Int. J. Appl. Math. Comput. Sci., 2000, Vol.10, No.3, 487–516

OPTIMAL SHAPE DESIGN FOR ELLIPTIC EQUATIONS VIA BIE–METHODS

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A special description of the boundary variation in a shape optimization problem is investigated. This, together with the use of a potential theory for the state, result in natural embedding of the problem in a Banach space. Therefore, standard differential calculus can be applied in order to prove the Fréchet-differentiability of the cost function for appropriately chosen data (sufficiently smooth). Moreover, necessary optimality conditions are obtained in a similar way as in other approaches, and are expressed in terms of an adjoint state for more regular data.

Keywords: optimal shape design, fundamental solution, boundary integral equation, first-order necessary condition

1. Introduction

Let us consider an optimization problem with a cost function $\mathcal{J}$ depending upon the shape of a bounded domain

$$\mathcal{J}(\Omega) = J(\Omega; u_\Omega(\cdot)) = \int_\Omega j(x, u_\Omega(x))\,\mathrm{d}x, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is the domain to be optimized, and u_Ω is the solution to a Dirichlet boundary value problem

$$\begin{aligned} \Delta u_\Omega &= f(x) \ \text{ in } \ \Omega, \\ u_\Omega|_\Gamma &= g(x) \ \text{ on } \ \Gamma = \partial\Omega. \end{aligned} \tag{2}$$

Here $f(\cdot)$, $g(\cdot)$ and $j(\cdot,\cdot)$ are sufficiently regular functions and, for convenience, we assume $\Omega \subset D \subseteq \mathbb{R}^2$, with a fixed closed 'hold all' D.

To deal with the problem, a related shape differential calculus should work well in connection with the solution method for the state equation. Especially, the velocity field method (see Sokołowski and Zolesio (1992) and references therein) is often used, combined with a a weak solution approach to the treatment of the state equation.

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In this paper, we shall study shape optimization problems for two-dimensional simply-connected bounded domains Ω, where the domains under consideration, satisfy a condition of star-shapeness with respect to a neighbourhood $U_\delta(x_0) = \{y \in \mathbb{R}^2 \mid |y - x_0| < \delta\}$, with some fixed $\delta > 0$. Without loss of generality, in the sequel we assume $x_0 = \mathbf{0}$.

The main advantage of this assumption is that the boundary $\Gamma = \partial\Omega$ of such domains can be described by a Lipschitz continuous function $r = r(\phi)$ of the polar angle ϕ (i.e.

$$\Gamma := \left\{ \gamma(\phi) = \begin{bmatrix} r(\phi)\cos\phi \\ r(\phi)\sin\phi \end{bmatrix} \,\middle|\, \phi \in [0, 2\pi] \right\},$$

cf. (Mazja, 1979)). In other words, the boundaries are graphs in polar coordinates and the description of domain or boundary perturbations is possible in the same way. More precisely, given a reference domain Ω associated with the boundary 'describing function' $r(\cdot)$, a sequence of perturbed domains Ω_ε is associated with $r_\varepsilon = r + \varepsilon r_1$ and a (directional) shape derivative is defined by

$$\mathrm{d}\mathcal{J}(\Omega)[r_1] = \mathrm{d}J(\Omega; u_\Omega)[r_1] = \lim_{\varepsilon \to 0} \frac{\mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega)}{\varepsilon},$$

where r_1 denotes the direction of variation, and Γ, $\Gamma_\varepsilon \in C^{k,\gamma}$ for a suitably chosen $k \in \mathbb{N}$, $\gamma \in [0,1]$. This domain (boundary) regularity is obviously equivalent to ('p' denotes 'periodic')

$$r(\cdot),\ r_1(\cdot) \in C_p^{k,\gamma}[0, 2\pi] := \left\{ r(\cdot) \in C^{k,\gamma}[0, 2\pi] | r^{(i)}(0) = r^{(i)}(2\pi), \quad i = 0(1)k \right\}. \quad (3)$$

Consequently, a Banach space embedding of the shape problem is possible, which allows us to apply standard differential calculus.

A potential theory is used for solving the state equation. This and the boundary description via polar coordinates allow the transformation of the double layer part of the solution representation as well as the related boundary integral equation (BIE) for the density into an integral and an integral equation over the interval $[0, 2\pi]$, respectively, where all the information about the shape is now contained in the kernels of the related integral operators. A set of (almost) explicit formulae is obtained, useful for 'straightforward' computation of the first-order directional derivatives of the cost function. However, for 'pure' boundary perturbation methods a 'complete' mapping between reference and perturbed domains (like for perturbation of identity or the general velocity field method) is not explicitly given. As a result, only the so-called local derivative of the state can be treated and the concept of material derivatives for the shape differentiation is not directly applicable. In particular, additional investigations are necessary to guarantee the existence of directional derivatives $\mathrm{d}\mathcal{J}[r_1]$, in polar coordinates formally given by

$$\mathrm{d}J\big(\Omega; u(\cdot)\big)[r_1] = \int_0^{2\pi} r r_1 j\big(\xi, g(\xi)\big)(\phi)\,\mathrm{d}\phi$$

$$+ \int_0^{2\pi}\int_0^{r(\alpha)} \frac{\partial j\big(\cdot, u(\cdot)\big)}{\partial u}\,\mathrm{d}u[r_1](\rho,\alpha)\rho\,\mathrm{d}\rho\,\mathrm{d}\alpha.$$

In order for that expression to make sense, we assume the local derivative $\mathrm{d}u[r_1]$ to be at least in $L_1(\Omega)$, but only a more technical second property (see Appendix A) completes the proof. Consequently, by showing the existence of the material derivative in a suitable weak sense (Sokołowski and Zolesio, 1992), the existence of the directional derivative for the cost follows immediately. For example, Gillaume and Mahsmoudi (1994) and Gillaume (1996) used material derivatives (combined with a perturbation-of-identity approach, cf. Murat and Simon (1976), Simon (1980), Pironneau (1983)) 'to differentiate with respect to the domain' and to apply well-known relationships with local derivatives, as well as integration by parts (the Green formula), to obtain intrinsic or boundary formulations for the derivative, which are helpful when dealing with integral-equation methods. Nevertheless, the approach presented here seems to be applicable to obtain such expressions, too. Whereas it is not important for most applications, this incidentally requires no differentiability assumption on the integrand j of the cost function with respect to the spatial variable x. Moreover, discussing shape derivatives of BIE-solutions ('density') and of related volume and double layer potentials is of some interest in its own right.

Especially, the derivative of the density can be viewed as a Fréchet derivative for the BIE 'transported to the parameter space'. Similarly, the Fréchet differentiability of the cost function can be shown based on explicit representations of $\mathrm{d}u$ inserted into $\mathrm{d}J$, even for some 'non-regular' case. Regularity is understood with respect to the regularity of the (local) state derivative $\mathrm{d}u$ and is given if all data (e.g. the boundary generating function $r \in C_p^2$, right-hand side $f \in C(D)$ and, in particular, boundary data $g \in C^{1,\gamma}(D)$) are smooth enough such that the characterization theorem for $\mathrm{d}u$ holds and ∇J is given as a complete boundary integral expression by introducing the adjoint state p (Section 4). Like for other methods (Gillaume and Masmoudi, 1994; Fujii, 1986; Pironneau, 1983; Sokołowski and Zolesio, 1992), this leads to 'standard' necessary optimality conditions for the optimization problem (1)–(2), in the paper also denoted by Problem (P).

It is not difficult to extend the method to star-shaped domains of higher dimensions, but also a combination of potential methods with boundary variation by smooth fields is possible for more general domains. Such methods are used by Potthast (1994a; 1994b) for the shape derivatives of the state in an inverse scattering problems. Obviously, boundary variation by smooth fields can be viewed as some perturbation of identity, 'reduced to the boundary'.

We conclude the introduction with some remarks regarding the contents and notation. In Section 2, the notation from potential theory and some basic facts are given. Section 3 contains the complete first-order shape calculus for the problem. In Section 4, we discuss a boundary integral representation formula for the gradient and a necessary condition in relation to other approaches. Some technical details are studied in Appendix B. In the paper, shape derivatives are usually denoted by $\mathrm{d}\cdot[r_1]$ or $\nabla\cdot[r_1]$ if they are Fréchet derivatives. Spatial gradients ∇_x and partial derivatives with respect to polar coordinates (especially $\partial(\cdot)/\partial\vec{r} = \langle\nabla_x(\cdot),\vec{e_r}\rangle$) or boundary normals $\partial(\cdot)/\partial n$ often occur in the formulae and should not be confused with shape derivatives.

