \dot{p})$, where $\dot{f} = \mathrm{d}(f_t \circ T_t)/\mathrm{d}t|_{t=0}$,

$$\begin{cases} \nu \int_\Omega A'(0) \nabla u \cdot \nabla v \, \mathrm{d}x + \nu \int_\Omega \nabla \dot{u} \cdot \nabla v \, \mathrm{d}x \\ \qquad + \int_\Omega (\dot{u} \cdot \nabla) u \cdot v \, \mathrm{d}x - \int_\Omega (u \cdot {}^t\mathrm{D}V(0) \nabla) u \cdot v \, \mathrm{d}x + \int_\Omega (u \cdot \nabla) \dot{u} \cdot v \, \mathrm{d}x \\ \qquad + \int_\Omega (u \cdot \nabla) u \cdot v \operatorname{div} V(0) \, \mathrm{d}x - \int_\Omega \dot{p} \operatorname{div} v \, \mathrm{d}x + \int_\Omega p \nabla v \cdot \mathrm{D}V(0) \, \mathrm{d}x_t \\ \qquad - \int_\Omega p \operatorname{div} v \operatorname{div} V(0) \, \mathrm{d}x = 0 \quad \forall v \in \mathcal{V}, \\ \int_\Omega q \operatorname{div} \dot{u} \, \mathrm{d}x - \int_\Omega q \, \nabla u \cdot \mathrm{D}V(0) \, \mathrm{d}x + \int_\Omega q \operatorname{div} u \operatorname{div} V(0) \, \mathrm{d}x = 0 \quad \forall q \in Q. \end{cases} \tag{13}$$

with $A'(0) = \operatorname{div} V(0) I - (\mathrm{D}V(0) + {}^t\mathrm{D}V(0))$.

Variational formulation of the adjoint state in Ω. The variational form $P_2(\Omega)$ of the state partial differential equation is considered as a constraint in the minimization Problem $\mathcal{P}_{\rm opt}$. We construct a Lagrange functional by introducing a multiplier function (u^*, p^*):

$$\begin{aligned}\mathcal{L}(\Omega, u, p, u^*, p^*) &= \mathcal{J}(\Omega) + \int_\Omega \nu \nabla u \cdot \nabla u^* \,\mathrm{d}x + \int_\Omega (u\cdot\nabla)u \cdot u^* \,\mathrm{d}x \\ &\quad + \int_\Omega p \operatorname{div} u^* \,\mathrm{d}x - \int_{\Gamma_2} g u^* \cdot n \,\mathrm{d}\sigma + \int_\Omega p^* \operatorname{div} u \,\mathrm{d}x.\end{aligned}$$

We use it in a classical fashion to write explicitly the variational form of the adjoint state solution (u^*, p^*) to the problem

$$\lim_{\theta\to 0} \frac{\partial \mathcal{L}}{\partial \theta}(\Omega, u+\theta\varphi, p+\theta\psi, u^*, p^*) = 0 \quad \forall \varphi \in \mathcal{V},\ \forall \psi \in Q.$$

Then we obtain the adjoint problem

$$P^*(\Omega): \begin{cases} \text{Find } (u^*, p^*) \in \mathcal{V}\times Q \text{ such that} \\ \nu \int_\Omega \nabla\varphi \cdot \nabla u^* \,\mathrm{d}x + \int_\Omega (\varphi\cdot\nabla)u\cdot u^* \,\mathrm{d}x + \int_\Omega (u\cdot\nabla)\varphi \cdot u^* \,\mathrm{d}x \\ \qquad - \int_\Omega p^* \operatorname{div}\varphi \,\mathrm{d}x = -\int_{\Gamma_0} \varphi\cdot n \,\mathrm{d}\sigma \quad \forall \varphi\in\mathcal{V}, \\ \int_\Omega \psi \operatorname{div} u^* \,\mathrm{d}x = 0 \quad \forall \psi \in Q. \end{cases} \tag{14}$$

In particular, for $\varphi = \dot{u}$

$$\begin{aligned}-\int_{\Gamma_0} \dot{u}\cdot n \,\mathrm{d}\sigma &= \nu \int_\Omega \nabla\dot{u}\cdot\nabla u^* \,\mathrm{d}x + \int_\Omega (\dot{u}\cdot\nabla)u\cdot u^* \,\mathrm{d}x \\ &\quad + \int_\Omega (u\cdot\nabla)\dot{u}\cdot u^* \,\mathrm{d}x - \int_\Omega p^* \operatorname{div}\dot{u} \,\mathrm{d}x\end{aligned} \tag{15}$$

and, setting $v = u^*$ in (13), we get the expression for the derivative of J in the direction of V:

$$\begin{aligned}\mathrm{D}\mathcal{J}(\Omega, V) &= \nu \int_\Omega A'(0)\nabla u\cdot\nabla u^* \,\mathrm{d}x \\ &\quad - \int_\Omega (u\cdot {}^t\mathrm{D}V(0)\nabla)u\cdot u^* \,\mathrm{d}x + \int_\Omega (u\cdot\nabla)u\cdot u^* \operatorname{div} V(0) \,\mathrm{d}x \\ &\quad + \int_\Omega p\,(\nabla u^* \cdot \mathrm{D}V(0)) \,\mathrm{d}x + \int_\Omega p^* \big(\nabla u\cdot \mathrm{D}V(0)\big) \,\mathrm{d}x\end{aligned} \tag{16}$$

Calculation of the deformation field. In the last section, we have seen that the introduction of the fictitious time t leads to the mapping

$$\Omega_t \to (u_t, p_t) \to j(t) \to \mathcal{J}(\Omega_t, u_t, p_t).$$

We have also computed the directional derivative with respect to the deformation created by a field V which belongs to an appropriate space.