2. Potential Theory, Domain Perturbations and Transformations into Polar Coordinates

It is well-known (Hackbusch, 1989; Günter, 1957; Michlin, 1978; Kress, 1989) that, using the fundamental solution to the Laplace operator, an integral representation for the solution u to the state equation of Problem (P) can be given as follows:

$$u_\Omega(x) = -\int_\Omega E(x,\xi)f(\xi)\,\mathrm{d}\xi + \int_\Gamma \frac{\partial E(x,\xi)}{\partial n_\xi}\mu(\xi)\,\mathrm{d}S_\xi$$
$$= V(f;x) + W(\mu;x), \quad x\in\Omega, \tag{4}$$

where $\mu(\cdot)$ satisfies the BIE

$$\frac{1}{2}\mu(x) - \int_\Gamma \frac{\partial E(x,\xi)}{\partial n_\xi}\mu(\xi)\,\mathrm{d}S_\xi = -g(x) - \int_\Omega E(x,\xi)f(\xi)\,\mathrm{d}\xi, \quad x\in\Gamma, \tag{5}$$

and the fundamental solution $E(x,\xi)$ in $\mathbb{R}^2$ is $E(x,\xi) = -(1/2\pi)\ln|x-\xi|$. The parts

$$V(f;x) = -\int_\Omega E(x,\xi)f(\xi)\,\mathrm{d}\xi \quad\text{and}\quad W(\mu;x) = \int_\Gamma \frac{\partial E(x,\xi)}{\partial n_\xi}\mu(\xi)\,\mathrm{d}S_\xi$$

are called the volume potential part and the double layer potential part of u_Ω, respectively. Furthermore, we introduce the boundary integral operator $\mathbf{K}$, defined by

$$(\mathbf{K}\mu)(x) = -\int_\Gamma \frac{\partial E(x,\xi)}{\partial n_\xi}\mu(\xi)\,\mathrm{d}S_\xi, \quad x\in\Gamma,$$

which implies a more compact notation for (5):

$$\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\mu = -g + V(f;\cdot)|_\Gamma. \tag{6}$$

Lemma 1. *If $f\in C(D)$ and $g\in C^1(D)$, then for every bounded domain $\Omega\subset D$ with $\Gamma\in C^2$ there exists a unique generalized or potential solution u_Ω of (2), which is given by (4) and, for arbitrary $\varepsilon\in(0,1)$ and $p\in[1,\infty)$, satisfies*

$$u_\Omega \in C^{1,1-\varepsilon}(\Omega)\cap C^{0,1-\varepsilon}(\bar{\Omega}) \quad\text{and}\quad u_\Omega\in W^{1,p}(\Omega). \tag{7}$$

Proof. In order to recall some known facts, we briefly sketch the proof. From the potential theory (Günter, 1957) it follows that $V(f;\cdot) \in C^{1,1-\varepsilon}(\mathbb{R}^2)$, which implies for the right hand side of the BIE (5)

$$-g(\cdot) - \int_\Omega E(\cdot,\xi) f(\xi)\,\mathrm{d}\xi \in C^1(\Gamma).$$

Therefore, for the unique solution $\mu \in C(\Gamma)$ (Hackbusch, 1989) we have $\mu \in C^1(\Gamma)$ because of the smoothness of the right-hand side and the lifting properties of the boundary integral operator K. More precisely, this holds for a C^2-boundary Γ and for every $\gamma \in (0,1)$: $K \in L(L^\infty(\Gamma), C^{0,\gamma}(\Gamma))$, cf. (Hackbusch, 1989), and $K \in L(C^{0,\gamma}(\Gamma), C^{1,\gamma}(\Gamma))$, cf. (Kress, 1989), respectively. Consequently, the C^1-regularity of the right-hand side carries over to the solution μ. Moreover, double layer potentials with C^1-densities satisfy $W(\mu;\cdot) \in C^\infty(\Omega) \cap C^{0,1-\varepsilon}(\bar{\Omega})$ for $\varepsilon > 0$ (Hackbusch, 1989; Kress, 1989), which implies the first relation of (7). For $u_\Omega \in W^{1,p}$, see Remark 1. ■

Remark 1. As a counter example, Günter (1957) shows that $f \in C(\bar{\Omega})$ guarantees neither $V(f;\cdot) \in C^2(\Omega)$ nor $V(f;\cdot) \in W^{2,\infty}(\Omega)$, but $V(f;\cdot) \in W^{2,p}(\Omega)$ is valid, see (Michlin, 1978). Moreover, $\mu \in C^1$ does not imply $W(\mu;\cdot) \in C^1(\bar{\Omega})$, whereas $\mu \in C^{1,\gamma} \Rightarrow W(\mu;\cdot) \in C^{1,\gamma}(\bar{\Omega})$ holds for $\Gamma \in C^2$ (Kress, 1989). Nevertheless, due to Hackbusch (1989) we have $|\nabla W(\mu;x)| \leq c|\ln[d(x,\Gamma)]|$, which implies $\nabla u_\Omega \in L_p(\Omega)$ for arbitrary $p \in [1,\infty)$.

Remark 2. The assumption $\Gamma \in C^2$ is equivalent to (cf. (3))

$$r(\cdot) \in C_p^2[0,2\pi] := \Big\{r(\cdot) \in C^2[0,2\pi] | r^{(i)}(0) = r^{(i)}(2\pi), \quad i = 0,1,2, \quad r(\phi) \geq \delta\Big\}.$$

Moreover, the unscaled and scaled outer normals to the boundary are respectively given by

$$\vec{a}(\phi) = \begin{pmatrix} r(\phi)\cos\phi + r'(\phi)\sin\phi \\ r(\phi)\sin\phi - r'(\phi)\cos\phi \end{pmatrix}, \quad \vec{n}(\phi) = \frac{1}{\sqrt{r^2(\phi) + r'^2(\phi)}}\vec{a}(\phi). \tag{8}$$

Due to the special form of the fundamental solution in $\mathbb{R}^2$, we get for the volume potential part

$$V(f;x) = \frac{1}{2\pi}\int_\Omega \ln|x-\xi| f(\xi)\,\mathrm{d}\xi = \int_0^{2\pi}\int_0^{r(\phi)} \ln\left|x - \rho\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}\right| f(\rho,\phi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi, \tag{9}$$

and for the double layer part (for polar coordinate transformations)

$$-W(\mu;x) = \frac{1}{2\pi}\int_\Gamma \frac{\partial \ln|x-\xi|}{\partial n_\xi}\mu(\xi)\,\mathrm{d}S_\xi = \frac{1}{2\pi}\int_\Gamma \langle\nabla_\xi \ln|x-\xi|, \vec{n}_\xi\rangle \mu(\xi)\,\mathrm{d}S_\xi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \langle\nabla_\xi \ln|x-\xi(\phi)|, \vec{a}(\phi)\rangle \mu(\xi(\phi))\,\mathrm{d}\phi = \frac{1}{2\pi}\int_0^{2\pi} \bar{K}(x,\phi)\mu(\phi)\,\mathrm{d}\phi,$$

where

$$\bar{K}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\phi\right) = \frac{\langle(\xi(\phi)-x),\vec{a}(\phi)\rangle}{|x-\xi(\phi)|^2}$$

$$= \frac{\left\langle\begin{pmatrix}r(\phi)\cos\phi - x_1\\ r(\phi)\sin\phi - x_2\end{pmatrix},\begin{pmatrix}r(\phi)\cos\phi + r'(\phi)\sin\phi\\ r(\phi)\sin\phi - r'(\phi)\cos\phi\end{pmatrix}\right\rangle}{\left(x_1 - r(\phi)\cos\phi\right)^2 + \left(x_2 - r(\phi)\sin\phi\right)^2}. \quad (10)$$

Therefore, (4) becomes

$$u(x) = V(f;x) - \frac{1}{2\pi}\int_0^{2\pi}\bar{K}(x,\phi)\mu(\phi)\,\mathrm{d}\phi, \quad x = \begin{pmatrix}x_1\\x_2\end{pmatrix} = \rho\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} \in \Omega.$$

In much the same way as for the BIE (5), we arrive at

$$\frac{1}{2}\mu(\alpha) + \frac{1}{2\pi}\int_0^{2\pi}K(\alpha,\phi)\mu(\phi)\,\mathrm{d}\phi = -g(\alpha) + V(f;\alpha), \quad \alpha \in [0,2\pi]. \quad (11)$$

Here we use the following notation for kernel functions of boundary integral operators:

$$\left.\begin{aligned} K(\alpha,\phi) &= d(\alpha,\phi)m(\alpha,\phi),\\ d(\alpha,\phi) &= r^2(\phi) - r(\phi)r(\alpha)\cos(\phi-\alpha) - r'(\phi)r(\alpha)\sin(\phi-\alpha),\\ m(\alpha,\phi) &= r^2(\phi) + r^2(\alpha) - 2r(\phi)r(\alpha)\cos(\phi-\alpha), \end{aligned}\right\} \quad (12)$$

and $g(\alpha) := g(x(\alpha)),\ V(f;\alpha) := V(f;x(\alpha))$.

Remark 3. For $r(\cdot) \in C^2$ the kernel K is regular i.e. K can be continuously extended to $[0,2\pi]\times[0,2\pi]$ with limiting values

$$K(\phi,\phi) = \frac{r'^2(\phi) + \frac{1}{2}\left(r^2(\phi) - r(\phi)r''(\phi)\right)}{r^2(\phi) + r'^2(\phi)} = \frac{1}{2}\kappa(\phi)l(\phi), \quad \phi \in [0,2\pi],$$

where $\kappa(\cdot)$ and $l(\cdot)$ denote the (local) curvature and the arc length of the boundary, respectively.

The admissible perturbed domains (or boundaries) Ω_ε are defined by (see the Introduction)

$$\Gamma_\varepsilon \Leftrightarrow r_\varepsilon(\phi) = r(\phi) + \varepsilon r_1(\phi), \quad r_1 \in C_p^2[0,2\pi],$$

$\varepsilon > 0$ being sufficiently small, provided that $r_\varepsilon(\phi) > \delta,\ \phi \in [0,2\pi]$. Moreover, the subscript ε is used in the sequel to denote the quantities related to Ω_ε. Thus we will use differential calculus in the Banach space $C_p^2[0,2\pi]$ for the study of Problem (P).

Remark 4. Since

$$\langle \vec{e}_r, \vec{n} \rangle = \frac{r}{\sqrt{r^2 + r'^2}} \geq c > 0,$$

$\vec{e}_r = \binom{\cos\phi}{\sin\phi}$ being the radial unit vector, the above perturbations are always regular, i.e. the perturbation field is tangential if and only if $r_1(\cdot) \equiv 0$. Moreover, an easy calculation shows the well-known fact (Sokołowski and Zolesio, 1992)

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\vec{n}_\varepsilon(\phi)|_{\varepsilon=0} = \frac{rr_1' - r'r_1}{r^2 + r'^2} \cdot \vec{\tau}(\phi) \perp \vec{n}(\phi), \quad \phi \in [0, 2\pi].$$

where $\vec{\tau}(\phi)$ denotes the unit tangential vector on Γ directed to increasing ϕ. In order to obtain the derivatives for the shape problem, some results on the differentiability of domain and boundary functionals are also used.

Lemma 2. *Let $f \in C(D)$ and $g \in C^1(D)$ be given. Then the functionals $J_1(\Omega) = J_1(r) = \int_\Omega f\,\mathrm{d}x$ and $J_2(\Omega) = J_2(r) = \int_\Gamma g\,\mathrm{d}S_\Gamma$ are Fréchet differentiable with respect to $C_p^1[0, 2\pi]$, where the derivatives are given by*

$$\nabla J_1(r)[r_1] = \int_0^{2\pi} r(\phi)r_1(\phi)f\big(r(\phi), \phi\big)\,\mathrm{d}\phi, \tag{13}$$

and

$$\nabla J_2(r)[r_1] = \int_0^{2\pi} r_1\sqrt{r^2 + r'^2}\frac{\partial g}{\partial \vec{r}}\big(r(\phi), \phi\big) + g\big(r(\phi), \phi\big)\frac{rr_1 + r'r_1'}{\sqrt{r^2 + r'^2}}\,\mathrm{d}\phi. \tag{14}$$

Remark 5. For the proof, see (Eppler, 1998a; 1999). Obviously, we have directional derivatives given by (13) and (14), respectively, linear and continuous with respect to r_1. Moreover, the related operator norm of the Gâteaux derivative depends continuously on the $C_p^1[0, 2\pi]$ norm of r. This ensures the continuous Fréchet differentiability of the functionals by standard arguments from functional analysis (Bögel and Tasche, 1974).

3. Derivatives for the Shape Problem

Owing to the character of potential solutions, we have three steps for the derivation of the corresponding formulae for derivatives. We start with the investigation of the BIE for the density, then we will discuss the variation of the state and, finally, the derivative of the cost function. According to Lemma 1, we assume $g \in C^1(D)$ and $f \in C(D)$ throughout this section.

3.1. Derivative of the Density

The BIE is transformed into an integral equation over the interval $[0, 2\pi]$, where all the information about the boundary is contained in the kernel and the right-hand side. Therefore, straightforward differentiation leads to the following result:

Theorem 1. *The mapping* $r(\cdot) \mapsto \mu(r;\cdot)$ *is Fréchet differentiable at* $r(\cdot)$ *as a mapping from* $C_p^2[0, 2\pi]$ *to* $C_p[0, 2\pi]$, *where the derivative* $\mathrm{d}\mu[r_1] \in C_p[0, 2\pi]$ *satisfies the following BIE (cf. (11)):*

$$\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right) \mathrm{d}\mu[r_1] = -\mathrm{d}g[r_1] + \mathrm{d}_\Gamma V[r_1](f;\cdot) - \mathrm{d}\mathbf{K}[r_1]\mu, \tag{15}$$

with $\mathrm{d}g[r_1]$, $\mathrm{d}_\Gamma V[r_1](f;\cdot)$ *and* $\mathrm{d}\mathbf{K}[r_1]\mu$ *defined as follows* ($\alpha \in [0, 2\pi]$)*:*

$$\mathrm{d}g[r_1](\alpha) = r_1(\alpha)\left\langle \nabla_x g\left(\begin{pmatrix} r(\alpha)\cos\alpha \\ r(\alpha)\sin\alpha \end{pmatrix}\right), \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}\right\rangle = r_1(\alpha)\frac{\partial g}{\partial \vec{r}}(\alpha),$$

$$\mathrm{d}_\Gamma V[r_1](f;\alpha) = \frac{1}{2\pi}\int_0^{2\pi} r(\phi)r_1(\phi)\ln\left|x(\alpha) - \xi(\phi)\right| f(\xi(\phi))\,\mathrm{d}\phi + r_1(\alpha)\frac{\partial V(f; x(\alpha))}{\partial \vec{r}}$$

and

$$(\mathrm{d}\mathbf{K}[r_1]\mu)(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}K[r_1](\alpha,\phi)\mu(\phi)\,\mathrm{d}\phi.$$

The kernel derivative $\mathrm{d}K[r_1](\alpha,\phi)$ *is given by (cf. (12))*

$$\mathrm{d}K[r_1](\alpha,\phi) = \frac{dd[r_1](\alpha,\phi)}{m(\alpha,\phi)} - K(\alpha,\phi)\cdot\frac{dm[r_1](\alpha,\phi)}{m(\alpha,\phi)},$$

where the shape derivatives of the numerator and denominator are respectively determined from

$$\begin{aligned} \mathrm{d}d[r_1](\alpha,\phi) &= 2r(\phi)r_1(\phi) - \left[r_1(\phi)r(\alpha) + r(\phi)r_1(\alpha)\right]\cos(\alpha-\phi) \\ &\quad + \left[r_1'(\phi)r(\alpha) + r'(\phi)r_1(\alpha)\right]\sin(\alpha-\phi), \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}m[r_1](\alpha,\phi) &= 2r(\phi)r_1(\phi) + 2r(\alpha)r_1(\alpha) \\ &\quad - 2\big(r_1(\phi)r(\alpha) + r(\phi)r_1(\alpha)\big)\cos(\alpha-\phi). \end{aligned}$$

Moreover, the shape derivative $\mathrm{d}\mathbf{K}$ *of the boundary integral operator* $\mathbf{K}$ *has a non-trivial kernel, i.e.* $\ker\{\mathrm{d}\mathbf{K}[r_1]\}$ *contains (for arbitrary* r*) the constant functions*

$$\mathrm{d}\mathbf{K}[r_1]\mathbf{1} = \mathbf{0}, \quad \textit{or} \quad \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}K[r_1](\alpha,\phi)\cdot 1\mathrm{d}\phi = 0, \quad \textit{for all} \quad \alpha \in [0,2\pi]. \tag{16}$$

Proof. We have $g \in C^1(D)$ and the kernel derivative $\mathrm{d}K[r_1](\cdot,\cdot)$ is regular with limiting values

$$\frac{\mathrm{d}d[r_1](\phi,\phi)}{m(\phi,\phi)} = \frac{r(\phi)r_1(\phi) + 2r'(\phi){r_1}'(\phi) - \frac{1}{2}\big(r(\phi){r_1}''(\phi) + r_1(\phi)r''(\phi)\big)}{r^2(\phi) + r'^2(\phi)},$$

and

$$\frac{\mathrm{d}m[r_1](\phi,\phi)}{m(\phi,\phi)} = \frac{2r(\phi)r_1(\phi) + 2{r_1}'(\phi)r'(\phi)}{r^2(\phi) + r'^2(\phi)}, \quad \phi \in [0,2\pi].$$

Therefore, formal differentiation of (11) leads to (15). In order to see the structure of $d_\Gamma V[r_1](f;\cdot)$, let us note that

$$\begin{aligned} \mathrm{d}_\Gamma V[r_1](f;\alpha) = \lim_{\varepsilon\to 0} & \frac{V_\varepsilon\big(f;x_\varepsilon(\alpha)\big) - V_\varepsilon\big(f;x(\alpha)\big)}{\varepsilon} \\ & + \frac{V_\varepsilon\big(f;x(\alpha)\big) - V\big(f;x(\alpha)\big)}{\varepsilon}. \end{aligned}$$

The weak singularity of the logarithmic kernel and $f \in C$ allows us to apply Lemma 1 and gives the first part of $d_\Gamma V$ from the second expression above. We obtain the second part by taking the limit as $\varepsilon \to 0$ in

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} f(y)\big[\ln|x_\varepsilon(\alpha) - y| - \ln|x(\alpha) - y|\big]\,\mathrm{d}y \\ &\qquad = r_1(\alpha)\int_{\Omega_\varepsilon} f(y)\frac{\langle x_\nu(\alpha) - y, \vec{e}_r(\alpha)\rangle}{|x_\nu(\alpha) - y|^2}\,\mathrm{d}y, \quad \nu \in (0,\varepsilon), \end{aligned}$$

because the related domain integrals are weakly singular and spatial differentiation can be performed under the integral sign as follows:

$$\frac{\partial V(f;x(\alpha))}{\partial \vec{r}} = \frac{1}{2\pi}\int_\Omega f(y)\langle\nabla_x \ln|x(\alpha) - y|, \vec{e}_r(\alpha)\rangle\,\mathrm{d}y. \tag{17}$$

Furthermore, the continuity of the right-hand side and the properties of $(\frac{1}{2}\mathbf{I} + \mathbf{K})$ imply the existence of the directional derivative $\mathrm{d}\mu[r_1] \in C_p[0,2\pi]$.