Using Hadamard's formula (Hadamard, 1969; Zolésio, 1981), we see that there exists a scalar distribution G on the surface such that

$$\mathrm{D}\mathcal{J}(\Omega, V) = \int_\Gamma G\, v \cdot n \,\mathrm{d}\sigma. \tag{17}$$

Note that our goal is to minimize the cost functional $J(\Omega_t)$ using some descent method. In order to do that in practice, we solve the following variational problem:

$$P_3(\Omega): \begin{cases} \text{Determine } (W, \Psi) \in \mathcal{H} \times Q \text{ such that} \\ \nu \displaystyle\int_\Omega \nabla W \cdot \nabla v \,\mathrm{d}x - \int_\Omega \Psi \operatorname{div} v \,\mathrm{d}x = -\int_{\Gamma_3} G\, v \cdot n \,\mathrm{d}\sigma \quad \forall v \in \mathcal{H}, \\ \displaystyle\int_\Omega q \operatorname{div} W \,\mathrm{d}x = 0 \quad \forall q \in Q, \end{cases} \tag{18}$$

where

$$\mathcal{H} = \left\{ v \in \mathbb{H}^1(\Omega) : v = 0 \text{ on } \Gamma \backslash \Gamma_3,\ v \cdot \tau = 0 \text{ on } \Gamma_3 \right\},$$

Γ_3 being the part of the boundary to be deformed.

Problem $P_3(\Omega)$ permits us to compute the descent direction in order to approximate Ω_t by the following Euler scheme:

$$x_t = x_0 + tW.$$

Here W is the solution to the elliptic problem

$$\begin{cases} \text{Find } W \in \mathcal{H} \cap \{v \in \mathbb{H}^1(\Omega) : \operatorname{div} v = 0\} \text{ such that} \\ \Delta W = 0 \text{ in } \Omega, \\ \dfrac{\partial W}{\partial n} = G \text{ on } \Gamma_3, \\ W = 0 \text{ on } \Gamma \backslash \Gamma_3. \end{cases}$$

W constitutes a lifting up of G in $\mathcal{H} \cap \{v \in \mathbb{H}^1(\Omega) : \operatorname{div} v = 0\}$.

5. Finite Element Approximation

We recall that our abstract optimal shape design problem is stated as follows:

$$\mathcal{P}_{\mathrm{opt}}: \begin{cases} \text{Find } \alpha^* \in \mathcal{U}_{\mathrm{ad}} \text{ such that} \\ J(\alpha^*) \le J(\alpha) \quad \forall \alpha \in \mathcal{U}_{\mathrm{ad}} \end{cases}$$

where

$$J(\alpha) = \int_{\Gamma_0} u(\alpha) \cdot n \,\mathrm{d}\sigma,$$

$\mathcal{U}_{\mathrm{ad}}$ is the set of admissible functions and $(u(\alpha), p(\alpha))$ denotes the solution to the variational problem

$$P_1(\Omega(\alpha)) : \begin{cases} a(u(\alpha), v) + c(u(\alpha), u(\alpha), v) - b(p(\alpha), v) = L(v) & \forall v \in \mathcal{V}, \\ b(q, u(\alpha)) = 0 \ \ \forall q \in Q, \end{cases}$$

where

$$\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < a, \ \ 0 < x_2 < \alpha(x_1)\}$$

and

$$\mathcal{U}_{\mathrm{ad}} = \Big\{\alpha \in C^{0.1}(0, a) \ : |\alpha(x) - \alpha(y)| \le k\,|x - y|\,, \\ 0 < b \le \alpha(x) \le c, \ \alpha(0) = \alpha(a) = d\Big\}.$$

The parts Γ_0, Γ_1 and Γ_2 of the boundary of $\Omega(\alpha)$ being fixed, we wish to minimize the cost functional with respect to the shape of Γ_3. We discretize the problem by the mixed finite element method using the $Q_2^{(9)}$–P_1 (biquadratic velocity, piecewise linear pressure) element on a quadrangular mesh. $Q_2^{(9)}$ is used to approximate each component of the velocity field u. It is a classical Lagrangian bilinear element with nine nodes. The pressure p is P_1 with three degrees of freedom, i.e. the value of p and its first derivatives at the barycenter. In practical calculations, the unknowns corresponding to the values of the velocity at the barycentre and those corresponding to the derivatives of the pressure are eliminated at the local level. The element $Q_2^{(9)}$–P_1 verifies the Babuška-Brezzi condition, so its order of convergence is $o(h^2)$. For more details concerning this element, see (Fortin and Fortin, 1984; 1985, p.912).

Our main assumptions are as follows:

- The nodes on $\bar{\Omega}_h \backslash \Gamma_{3h}$ are fixed;
- $\Gamma_{3h} = \{(x_1, x_2) : x_2 = \alpha_h(x_1), \ x_1 \in (0, a)$ and α_h is a piecewise linear function$\}$;
- $\mathcal{T}_h$ is regular with respect to h such that:
  - for any fixed $h > 0$, $\mathcal{T}_h(\alpha_h)$ depends continuously on $\alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$,
  - for any fixed $h > 0$, $\mathcal{T}_h(\alpha_h)$'s are topologically equivalent to $\alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$, i.e. the number of nodes from $\mathcal{T}_h(\alpha_h)$ is the same for all $\alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$ and the nodes still have the same neighbours;

– the family $\{\mathcal{T}_h(\alpha_h)\}$, $h \to 0^+$, is uniformly regular with respect to $h > 0$, $\alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$, i.e. a constant $\theta_0 > 0$ exists such that $\theta_h(\alpha_h) \geq \theta_0 \quad \forall h > 0$, $\forall \alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$, where $\theta_h(\alpha_h)$ is the minimal interior angle of all triangles from $\mathcal{T}_h(\alpha_h)$.