To show Gâteaux differentiability, we only need an estimate of the right-hand side of (15), because $(\frac{1}{2}\mathbf{I}+\mathbf{K})$ is independent of r_1 and so is the norm of its inverse.

This is immediately possible for the first two parts $\mathrm{d}g[r_1]$ and $\mathrm{d}_\Gamma V[r_1](f;\cdot)$, in both the cases with respect to the C-norm of r_1. Using the formulae for limiting values of $\mathrm{d}\mathbf{K}[r_1]$, an estimate of the third part with respect to the C^2-norm of r_1 is also valid and we obtain

$$\|\mathrm{d}\mu[r_1]\|_C \le c(r)\|r_1\|_{C^2}, \quad \text{for all admissible } r \in C_p^2[0,2\pi].$$

For the continuous dependence of the norm $c(r)$ on r, we only remark that the estimate

$$\sqrt{r^2(\phi)+r'^2(\phi)} \ge \frac{\delta}{2}, \quad \phi \in [0,2\pi],$$

holds uniformly in a sufficiently small neighbourhood of every admissible r. Consequently, the Fréchet differentiability of the mapping $r(\cdot) \mapsto \mu(r;\cdot)$ follows by the same arguments as for the functionals.

To deduce (16), we recall the well-known identity

$$-\int_\Gamma \frac{\partial E(x,\xi)}{\partial n_\xi}(\xi)\,\mathrm{d}S_\xi = \frac{1}{2\pi}\int_0^{2\pi} K(\alpha,\phi)\,\mathrm{d}\phi = \frac{1}{2}, \quad x(\alpha) \in \Gamma, \quad (\alpha \in [0,2\pi], \text{ resp.}),$$

which is valid uniformly for all the boundaries under consideration. Due to the regularity of $dK[r_1](\cdot,\cdot)$, differentiation with respect to the boundary variation is possible under the integral sign and leads to (16). ∎

Remark 6. With the aid of (16), rewriting $\mathrm{d}\mathbf{K}[r_1]\mu$ as

$$(\mathrm{d}\mathbf{K}[r_1]\mu)(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} dK[r_1](\alpha,\phi)\big(\mu(\phi)-\mu(\alpha)\big)\,\mathrm{d}\phi,$$

and using $\mu \in C^1$, we obtain in fact a C^1-estimate for the norm of the Gâteaux derivative. Altogether, we have $\|\mathrm{d}\mu[r_1]\|_C \le c\|r_1\|_{C^1}$, $c = c(\|r\|_{C^2})$.

Remark 7. The regularity of $dK[r_1](\cdot,\cdot)$ and the property (16) are similar to results of Potthast (1994b) (cf. Ch. 3.2 (examples) and the proof of Thm. 3.17). Moreover, the integral operator $\mathrm{d}\mathbf{K}[r_1]$ is bounded from $C^{0,\alpha}[0,2\pi]$ to $C^{1,\alpha}[0,2\pi]$, and $\mathrm{d}\mu[r_1]$ can be expressed in accordance with Potthast (1994a) as

$$\begin{aligned}
\mathrm{d}\mu[r_1] &= \left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\left\{-\mathrm{d}g[r_1]+\mathrm{d}_\Gamma V[r_1](f;\cdot)-\mathrm{d}\mathbf{K}[r_1]\mu\right\} \\
&= \left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\left\{-\mathrm{d}g[r_1]+\mathrm{d}_\Gamma V[r_1](f;\cdot)\right\} \\
&\quad - \left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\mathrm{d}\mathbf{K}[r_1]\left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\left\{-g+V(f;\cdot)|_\Gamma\right\}.
\end{aligned}$$

3.2. Derivative of the State

One difficulty in using boundary variational approaches in shape optimization is that the solutions u_ε to the state equation for perturbed domains are not comparable to u in the whole Ω, because they are defined on different domains $((\Omega_\varepsilon \setminus \Omega) \cup (\Omega \setminus \Omega_\varepsilon) \neq \emptyset)$. Therefore, when dealing with local derivatives only, the Fréchet differentiability of the state, e.g. with respect to $C(\bar{\Omega})$ or $L_p(\Omega)$ for a suitable $p \geq 1$, is not possible. Only particular results can be obtained by embedding into a Banach space, defined on compact subsets of Ω (cf. (Potthast, 1994b) or Corollary 1 below). However, this is not sufficient for the investigation of our cost function.

Remark 8. To avoid this problem, the material derivative concept was developed (Sokołowski and Zolesio, 1992), applicable if an explicit mapping between reference and perturbed domains is given (perturbation of identity, cf. Murat and Simon, 1976) or if such a mapping is constructed more implicitly (the velocity field method). However, material derivatives contain not only 'shape variational' parts of the solution but also some 'transport of domains'.

Nevertheless, a shape directional derivative exists pointwise on Ω.

Theorem 2. *The directional derivative* $\mathrm{d}u[r_1]$ *of the state* u *exists for all* $x \in \Omega$ *and is given by*

$$\mathrm{d}u[r_1](x) = \mathrm{d}V[r_1](f;x) + W(\mathrm{d}\mu[r_1];x) + \mathrm{d}W[r_1](\mu;x), \tag{18}$$

where $\mathrm{d}V[r_1](f;x)$, $W(\mathrm{d}\mu[r_1];x)$ *and* $\mathrm{d}W[r_1](\mu;x)$ *are respectively defined as* $(x \in \Omega,\ \phi \in [0, 2\pi])$

$$\mathrm{d}V[r_1](f;x) = \frac{1}{2\pi} \int_0^{2\pi} r(\phi) r_1(\phi) \ln|x - \xi(\phi)| f(\xi(\phi))\, \mathrm{d}\phi,$$

$$W(\mathrm{d}\mu[r_1];x) = -\frac{1}{2\pi} \int_0^{2\pi} \bar{K}(x,\phi)\, \mathrm{d}\mu[r_1](\phi)\, \mathrm{d}\phi,$$

$$\mathrm{d}W[r_1](\mu;x) = -\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\bar{K}[r_1](x,\phi)\mu(\phi)\, \mathrm{d}\phi.$$

where $\mathrm{d}\mu[r_1]$ *satisfies (15). The kernel derivative* $\mathrm{d}\bar{K}(r)[r_1](x,\phi)$ *is the derivative of* $\bar{K}(r)(x,\phi)$ *in direction* r_1

$$\mathrm{d}\bar{K}[r_1](x,\phi) = \left\langle \nabla_\xi^2 \ln|x - \xi(\phi)| \cdot \begin{pmatrix} r_1(\phi)\cos\phi \\ r_1(\phi)\sin\phi \end{pmatrix}, \vec{a}(\phi) \right\rangle + \left\langle \nabla_\xi \ln|x - \xi(\phi)|,\ \mathrm{d}\vec{a}(\phi) \right\rangle.$$

Moreover, as is the case of (16), we have

$$\mathrm{d}W[r_1](\mathbf{1};\cdot) \equiv \mathbf{0}, \quad \textit{or} \quad \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\bar{K}[r_1](x,\phi)\cdot 1\,\mathrm{d}\phi = 0, \quad \textit{for all} \quad x \in \Omega. \tag{19}$$

Proof. The shape derivative $\mathrm{d}\vec{a}(\phi)$ of the (unscaled) normal is obviously given by

$$\mathrm{d}\vec{a}(\phi) = \begin{pmatrix} r_1(\phi)\cos\phi + r_1'(\phi)\sin\phi \\ r_1(\phi)\sin\phi - r_1'(\phi)\cos\phi \end{pmatrix} = r_1(\phi)\vec{e}_r(\phi) - r_1'\vec{e}_\varphi(\phi), \quad \phi \in [0,2\pi].$$