Here

$$\mathcal{U}_{\mathrm{ad}}^h = \left\{ \begin{array}{l} \alpha_h \in C^0(0,a) : \alpha_h \text{ is piecewise linear such that} \\ b \leq \alpha_h \leq c, \;\; \alpha_h(0) = \alpha_h(a) = d \;\text{ and }\; \left| \dfrac{\alpha_h(a_i) - \alpha_h(a_{i-1})}{a_i - a_{i-1}} \right| \leq k \end{array} \right\},$$

and (a_i) are the nodes of $\mathcal{T}_h$ on $(0,a)$.

In fact, those hypotheses are required to prove the continuity of finite element solutions with respect to discrete shapes.

The discrete problem is formulated as follows:

$$P(\alpha_h)_h : \left\{ \begin{array}{l} \text{Find } (u_h, p_h) \in \mathcal{V}_h \times Q_h \text{ such that} \\ a(u_h, v_h) + c(u_h, u_h, v_h) - b_h(v_h, p_h) = L(v_h) \quad \forall v_h \in \mathcal{V}_h(\alpha_h), \\ b_h(u_h, q_h) = 0 \quad \forall q_h \in Q_h, \end{array} \right.$$

where

$$\mathcal{V}_h = \left\{ \begin{array}{l} v_h \in C^0\big(\Omega_h(\alpha_h)\big)^2 : v_h|_K \in Q_2(K)^2 \;\forall K \in \mathcal{T}_h(\alpha_h) \text{ such that} \\ v_h = 0 \quad \text{on } \Gamma_{1h} \cup \Gamma_{3h} \\ v_h \cdot \tau = 0 \quad \text{on } \Gamma_{0h} \cup \Gamma_{2h} \end{array} \right\},$$

$$Q_h = \big\{ q_h : q_h|_K \in P_1 \;\forall K \in \mathcal{T}_h(\alpha_h) \big\}$$

and

$$b_h(v_h, q_h) = \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div} v_h \, q_h \, dx \quad \forall (v_h, q_h) \in \mathcal{V}_h \times Q_h.$$

For fixed h, let us prove that $P(\alpha_h)_h$ has a solution u_h which depends continuously on α_h.

Theorem 3. *Under the hypothesis of the existence theorem in the continuous case, there exists a unique solution $u_h(\alpha_h)$ to $P(\alpha_h)_h$ in the ball M. Moreover, $u_h(\alpha_h)$ depends continuously on α_h.*

Proof. Let us first prove that, when $u_h(\alpha_h)$ exists, it depends continuously on α_h. Let $u_{0h} \in \mathcal{V}_{1h}(\alpha_h)$ and K_h be defined by

$$K_h : \left\{ \begin{array}{ccc} \mathcal{V}_{1h} & \longrightarrow & \mathcal{V}_{1h} \\ u_{0h} & \longmapsto & u_h(\alpha_h) \end{array} \right.,$$

where

$$\mathcal{V}_{1h} = \{v_h \in \mathcal{V}_h : b_h(v_h, q_h) = 0 \quad \forall q_h \in Q_h\},$$

$u_h(\alpha_h)$ is a solution to the problem

$$P_{u_{0h}}(\alpha_h)_h : \quad a(u_h(\alpha_h), v_h) = L(v_h) - c(u_{0h}, u_{0h}, v_h) \quad \forall v_h \in \mathcal{V}_{1h}(\alpha_h).$$

We identify an element of $\mathcal{V}_{1h}$ with an element of $\mathbb{R}^{2N}$, where N is the number of nodes, i.e. $u_h(\alpha_h) \sim x(\alpha_h)$ where $x(\alpha_h)_i$ is the value of $u_h(\alpha_h)$ at node N_i.

We have to solve the linear problem

$$A(\alpha_h)x(\alpha_h) = L(\alpha_h)$$

which has a solution owing to the ellipticity of $A(\alpha_h)$. The existence and uniqueness of the solution to the non-linear problem P_h are proved in the same way as in the continuous case: we check that K_h is a contraction operator in M. Moreover, $u_h(\alpha_h)$ depends continuously on α_h due to the hypothesis verified by $\mathcal{T}_h(\alpha_h)$.

For the existence of p_h, it is sufficient to remark that the inf-sup condition is satisfied for the $Q_2^{(9)}$-approximation of the velocity and P_1-approximation of the pressure:

$$\inf_{q \in Q_h} \sup_{\mathcal{V}_h} \frac{b_h(v_h, q_h)}{\|v_h\|_1 \, \|q_h\|_0} \geq k,$$

where k is independent of h. This condition is sufficient to prove the existence of $p_h(\alpha_h)$ and to prove the convergence of (u_h, p_h) to (u, p). For more details, see (Brezzi and Fortin, 1991; Girault and Raviart, 1979; Fortin, 1977). ■

The discrete cost functional is

$$J_h(\alpha_h) = \int_{\Gamma_{0h}} u_h(\alpha_h) \cdot n \, \mathrm{d}\sigma.$$

Finally, the discrete optimization problem is

$$\mathcal{P}^h_{\mathrm{opt}} : \begin{cases} \text{Find } \alpha_h^* \in \mathcal{U}^h_{\mathrm{ad}} \text{ such that} \\ J_h(\alpha_h^*) \leq J_h(\alpha_h) \quad \forall \alpha_h \in \mathcal{U}^h_{\mathrm{ad}}. \end{cases}$$

Let $\varphi_{t,i}$ be the basis function in $Q_2(K)$ associated with the node $N_{t,i} = \mathcal{F}_t(N_i)$, where $\mathcal{F}_t$ is a perturbation of the identity, i.e.

$$\mathcal{F}_t(x_1, x_2) = (x_1, x_2) + t\, W_h(x_1, x_2),$$

W_h being the approximation of the deformation vector field W. Since

$$\varphi_{t,i}(x_t) = \varphi_{t,i}(\mathcal{F}_t(x)) = \varphi_i(x) \quad \forall x \in \Omega_h,$$

$L(\alpha_h)$ and $A(\alpha_h)$ depend continuously on α_h.

Note that $x(\alpha_h)$ is in a compact set of $\mathbb{R}^{2N}$ and $J_h(\alpha_h) = \int_{\Gamma_{0h}} u_h(\alpha_h) \cdot n \, d\sigma_h = \int_{\Gamma_{0h}} \sum_i x(\alpha_h)_i \varphi_i \cdot n \, d\sigma_h$ is continuous with respect to α_h, which proves the existence of a solution to Problem $\mathcal{P}^h_{\text{opt}}$.

We will study the relation between $\mathcal{P}_{\text{opt}}$ and $\mathcal{P}^h_{\text{opt}}$ as $h \to 0^+$. To say that $\mathcal{P}^h_{\text{opt}}$ is an approximation of $\mathcal{P}_{\text{opt}}$, we must prove that the solutions to $\mathcal{P}^h_{\text{opt}}$ (at least some of them) are close, in some sense, to those of $\mathcal{P}_{\text{opt}}$. Let us introduce the following assumptions:

A(i) For any $\alpha \in \mathcal{U}_{\text{ad}}$ and $\Omega = \Omega(\alpha)$, there exists $\alpha_h \in \mathcal{U}^h_{\text{ad}}$ and $\Omega_h = \Omega_h(\alpha_h)$ such that

$$\text{`}\Omega_h \longrightarrow \Omega\text{'} \quad \text{as} \quad h \longrightarrow 0^+.$$

A(ii) For every sequence $\{\Omega_h, u_h(\Omega_h)\}$ where $\Omega_h = \Omega_h(\alpha_h)$ and $\alpha_h \in \mathcal{U}^h_{\text{ad}}$, there exits a subsequence $\{\Omega_{hj}, u_{hj}(\Omega_{hj})\}$ and $\{\Omega, u(\Omega)\}$ such that

$$\text{`}\Omega_{hj} \longrightarrow \Omega\text{'}, \quad \text{`}u_{hj}(\Omega_{hj}) \longrightarrow u(\Omega)\text{'} \quad \text{as} \quad j \longrightarrow \infty.$$

A(iii) If $\alpha_h \in \mathcal{U}^h_{\text{ad}}$ and $\alpha \in \mathcal{U}_{\text{ad}}$ with '$\Omega_h \to \Omega$' as $h \to 0^+$ and if $u_h(\Omega_h) \in \mathcal{V}_{1h}(\Omega_h)$, $u(\Omega) \in \mathcal{V}_1(\Omega)$ with '$u_h(\Omega_h) \to u(\Omega)$' as $h \to 0^+$, then

$$\lim_{h \to 0+} J_h(\alpha_h) = J(\alpha).$$

Theorem 4. *Let Assumptions A(i)–A(iii) be satisfied,* $\alpha^*_h \in \mathcal{U}^h_{\text{ad}}$ *be a solution to* $\mathcal{P}^h_{\text{opt}}$ *and* $u_h(\alpha^*_h)$ *be a solution to* $P(\alpha^*_h)_h$. *Then there exists a subsequence* $\{\alpha^*_{hj}\}$, $\{u_{hj}(\alpha^*_{hj})\}$ *and elements* $\alpha^* \in \mathcal{U}_{\text{ad}}$, $u \in \mathcal{V}_1(\alpha^*)$ *such that*

$$\alpha^*_{hj} \underset{j \to \infty}{\rightrightarrows} \alpha^* \quad \textit{in} \quad (0, a),$$

$$\text{`}u_{hj}(\alpha_{hj}) \underset{j \to \infty}{\longrightarrow} u\text{'}.$$

Moreover, α^* *is a solution to* $\mathcal{P}_{\text{opt}}$ *and* $u = u(\alpha^*)$ *solves* $P_2(\alpha^*)$.

For the proof, see (Haslinger and Neittaanmäki, 1988, Th. 2.3). In the following, we check that A(i)–A(iii) are satisfied:

- From Bégis and Glowinski (1975) it follows that for any $\alpha \in \mathcal{U}_{\text{ad}}$, there exists $\{\alpha_h\}_h \in \mathcal{U}^h_{\text{ad}}$ such that $\alpha_h \rightrightarrows \alpha$ in $(0, a)$ as $h \to 0^+$, i.e. $A(i)$ is satisfied.

- Suppose now that $\alpha_h \rightrightarrows \alpha$ in $(0, a)$ as $h \to 0^+$. Then a sequence of solutions $\{u_h(\alpha_h)\}_h$ to the problems $P(\alpha_h)_h$ exists in M. We can then extract a subsequence of $\{\tilde{u}_h(\alpha_h)\}_h$ ($\tilde{u}_h$ is the extension of u_h to $\hat{\Omega}$ by 0), again denoted by $\{\tilde{u}_h(\alpha_h)\}_h$, that weakly converges towards $\tilde{u}$ in $\mathbb{H}^1(\hat{\Omega})$. It is proved in the continuous case that $\tilde{u}\,|_{\Omega(\alpha)} \in \mathcal{V}_1(\alpha)$. Let us prove that $\tilde{u}\,|_{\Omega(\alpha)}$ solves $P(\alpha)$. For all v_h in $\mathcal{V}_{1h}(\alpha_h)$, we have