For the volume potential part we obtain the derivative similar to Lemma 2, because the logarithmic singularity of the kernel of $V(f;\cdot)$ causes no additional difficulty ($x \in \Omega$ is fixed). Furthermore, differentiation of the double layer potential part leads directly to the second and third parts. By using polar coordinates for x ($x_1 = \rho\cos\alpha$, $x_2 = \rho\sin\alpha$), the kernel $\bar{K}(x,\phi) = \bar{K}(\rho,\alpha,\phi)$ and the derivative $\mathrm{d}\bar{K}[r_1](x,\phi) = \mathrm{d}\bar{K}[r_1](\rho,\alpha,\phi)$ can be written down as

$$\bar{K}(\rho,\alpha,\phi) = \frac{r^2(\phi) - r(\phi)\rho\cos(\phi-\alpha) - r'(\phi)\rho\sin(\phi-\alpha)}{r^2(\phi)+\rho^2 - 2r(\phi)\rho\cos(\phi-\alpha)},$$

and

$$\mathrm{d}\bar{K}[r_1](\rho,\alpha,\phi) = \frac{2r(\phi)r_1(\phi) - r_1(\phi)\rho\cos(\phi-\alpha) - r_1'(\phi)\rho\sin(\phi-\alpha)}{r^2(\phi)+\rho^2-2r(\phi)\rho\cos(\phi-\alpha)}$$
$$- \bar{K}(\rho,\alpha,\phi)\cdot\frac{2r(\phi)r_1(\phi) - 2r_1(\phi)\rho\cos(\phi-\alpha)}{r^2(\phi)+\rho^2-2r(\phi)\rho\cos(\phi-\alpha)},$$

respectively. The basic identity for (19) is (just as in Section 3.1)

$$-\int_\Gamma \frac{\partial E(x,\xi)}{\partial n_\xi}(\xi)\,\mathrm{d}S_\xi = \frac{1}{2\pi}\int_0^{2\pi}\bar{K}(\rho,\alpha,\phi)\,\mathrm{d}\phi = 1, \quad x = (\rho,\alpha)^T \in \Omega, \quad \alpha \in [0,2\pi],$$

which is valid uniformly for all the boundaries under consideration. By $\operatorname{dist}(x,\Gamma) > 0$, $x \in \Omega$, the kernel $\mathrm{d}\bar{K}[r_1]$ is regular and differentiation with respect to the boundary variation under the integral sign leads to (19). ■

Remark 9. $\mathrm{d}V[r_1](f;\cdot)$ can be interpreted as a single-layer potential with density $f(\xi)\cdot\langle r_1\vec{e}_r, \vec{n}_\xi\rangle$. Due to the continuity property of the single-layer potential on the boundary, we have only the first part of $\mathrm{d}_\Gamma V[r_1](f;\cdot)$ (cf. Section 3.1) as limiting values

$$\lim \mathrm{d}V[r_1](f;x) = \frac{1}{2\pi}\int_0^{2\pi} r(\phi)r_1(\phi)\ln|x(\alpha)-\xi(\phi)|f(\xi(\phi))\,\mathrm{d}\phi, \quad \text{as} \quad x \to \xi(\alpha) \in \Gamma,$$

which implies $\mathrm{d}_\Gamma V[r_1](f;\cdot) \neq \mathrm{d}V[r_1](f;\cdot)|_\Gamma$.

Remark 10. Whereas $\lim_{x\to x(\alpha)} \bar{K}(x,\phi) = K(\alpha,\phi)$ holds for $\alpha \neq \phi$, the same relation for the differentiated kernels is not true, i.e. $\lim \mathrm{d}\bar{K}[r_1](x,\phi) \neq \mathrm{d}K[r_1](\alpha,\phi)$. Moreover, in contrast to the regularity of $\mathrm{d}K[r_1]$, the kernel $\mathrm{d}\bar{K}[r_1]$ has a singularity of order 2 on the boundary for $(\rho,\alpha)^T \to (r(\phi),\phi)^T$. More precisely,

$$|\mathrm{d}\bar{K}[r_1](x,\phi)| = |\mathrm{d}\bar{K}[r_1](\rho,\alpha,\phi)| \leq \frac{c}{|x-\xi(\phi)|^2}$$
$$= \frac{c}{r^2(\phi)+\rho^2-2r(\phi)\rho\cos(\phi-\alpha)}.$$

Hence, the singularity of the differentiated kernels for the solution representation increases by one order for each step of differentiation.

Obviously, this singularity influences essentially the behaviour of $\mathrm{d}u[r_1]$ close to Γ. In particular, neither continuity nor boundedness on Ω can be expected without additional assumptions. Together with the non-comparability of perturbed solutions u_ε with u on $\bar{\Omega}$, this leads to problems with the investigation of the derivative of the cost function. Before discussing this in more detail, we present some immediate consequences.

Corollary 1. *The mapping $r(\cdot) \mapsto u(r;\cdot)$ is Fréchet differentiable as a mapping from $C_p^2[0,2\pi]$ to $C(K)$ for every compact subset $K \subset \Omega$.*

Proof. Since $\mathrm{d}(K,\Gamma) > 0$, we have $K \subset \Omega_\varepsilon \cap \Omega$ (for sufficiently small ε) as an essential assumption for the investigation of Fréchet differentiability. Discussing the three parts of $\mathrm{d}u[r_1]$ (cf. (18)) simultaneously, we immediately observe the linearity. A uniform estimate (in order to have a Gâteaux derivative) is directly possible for the first part $(|\mathrm{d}V[r_1](f;x)| \leq \|r_1\|_C |(1/2\pi)\int_0^{2\pi} r(\phi)\ln|x-\xi(\phi)| f(\xi(\phi))\,\mathrm{d}\phi|)$ and can be obtained for the second part by the results of Section 3.1. By using $d(K,\Gamma) > 0$, a similar estimate holds for the third part. Moreover, we have $K \subset \Omega_\varepsilon \cap \Omega_{\bar{r}}$ (for $\|\bar{r}-r\|_{C^2} < d(K,\Gamma)/2$), and the norm estimates depend continuously on $\bar{r} \in U_\delta(r)$. ∎

Corollary 2. *The directional derivative* $\mathrm{d}u[r_1]$ *satisfies the Laplace equation in Ω (in a classical sense).*

Proof. From $f \in C$ and $g \in C^1$ (hence $\mu \in C^1$ and $\mathrm{d}\mu[r_1] \in C$), we deduce that $\mathrm{d}u[r_1] \in C^2(\Omega)$. Moreover,

$$\Delta_x \,\mathrm{d}V[r_1](f;x) = 0 \quad\text{and}\quad \Delta_x W(\mathrm{d}\mu[r_1];x) = 0$$

hold for all $x \in \Omega$, because these are a single and a double-layer potential, respectively. The third part contains partial derivatives of the fundamental solution (combined with at least continuous 'densities'), implying also $\Delta_x \,\mathrm{d}W[r_1](\mu;x) = 0$ for $x \in \Omega$. ∎

Remark 11. In general, neither the existence nor information about concrete values of $\mathrm{d}u[r_1]|_\Gamma$ are known without additional assumptions.

Remark 12. In much the same way as in (Potthast, 1994b), an abstract formula for $\mathrm{d}u[r_1]$ is valid. Namely, from $u = V(f;\cdot) + W(((1/2)\mathbf{I}+\mathbf{K})^{-1}\{-g+V(f;\cdot)|_\Gamma\};\cdot)$ we conclude that

$$\mathrm{d}u[r_1] = \mathrm{d}V[r_1](f;\cdot) + \mathrm{d}W[r_1]\left(\left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\{-g+V(f;\cdot)|_\Gamma\};\cdot\right)$$
$$+W\left(\left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\{-\mathrm{d}g[r_1]+\mathrm{d}_\Gamma V[r_1](f;\cdot)\}\right.$$
$$\left.-\left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\mathrm{d}\mathbf{K}[r_1]\left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\{-g+V(f;\cdot)|_\Gamma\};\cdot\right).$$

The last result of this subsection will be essential for the investigation of the cost function.

Theorem 3. *The shape derivative* $\mathrm{d}u[r_1]$ *satisfies*

(i) $\mathrm{d}u(\cdot) \in L_p(\Omega)$ *for* $p < 2$,

(ii) $$\int\limits_{\Omega_\varepsilon\cap\Omega} |u_\varepsilon(x) - u(x) - \varepsilon\,\mathrm{d}u[r_1](x)|\,\mathrm{d}x = o(\varepsilon).$$

Proof. Using (18), we immediately get

$$\mathrm{d}V[r_1](f;\cdot) + W(\mathrm{d}\mu[r_1];\cdot) \in C(\bar{\Omega}),$$

because this is the sum of single- and double-layer potentials with continuous densities (cf. the proof of Corollary 2). For the strong singular part we apply (19) and get

$$\begin{aligned}\mathrm{d}W[r_1](\mu;x) &= \mathrm{d}W[r_1](\mu;\rho,\alpha)\\ &= -\frac{1}{2\pi}\int\limits_0^{2\pi}\mathrm{d}\bar{K}[r_1](\rho,\alpha,\phi)[\mu(\phi)-\mu(\alpha)]\,\mathrm{d}\phi.\end{aligned}$$

Since $\langle\vec{e}_r,\vec{n}\rangle \geq c > 0$ (see Remark 4) and $\mu \in C^1$, we have the estimates

$$\begin{aligned}|\mu(\alpha)-\mu(\phi)| &= |\mu(\xi(\alpha)) - \mu(\xi(\phi))| \leq c|\xi(\alpha)-\xi(\phi)|\\ &\leq k|x-\xi(\phi)|, \quad x=(\rho,\alpha)\in\Omega,\end{aligned}$$

provided that $|\alpha-\phi|$ is sufficiently small. Due to the structure of $\mathrm{d}\bar{K}[r_1](\rho,\alpha,\phi)$ (cf. the proof of Theorem 2), this reduces the strong singularity of order 2 to a singularity of order 1, which is integrable for two-dimensional domains up to an order of $p < 2$. The weakly singular part of $\mathrm{d}\bar{K}[r_1](\rho,\alpha,\phi)$ becomes regular by the additional $[\mu(\phi)-\mu(\alpha)]$ term.

The proof of relation (ii) is rather technical and uses extensively estimates of singularities of potential kernels and its spatial derivatives. Therefore we present the detailed investigation separately in Appendix A. We only remark that the volume and double-layer potential part of the solution satisfy independently

$$\text{(iia)} \qquad \int_{\Omega_\varepsilon \cap \Omega} |V_\varepsilon(f;x) - V(f;x) - \varepsilon\, \mathrm{d}V[r_1](f;x)|\, \mathrm{d}x = o(\varepsilon),$$

$$\text{(iib)} \qquad \int_{\Omega_\varepsilon \cap \Omega} |W_\varepsilon(\mu_\varepsilon;x) - W(\mu;x) - \varepsilon\,\{W(d\mu[r_1];x) + \mathrm{d}W[r_1](\mu;x)\}\ \mathrm{d}x = o(\varepsilon),$$

where the volume potential part allows, in fact, the estimate (uniformly for $x \in \Omega_\varepsilon \cap \Omega$)

$$|V_\varepsilon(f;x) - V(f;x) - \varepsilon\, \mathrm{d}V[r_1](f;x)| = o(\varepsilon). \qquad \blacksquare$$

Remark 13. Relation (ii) means that $\mathrm{d}u[r_1]$ is a *uniform* L_1-directional derivative of the solution on *every* compact subset of Ω (which is not directly clear from the pointwise existence of $\mathrm{d}u[r_1](x)$). Moreover, this result is in some sense similar to the existence of a strong material L_1-derivative for the velocity method.

Remark 14. The result (ii) can be particularly improved (see the last remark in the proof), but this is not necessary for the existence of a directional derivative of the cost function. Therefore, a more careful discussion is only contained in the Appendix.

3.3. Derivative of the Cost Function

Similarly to other cases of composed functionals, one may argue in a ‘straightforward’ sense as follows: The solution to the BIE (the density) has a Fréchet derivative, hence the solution to the state equation is Fréchet differentiable, so the same must hold true for the cost function. This argumentation fails, however, as we have already discussed. Nevertheless, the Fréchet differentiability of the cost function can be shown.

Theorem 4. *In addition to $f \in C(D)$ and $g \in C^1(D)$, we assume that $j(\cdot,\cdot)$ is continuous and has a continuous partial derivative $\partial j/\partial u$ on $D \times \mathbb{R}$. Then the cost function is Fréchet differentiable and the derivative $\nabla \mathcal{J}(r)[r_1] = \nabla J(\Omega; u(\cdot))[r_1]$ is given in polar coordinates by*

$$\nabla \mathcal{J}[r_1] = \int_0^{2\pi} r(\phi) r_1(\phi) j\big(\xi(\phi), g(\xi(\phi))\big)\, \mathrm{d}\phi$$

$$+ \int_0^{2\pi} \int_0^{r(\alpha)} \frac{\partial}{\partial u} j\big(\rho, \alpha, u(\rho,\alpha)\big)\, \mathrm{d}u[r_1](\rho,\alpha) \rho\, \mathrm{d}\rho\, \mathrm{d}\alpha,$$

where the (formally) strongly singular part

$$\int_{\Omega} j_u^0(x)\, \mathrm{d}W[r_1](\mu;x)\, \mathrm{d}x$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} \int_0^{r(\alpha)} j_u^0(\rho,\alpha) \int_0^{2\pi} \mathrm{d}\bar{K}_0[r_1](\rho,\alpha,\phi)\mu_0(\phi)\, \mathrm{d}\phi \rho\, \mathrm{d}\rho\, \mathrm{d}\alpha,$$

(here $j_u^0(x) = j_u^0(\rho,\alpha) = \partial j(\rho,\alpha,u(\rho,\alpha))/\partial u$) can be redefined by

$$\int_{\Omega} j_u^0(x)\, \mathrm{d}W[r_1](\mu;x)\, \mathrm{d}x$$

$$= -1/2\pi \int_0^{2\pi} \int_0^{r(\phi)} j_u^0(\rho,\alpha) \int_0^{2\pi} \mathrm{d}\bar{K}[r_1](\rho,\alpha,\phi)\,[\mu(\phi)-\mu(\alpha)]\; \mathrm{d}\phi\; \rho\, \mathrm{d}\rho\, \mathrm{d}\alpha,$$

as a weakly singular integral.

Proof. At first we fix the direction r_1, show the existence of the directional derivative

$$\mathrm{d}\mathcal{J}[r_1] = \lim_{\varepsilon\to 0} \frac{J(\Omega_\varepsilon;u_\varepsilon) - J(\Omega;u)}{\varepsilon}$$

and compute it. To this end, we split $J(\Omega_\varepsilon;u_\varepsilon) - J(\Omega_0;u)$ as follows:

$$J(\Omega_\varepsilon;u_\varepsilon) - J(\Omega;u) = I_1 + I_2 - I_3,$$

with

$$I_1 = \int_{\Omega_\varepsilon\cap\Omega} \left[j\big(x,u_\varepsilon(x)\big) - j\big(x,u(x)\big)\right] \mathrm{d}x,$$

$$I_2 = \int_{\Omega_\varepsilon\setminus\Omega} j\big(x,u_\varepsilon(x)\big)\, \mathrm{d}x,$$

$$I_3 = \int_{\Omega\setminus\Omega_\varepsilon} j\big(x,u(x)\big)\, \mathrm{d}x.$$

Although the situation for I_2 and I_3 is not equivalent to Lemma 2, we are able to proceed similarly to the proof by using the boundary condition for u_ε on Γ_ε and for u on Γ. We arrive at the first part of (20) with an estimate of $o(\varepsilon)$ for the remainder, because $g_\varepsilon = g|_{\Gamma_\varepsilon} \to g = g|_\Gamma$ uniformly as $\varepsilon \to 0$.

Due to relation (i) of Theorem 3, the second part of (20) makes sense in the way discussed above. Moreover, this part is connected with $\lim\limits_{\varepsilon\to 0} I_1/\varepsilon$. Now the existence of this limit follows by the assumption on j, the properties of u and from relation (ii) of Theorem 3. Therefore $\mathrm{d}J[r_1]$ exists for arbitrary $r_1 \in C_p^2$ and is given by (20).

Fréchet differentiability is ensured along the lines of our standard arguments, i.e.

(i) $\mathrm{d}\mathcal{J}(r)[r_1]$ is linear and continuous (with respect to r_1) (hence it is a Gâteaux derivative),

(ii) the norm of the Gâteaux derivative depends continuously on r, because (20) contains only (at most) weak singular integrals which are continuous with respect to r. ■

In order to simplify the gradient and the standard necessary condition

$$\nabla\mathcal{J}(r)[r_1] \geq 0 \ \text{ for all admissible } \ r_1 \in C_p^2[0, 2\pi],$$

by introducing an adjoint state, it is useful to assume more regular data (especially for the boundary-value field g).

4. Necessary Optimality Conditions for More Regular Data

From $f \in C$ and $g \in C^1$, we can only conclude that $u \in C^{0,1-\varepsilon}(\bar{\Omega})$ and $\mathrm{d}u[r_1] \in L_p(\Omega)$ (but not $u \in C^1(\bar{\Omega})$ and $\mathrm{d}u[r_1] \in C(\bar{\Omega})$). To prove the characterization theorem about $\mathrm{d}u[r_1]$ as the solution to a related boundary-value problem of Dirichlet type, more regularity of the data field g is necessary.

Lemma 1. *For $g \in C^{1,\alpha}(D)$ the directional derivative $\mathrm{d}u[r_1]$ satisfies*

$$\Delta\, \mathrm{d}u[r_1] = 0 \quad in \ \ \Omega$$

$$\mathrm{d}u[r_1](r(\phi), \phi) = r_1(\phi)\left[\frac{\partial g}{\partial \vec{r}}(r(\phi), \phi) - \frac{\partial u}{\partial \vec{r}}(r(\phi), \phi)\right] \quad on \ \ \Gamma = \partial\Omega. \quad (20)$$

Proof. Since $g \in C^{1,\gamma}(D)$, we have $u \in C^{1,\gamma}(\Omega)$ and $u_\varepsilon \in C^{1,\gamma}(\Omega_\varepsilon)$. Therefore the right-hand side of the boundary condition in (20) is well-defined as an element of $C^{0,\gamma}(\Gamma)$. Moreover, we can conclude, in addition to Corollary 2, that $\mathrm{d}u[r_1] \in C^{0,\gamma}(\bar{\Omega})$ (see Appendix B). It remains to compute the boundary values in the sense of

$$\mathrm{d}u[r_1](r(\phi), \phi) = \lim_{\varepsilon\to 0} \frac{u_\varepsilon\big(x_\varepsilon(\phi)\big) - u\big(x_\varepsilon(\phi)\big)}{\varepsilon}, \quad x_\varepsilon(\phi) \in \partial(\Omega \cap \Omega_\varepsilon).$$

We follow Pironneau (1983) and split $[0, 2\pi]$ into three sets M_1^+, M_1^- and M_1^0 according to the sign of r_1:

$$M_1^+ = \{\phi \in [0, 2\pi] | r_1(\phi) > 0\}, \quad M_1^- = \{\phi \in [0, 2\pi] | r_1(\phi) < 0\},$$
$$M_1^0 = \{\phi \in [0, 2\pi] | r_1(\phi) = 0\}.$$