$$\nu \int_{\Omega(\alpha_h)_h} \nabla u_h(\alpha_h) \cdot \nabla v_h \, dx + \int_{\Omega_h} (u_h(\alpha_h) \cdot \nabla) u_h(\alpha_h) v_h \, dx = L(v_h),$$

where $v_h = \Pi_h v$, $v \in \mathcal{V}_1(\alpha)$. Π_h is the linear interpolation operator introduced in (Fortin, 1977, p. 346). The main property of the operator Π_h is that it is a uniformly continuous operator from $\mathcal{V}$ into $\mathcal{V}_h$, and it verifies

$$b(v - v_h, q_h) = 0 \quad \forall q_h \in Q_h.$$

For the case of the element $Q_2^{(9)}\text{–}P_1$ used for our discretisation, this operator can be constructed based on the Babuška-Brezzi condition (Fortin, 1977, p. 346). Using the fact that $\Pi_h v \to v$ in $\mathbb{H}^1(\hat{\Omega})$ strongly and passing to the limit in each term, we obtain

$$\nu \int_{\Omega(\alpha)} \nabla u(\alpha) \cdot \nabla v \, dx + \int_{\Omega} (u(\alpha) \cdot \nabla) u(\alpha) v \, dx = L(v) \quad \forall v \in \mathcal{V}_1(\alpha).$$

- We have '$\Omega_h \xrightarrow[h \to 0^+]{} \Omega$' and the trace mapping from $\mathbb{H}^1(\hat{\Omega})$ to $(L^2(\partial\hat{\Omega}))^2$ being compact, which implies

$$J_h(\alpha_h) \xrightarrow[h \to 0^+]{} J(\alpha).$$

We can thus conclude that $\mathcal{P}^h_{\text{opt}}$ is an approximation of $\mathcal{P}_{\text{opt}}$ for a small h.

6. Numerical Experiments

Algorithm

Given Ω_0 and ρ

for $n = 0$

Step 1: Solve the state problem $P(\Omega_n)$, compute $J(\Omega_n)$ and check whether $J(\Omega_n) < J(\Omega_{n-1})$.

Step 2: Solve the adjoint state problem (14).

Step 3: Compute the deformation field W_n by solving (18).

Step 4: If $\|W_n\| \le \varepsilon$, then $\Omega_n = \Omega_{\text{opt}}$. Stop. Otherwise, set $\Omega_{n+1} = \Omega_n + \rho W_n$ and go to Step 1.

In the following, we present numerical results obtained by using the above algorithm. Figures 3–6 represent, respectively, the velocity field and the pressure lines for the state and the adjoint state. We show in Figs. 7–10 the optimal shape obtained and the behaviour of the cost function for $\nu = 1,\ 0.3$ and $g = 1$. For the case of $\nu = 1$ and $g = 2,\ 6$, the results are shown in Figs. 11–14.

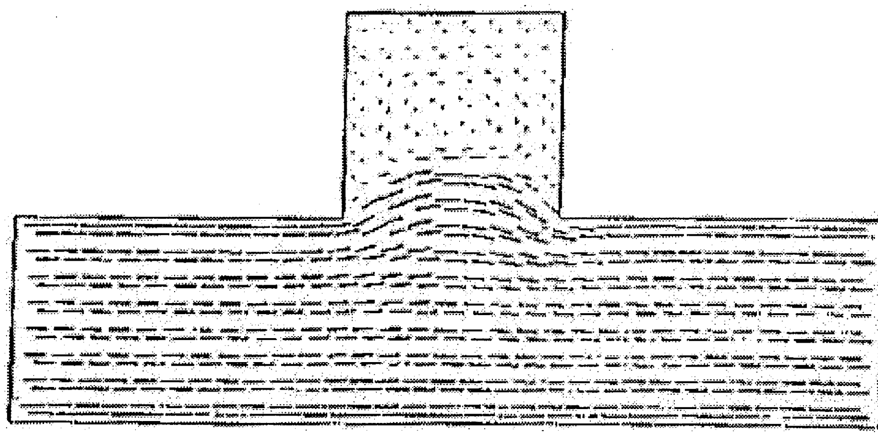

Fig. 3. Velocity field for the state (viscosity = 1).

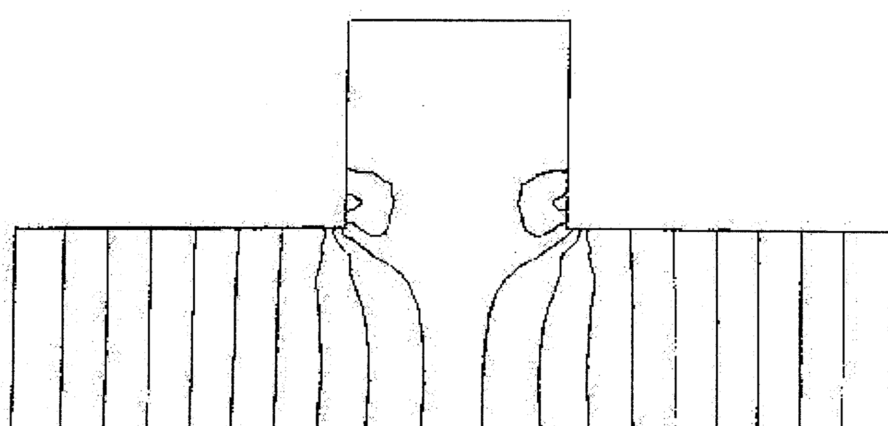

Fig. 4. Pressure lines for the state (viscosity = 1).

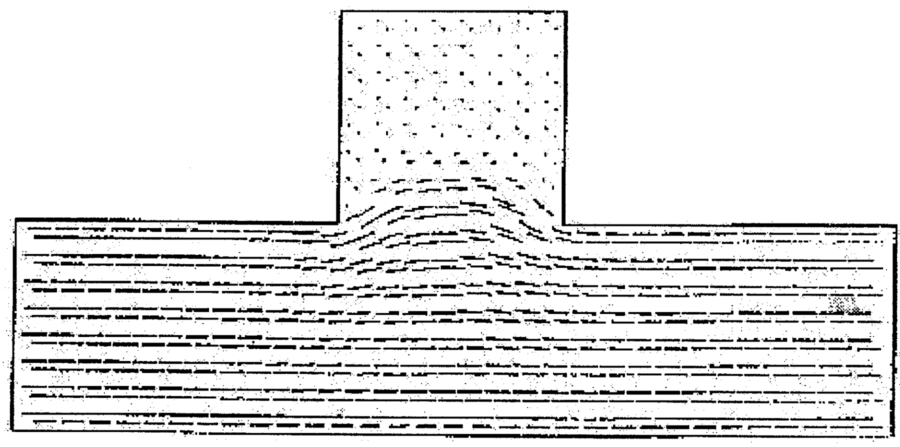

Fig. 5. Velocity field for the adjoint state (viscosity = 1).

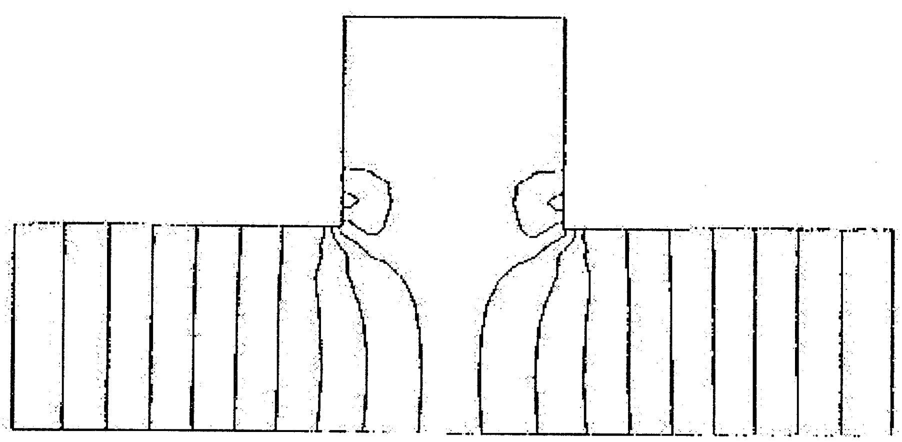

Fig. 6. Pressure lines for the adjoint state (viscosity = 1).

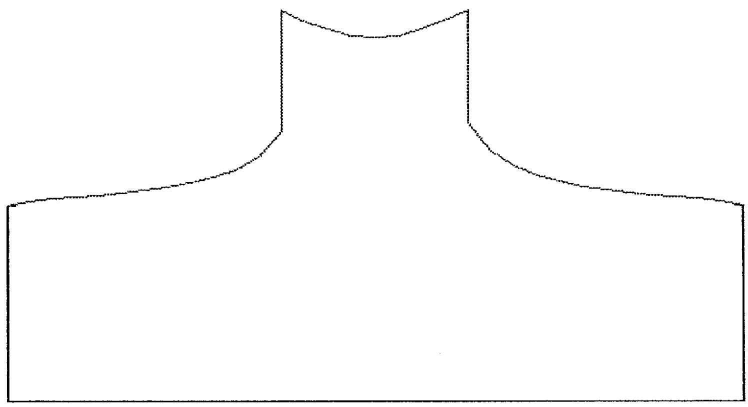

Fig. 7. Optimal shape ($v = 1, \; g = 1$).

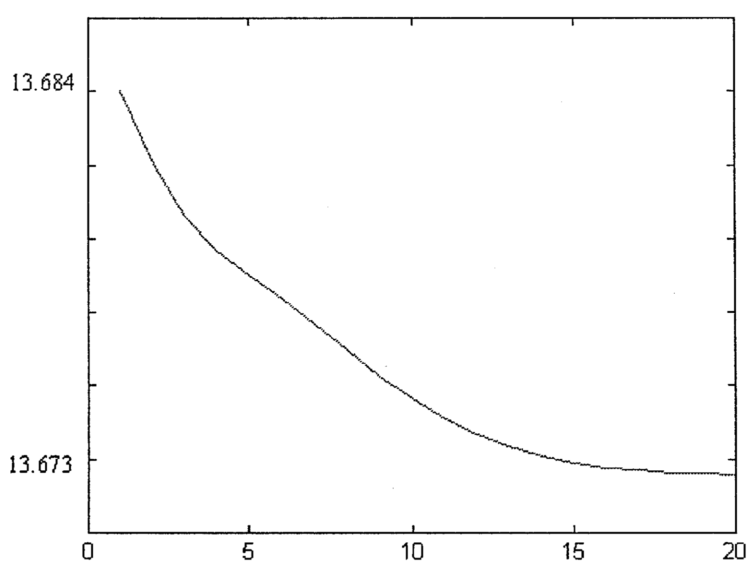

Fig. 8. Flow leakage losses at the input with respect to the number of iterations.