We obtain

$$\begin{aligned} u_\varepsilon\big(r(\phi), \phi\big) - u\big(r(\phi), \phi\big) &= u_\varepsilon\big(r(\phi), \phi\big) - u_\varepsilon\big(r_\varepsilon(\phi), \phi\big) \\ &\quad + g\big(r_\varepsilon(\phi), \phi\big) - g\big(r(\phi), \phi\big) \\ &= \varepsilon r_1(\phi)\left[\frac{\partial g}{\partial \vec{r}} - \frac{\partial u_\varepsilon}{\partial \vec{r}}\bigg|_{r+\theta r_1}\right], \quad \phi \in M_1^+, \end{aligned}$$

and

$$\begin{aligned} u_\varepsilon\big(r_\varepsilon(\phi),\phi\big) - u\big(r_\varepsilon(\phi),\phi\big) &= u\big(r(\phi),\phi\big) - u\big(r_\varepsilon(\phi),\phi\big) \\ &\quad + g\big(r_\varepsilon(\phi),\phi\big) - g\big(r(\phi),\phi\big) \\ &= \varepsilon r_1(\phi)\left[\frac{\partial g}{\partial \vec r} - \frac{\partial u}{\partial \vec r}\bigg|_{r+\theta r_1}\right], \quad \phi \in M_1^-. \end{aligned}$$

For $\phi \in M_1^0$ the definition of the boundary values can be formally extended, since $u_\varepsilon(x_\varepsilon) - u(x_\varepsilon) = g(x_\varepsilon) - g(x_\varepsilon) = 0.$ ■

Remark 15. Owing to the Dirichlet condition for u (i.e. $u - g = 0$ on Γ), we have

$$\begin{aligned} r_1\left[\frac{\partial g}{\partial \vec r} - \frac{\partial u}{\partial \vec r}\right] &= \langle r_1\vec e_r, \vec n\rangle\left[\frac{\partial g}{\partial n} - \frac{\partial u}{\partial n}\right] \\ &= \frac{r_1(\phi)r(\phi)}{\sqrt{r^2(\phi)+r'^2(\phi)}}\left[\frac{\partial g}{\partial n}(r(\phi),\phi) - \frac{\partial u}{\partial n}(r(\phi),\phi)\right]. \end{aligned}$$

Furthermore, $du[r_1] \in C^{0,\gamma}(\bar\Omega)$ is a classical solution to the Laplace equation, which can be expressed in terms of a double-layer potential only, where the associated density satisfies a BIE with the boundary values of (20) as the right-hand side.

The adjoint state p is defined as the solution to the following boundary-value problem:

$$\begin{cases} -\Delta p = j_u^0(\cdot,\cdot) & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{21}$$

Remark 16. The adjoint state can be expressed in terms of a potential representation, too. Moreover, according to the continuity of our right-hand side j_u^0, we have $p \in C^{1,\gamma}(\bar\Omega)$. Substituting Δp into $\nabla\mathcal{J}(r)[r_1]$ and integrating twice by parts, we obtain

$$\begin{aligned} \nabla\mathcal{J}[r_1] &= \int_\Omega -\Delta p\, \mathrm{d}u[r_1]\,\mathrm{d}x + \int_\Gamma \langle r_1\vec e_r, \vec n\rangle j_u^0(x)\,\mathrm{d}S_\Gamma \\ &= \int_\Omega \langle \nabla_x p, \nabla\, \mathrm{d}u[r_1]\rangle\,\mathrm{d}x - \int_\Gamma \frac{\partial p}{\partial n}\mathrm{d}u[r_1]\,\mathrm{d}S_\Gamma + \int_\Gamma \langle r_1\vec e_r, \vec n\rangle\, j_u^0(x)\,\mathrm{d}S_\Gamma \\ &= \int_\Omega -p\,\Delta\, \mathrm{d}u[r_1]\,\mathrm{d}x - \int_\Gamma \frac{\partial p}{\partial n}\mathrm{d}u[r_1]\,\mathrm{d}S_\Gamma \\ &\quad + \int_\Gamma \frac{\partial\, \mathrm{d}u[r_1]}{\partial n} p\,\mathrm{d}S_\Gamma + \int_\Gamma \langle r_1\vec e_r, \vec n\rangle\, j_u^0(x)\,\mathrm{d}S_\Gamma \end{aligned}$$

$$= \int_{\Gamma} \langle r_1 \vec{e}_r, \vec{n} \rangle \left[j_u^0(x) - \frac{\partial p}{\partial n} \left(\frac{\partial g}{\partial n} - \frac{\partial u}{\partial n} \right) \right] \mathrm{d}S_\Gamma$$

$$= \int_0^{2\pi} r_1(\phi) r(\phi) \left[j_u^0(r(\phi), \phi) - \frac{\partial p}{\partial n}(r(\phi), \phi) \left(\frac{\partial g}{\partial n}(r(\phi), \phi) - \frac{\partial u}{\partial n}(r(\phi), \phi) \right) \right] \mathrm{d}\phi.$$

Although the explicit transformation requires the existence of $\partial \mathrm{d}u/\partial n$ at least in some weak sense, the result for $\nabla \mathcal{J}[r_1]$ is valid even for $\mathrm{d}u[r_1] \in C^{0,\gamma}(\bar{\Omega})$. This can be seen as follows: We take a sequence $\{g_n\} \subset C^{1,\gamma}(\Gamma)$ with $g_n \to \mathrm{d}u[r_1]|_\Gamma$ in the sense of $C(\Gamma)$. Due to the maximum principle, for the associated solutions $u_n \in C^{1,\gamma}(\bar{\Omega})$ we have the property $u_n \to \mathrm{d}u[r_1]$ in the sense of $C(\bar{\Omega})$. Therefore

$$\int_{\Omega} -\Delta p \, u_n \, \mathrm{d}x + \int_{\Gamma} \langle r_1 \vec{e}_r, \vec{n} \rangle \, j_u^0(x) \, \mathrm{d}S_\Gamma \to \nabla \mathcal{J}[r_1],$$

and

$$\int_{\Gamma} \langle r_1 \vec{e}_r, \vec{n} \rangle \frac{\partial p}{\partial n} u_n \, \mathrm{d}S_\Gamma \to \int_{\Gamma} \langle r_1 \vec{e}_r, \vec{n} \rangle \frac{\partial p}{\partial n} \left(\frac{\partial g}{\partial n} - \frac{\partial u}{\partial n} \right) \mathrm{d}S_\Gamma,$$

which implies

$$\nabla \mathcal{J}[r_1] = \int_{\Gamma} \langle r_1 \vec{e}_r, \vec{n} \rangle \left[j_u^0(x) - \frac{\partial p}{\partial n} \left(\frac{\partial g}{\partial n} - \frac{\partial u}{\partial n} \right) \right] \mathrm{d}S_\Gamma.$$

Similar results are also known for several approaches (Gillaume and Masmoudi, 1994; Fujii, 1986; Pironneau, 1983; Sokołowski and Zolesio, 1992). As a conclusion, we obtain the necessary optimality condition in the case of a free minimum (all $r_1 \in C^2[0, 2\pi]$ are admissible).

Corollary 3. *Let $\Omega \in C^2$ be a local optimum for all the domains which are star-shaped with respect to a neighbourhood $U_\delta(\mathbf{0})$. If all $r_1 \in C_p^2[0, 2\pi]$, admissible without any additional restriction, then the function*

$$j_u^0(x) - \frac{\partial p}{\partial n} \left(\frac{\partial g}{\partial n} - \frac{\partial u}{\partial n} \right)$$

must vanish on the boundary Γ, i.e.

$$j_u^0(r(\phi), \phi) - \frac{\partial p}{\partial n}(r(\phi), \phi) \left(\frac{\partial g}{\partial n}(r(\phi), \phi) - \frac{\partial u}{\partial n}(r(\phi), \phi) \right) = 0, \quad \phi \in [0, 2\pi].$$

Remark 17. For some applications of shape optimization methods, see the monographs by Haslinger and Neittanmäki (1988), Khludnev and Sokołowski (1997), Pironneau (1983) and Sokołowski and Zolesio (1992). Few additional references are (Fujii and Goto, 1994l; Henrot and Michel, 1991; Kirsch, 1993; Leal and Mota Soares, 1990; Ring, 1995; Roche and Sokołowski, 1996).

Remark 18. The approach presented here is also useful for the computation of higher-order derivatives (Eppler, 2000b). It is of some interest while studying sufficient optimality conditions for shape optimization problems (Eppler, 2000a).