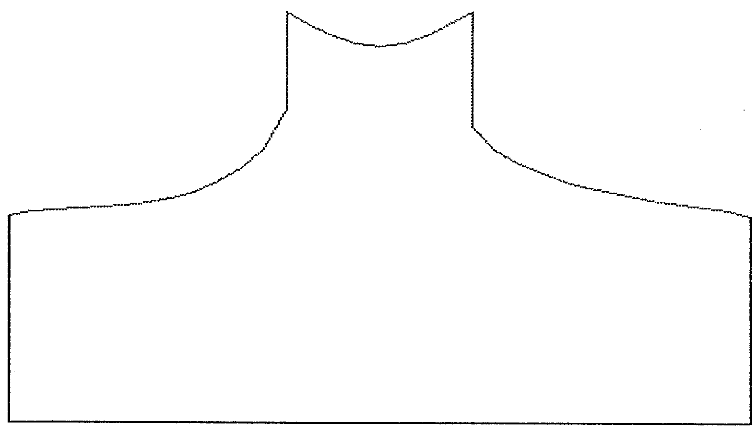

Fig. 9. Optimal shape ($v = 0.3,\quad g = 1$).

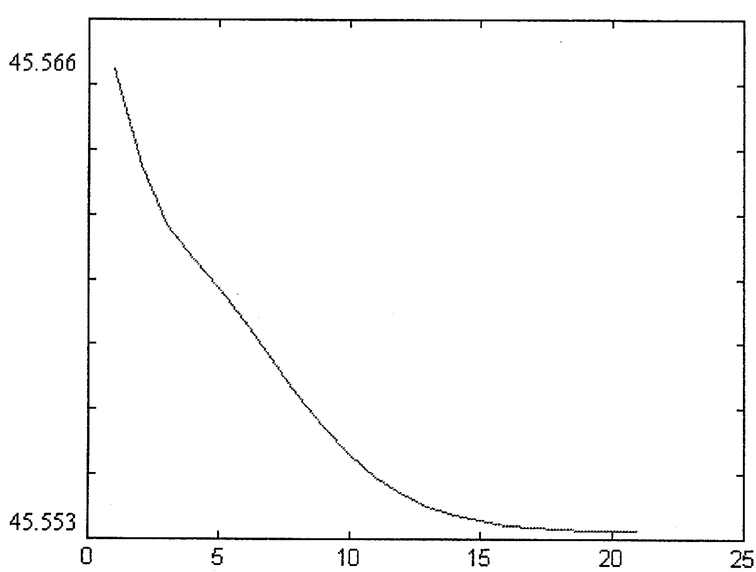

Fig. 10. Flow leakage losses at the input with respect to the number of iterations.

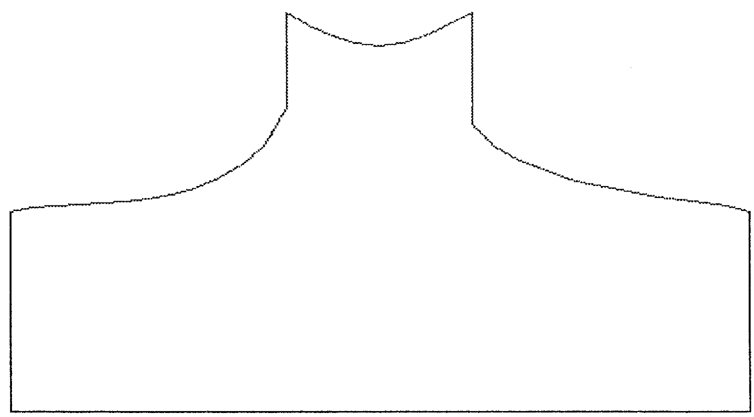

Fig. 11. Optimal shape ($v = 1, \quad g = 2$).

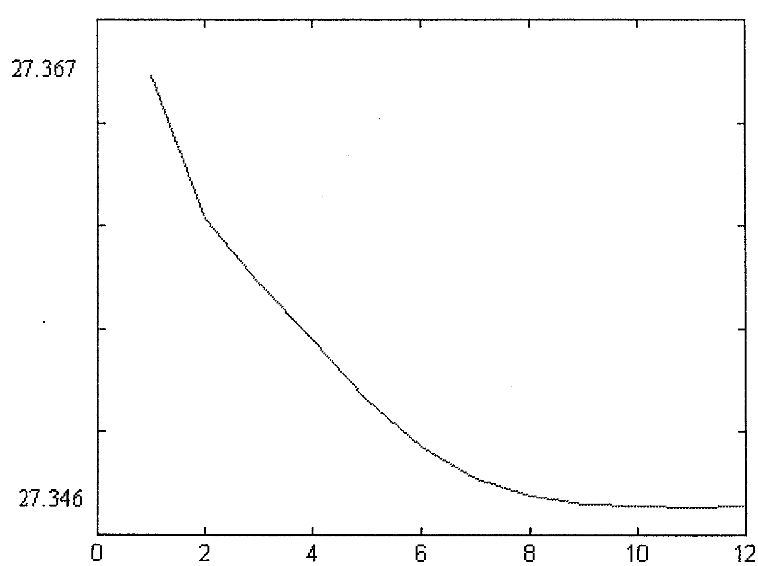

Fig. 12. Flow leakage losses at the input with respect to the number of iterations.

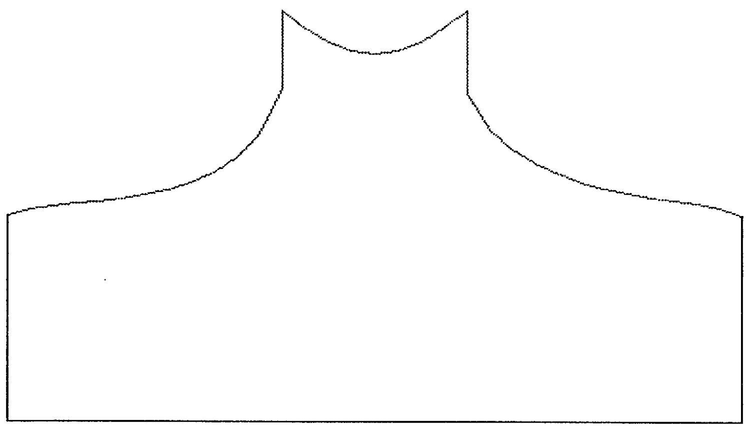

Fig. 13. Optimal shape ($v = 1, \ g = 6$).

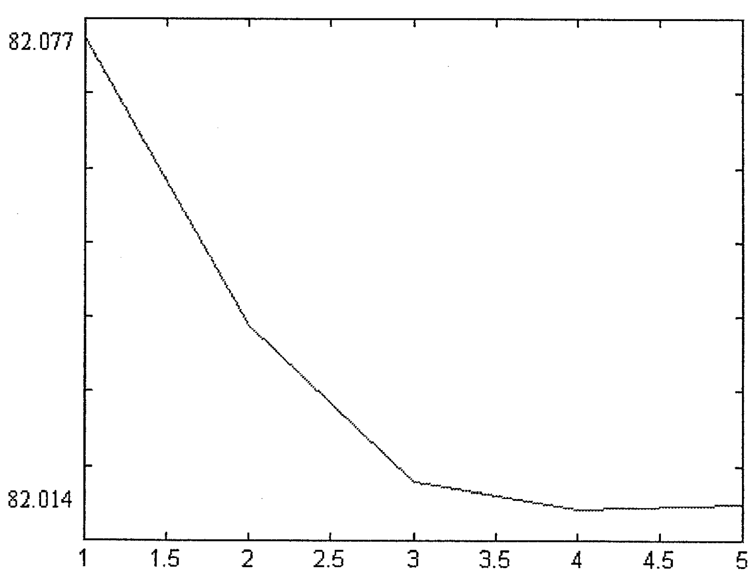

Fig. 14. Flow leakage losses at the input with respect to the number of iterations.

7. Conclusion

Numerical tests have been carried out for different values of the viscosity ν and g which corresponds to the pressure gap between the entry and the exit of the seal.

The optimal shape constitutes an opening (of the turbine side) with an angle increasing with $1/\nu$ and g which tends to trap the water in the cavity. The deformation obtained has the particularity of creating a barrage effect which is much important than the one of the square cavity used as the initial domain. Event though the cost values corresponding to each case are different, the relative gain remains virtually the same.

References

Azelmat K. (1997): *Optimisation de forme d'un labyrinthe à créneaux.* – Thèse de troisième cycle, Université Mohammed V, Rabat, Morocco.

Bégis D. and Glowinski R. (1975): *Application de la méthode des éléments finish á l'approximation d'un probléme de domaine optimal. Méthodes de résolution des probléms approchés.* — Appl. Math. Optimisation, Vol.2, No.2.

Brezzi K. and Fortin M. (1991): *Mixed and Hybrid Finite Element Methods.* — Springer Series in Comput. Math., Vol.15, Springer-Verlag.

Fortin M. (1977): *An analysis of the convergence of mixed finite element methods.* — R.A.I.R.O. Numer. Anal., Vol.11, No.4, pp.341–354.

Fortin M. and Fortin A. (1984): *Newer and newer elements for incompressible flows*, In: Finite Elements in Fluids (J.T. Oden, Ed.). — Vol.6, Wiley.

Fortin M. and Fortin A. (1985): *Experiments with several elements for viscous incompressible flows.* – Int. J. Numer. Methods in Fluids, Vol.5, pp.911–928.

Girault V. and Raviart P.A. (1979): *Finite Element Approximation of Navier-Stokes Equations.* — Lecture Notes in Math., Berlin: Springer-Verlag.

Gunzburger M. and Hou M. (1992): *Treating inhomogeneous essential boundary conditions in finite element methods and the calculation of boundary stress.* — SIAM J. Numer. Anal., Vol.29, pp.390–424.

Hadamard J. (1969): *Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées.* — Oeuvres de J. Hadamard, Vol.II (Original reference: Mémoire des Savants Etrangers. 33 (1907), Mémoire couronné par l'Académie des Sciences), C.N.R.S, Paris, pp.515–641.

Haslinger J. and Neittaanmäki P. (1988): *Finite Element Approximation for Optimal Shape Design, Theory and Application.* — Wiley.

Lions J.L. and Magenes E. (1972): *Problèmes aux Limites non Homogènes et Application.* — Paris: Dunod, Vol.1.

Orlt M. and Sändig A.M. (1994): *Regularity of viscous Navier-Stokes flows in nonsmooth domains*, In: Boundary Value Problems and Integral Equations in Nonsmooth Domains (M. Costabel, M. Dauge and S. Nicaise, Eds.). — Lecture Notes in Pure and Applied Mathematics, Vol.167, New York: Marcel Dekker.

Samouh N. (1992): *Résolution par eléments finis d'un problème d'optimisation de forme en hydrodynamique.* — Thèse de troisième cycle, Université Mohamed V, Rabat, Morocco.

Tartar L. (1974): *Sur l'etude directe d'equations non linéaires intervenant en contrôle optimal.* — J. Func. Anal., Vol.17, pp.1–47.

Temam R. (1983): *Navier-Stokes Equations and Nonlinear Functional Analysis.* — Philadelphia: SIAM.

Zolésio J.P. (1981): *The material derivative (or speed method) for shape optimization*, In: Optimization of Parameter Structures, Vol.II (E.J. Haug and J. Cea, Eds.). — Alphen aan den Rijn, The Netherlands: Sijthoff, pp.1098–1151.

Received: 10 March 1998
Revised: 1 April 1999
Re-revised: 24 February 2000