Appendix

A. Proof of Relation (ii) of Theorem 3

As was already discussed, the volume and double-layer potential parts of the solution will be investigated separately. We have to show that

$$\text{(iia)} \qquad \int\limits_{\Omega_\varepsilon \cap \Omega} |V_\varepsilon(f;x) - V(f;x) - \varepsilon \, \mathrm{d}V[r_1](f;x)| \, \mathrm{d}x = o(\varepsilon)$$

$$\text{(iib)} \qquad \int\limits_{\Omega_\varepsilon \cap \Omega} |W_\varepsilon(\mu_\varepsilon;x) - W(\mu;x) - \varepsilon \left\{ W(\mathrm{d}\mu[r_1];x) + \mathrm{d}W[r_1](\mu;x) \right\} | \, \mathrm{d}x = o(\varepsilon).$$

For the proof of (iia), we define $e_1(\varepsilon;x) = 2\pi[V_\varepsilon(f;x) - V(f;x) - \varepsilon \, \mathrm{d}V[r_1](f;x)]$ and rewrite the remainder as

$$\begin{aligned}
e_1(\varepsilon;x) &= \int\limits_{\Omega_\varepsilon \setminus \Omega} \ln|x-y| f(y) \, \mathrm{d}y - \int\limits_{\Omega \setminus \Omega_\varepsilon} \ln|x-y| f(y) \, \mathrm{d}y \\
&\quad - \varepsilon \int\limits_0^{2\pi} r(\phi) r_1(\phi) \ln|x - \xi(\phi)| f\big(\xi(\phi)\big) \, \mathrm{d}\phi \\
&= \int\limits_0^{2\pi} \int\limits_{r(\phi)}^{r_\varepsilon(\phi)} \ln|x - y(\rho,\phi) f\big(y(\rho,\phi)\big) \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \\
&\quad - \varepsilon \int\limits_0^{2\pi} r(\phi) r_1(\phi) \ln|x - \xi(\phi)| f\big(\xi(\phi)\big) \, \mathrm{d}\phi \\
&= \int\limits_0^{2\pi} r_1(\phi) \int\limits_0^{\varepsilon} \big[r(\phi) + \tau r_1(\phi)\big] \ln|x - y(\tau,\phi)| \\
&\quad \times f\big(y(\tau,\phi)\big) - r(\phi) \ln|x - \xi(\phi)| f\big(\xi(\phi)\big) \, \mathrm{d}\tau \, \mathrm{d}\phi \\
&= \int\limits_0^{2\pi} r_1(\phi) r(\phi) \int\limits_0^{\varepsilon} \Big[\ln|x - y(\tau,\phi)| f\big(y(\tau,\phi)\big) - \ln|x - \xi(\phi)| f\big(\xi(\phi)\big) \Big] \, \mathrm{d}\tau \, \mathrm{d}\phi \\
&\quad + \int\limits_0^{2\pi} r_1^2(\phi) \int\limits_0^{\varepsilon} \tau \ln|x - y(\tau,\phi)| f\big(y(\tau,\phi)\big) \, \mathrm{d}\tau \, \mathrm{d}\phi
\end{aligned}$$

$$= \int_0^{2\pi} r_1(\phi) r(\phi) \int_0^{\varepsilon} \Big[ln|x - y(\tau,\phi)| - \ln|x - \xi(\phi)| \Big] f\big(y(\tau,\phi)\big)\, \mathrm{d}\tau\, \mathrm{d}\phi$$

$$+ \int_0^{2\pi} r_1(\phi) r(\phi) \int_0^{\varepsilon} \ln|x - \xi(\phi)| \Big[f\big(y(\tau,\phi)\big) - f\big(\xi(\phi)\big) \Big]\, \mathrm{d}\tau\, \mathrm{d}\phi$$

$$+ \int_0^{2\pi} r_1^2(\phi) \int_0^{\varepsilon} \tau \ln|x - y(\tau,\phi)| f\big(y(\tau,\phi)\big)\, \mathrm{d}\tau\, \mathrm{d}\phi, \qquad x \in \Omega_\varepsilon \cap \Omega$$

Now we additionally show that (also for continuous f only) the remainder satisfies an estimate of the type $|e_1(\varepsilon, x)| \le co(\varepsilon)$ *uniformly* for all $x \in \Omega_\varepsilon \cap \Omega$ (i.e. c does not depend on x and ε). This is in fact a uniform C-type estimate implying clearly a related estimate for the L_1-norm. We have

$$|e_1(\varepsilon, x)| = \big| V_\varepsilon(f;x) - V(f;x) - \varepsilon\, \mathrm{d}V[r_1](f;x) \big|\, \mathrm{d}x$$

$$\le \|f\|\, \|r r_1\| \int_0^{2\pi} \int_0^{\varepsilon} \big| \ln|x(\rho,\alpha) - y(\tau,\phi)| - \ln|x(\rho,\alpha) - \xi(\phi)| \big|\, \mathrm{d}\tau\, \mathrm{d}\phi$$

$$+ \|r r_1\| \varepsilon \delta_f(\varepsilon) \int_0^{2\pi} |\ln|x(\rho,\alpha) - \xi(\phi)||\, \mathrm{d}\phi$$

$$+ \varepsilon \|f\| \|r_1\|^2 \int_0^{2\pi} \int_0^{\varepsilon} |\ln|x(\rho,\alpha) - y(\tau,\phi)||\, \mathrm{d}\tau\, \mathrm{d}\phi$$

$$\le c(f, r, r_1) \int_0^{2\pi} \int_0^{\varepsilon} \big| \ln|x(\rho,\alpha) - y(\tau,\phi)| - \ln|x(\rho,\alpha) - \xi(\phi)| \big|\, \mathrm{d}\tau\, \mathrm{d}\phi$$

$$+ c_1(\Omega) \Big[\varepsilon \delta_f(\varepsilon) \|r r_1\| + \varepsilon^2 \|f\| \|r_1\|^2 \Big],$$

where $\delta_f(\cdot)$ is the continuity modulus of f, because $\int_0^{2\pi} |\ln|x(\rho,\alpha) - \xi(\phi)||\, \mathrm{d}\phi \le c_1(\Omega)$ and $\int_0^{2\pi} |\ln|x(\rho,\alpha) - y(\tau,\phi)||\, \mathrm{d}\phi \le c_1(\Omega)$, *uniformly* for all $x \in \Omega$ and $\tau \in (0, \varepsilon)$ (these are at most weakly singular integrals).

Moreover, we have the estimate (uniformly with respect to τ, $\xi_t(\phi) := \xi(\phi) + t[\xi_\varepsilon(\phi) - \xi(\phi)]$, $t \in [0,1]$)

$$\big| \ln|x - y_\tau| - \ln|x - \xi| \big| = \big| \big\langle \nabla \ln\big(x - \xi + \eta[\xi_0 - y_\tau]\big), y_\tau - \xi_0 \big\rangle \big|$$

$$\le \frac{|\xi_\varepsilon(\phi) - \xi_0(\phi)|}{\min|x - \xi_t(\phi)|}.$$

Introducing the set

$$B_\varepsilon(\phi) := \left\{ x \in \Omega_\varepsilon \cap \Omega | \min\left\{|x-\xi_\varepsilon|, |x-\xi|\right\} \geq \frac{\sqrt{5}}{2}\varepsilon|r_1(\phi)| = \frac{\sqrt{5}}{2}|\xi_\varepsilon(\phi)-\xi(\phi)| \right\},$$

we have

$$\begin{aligned} &|x-\xi_t| \geq |\xi_\varepsilon - \xi_0|, \\ &|x-\xi_0| \leq |\xi_\varepsilon - \xi_0| + |x-\xi_t| \leq 2|x-\xi_t|. \end{aligned} \tag{A1}$$

As a result, for all $x = x(\rho,\alpha) \in \Omega_\varepsilon \cap \Omega$ with $\mathrm{d}(x,\Gamma_\varepsilon^0) \geq \varepsilon\sqrt{5}/2\|r_1\|_C$ (where $\partial(\Omega_\varepsilon \cap \Omega) := \Gamma_\varepsilon^0 = \{\xi_\varepsilon^0(\phi) = (r_\varepsilon^0(\phi))\binom{\cos\phi}{\sin\phi},\ \phi \in [0,2\pi]\}$, $r_\varepsilon^0(\phi) = \min\{r(\phi), r_\varepsilon(\phi)\} = r + \varepsilon(1-\mathrm{sgn}(r_1))r_1/2\}$, we have

$$\int_0^{2\pi} |\ln|x(\rho,\alpha) - y(\tau,\phi)| - \ln|x(\rho,\alpha)-\xi(\phi)||\,\mathrm{d}\phi$$

$$\leq 2\int_0^{2\pi} \frac{|\xi_\varepsilon(\phi)-\xi(\phi)|}{|x-\xi(\phi)|}\,\mathrm{d}\phi \leq 2\varepsilon^\lambda \int_0^{2\pi} \frac{\mathrm{d}\phi}{|x-\xi(\phi)|^\lambda},$$

where the last integral is at most weakly singular and therefore uniformly bounded.

For x 'close' to the boundary of $\Omega_\varepsilon \cap \Omega$ $(d(x,\Gamma_\varepsilon^0) < \varepsilon\sqrt{5}\|r_1\|_C/2)$, the situation is more complicated, because the estimates above do not hold for all $\phi \in [0,2\pi]$. Therefore, we split the interval $[0,2\pi]$ into $A_\varepsilon(\alpha) := \{\phi \in [0,2\pi]| \min\{|x-\xi_\varepsilon(\phi)|, |x-\xi(\phi)|\} \geq \varepsilon\sqrt{5}\|r_1\|_C/2\}$ and $[0,2\pi]\setminus A_\varepsilon(\alpha)$. On the other set (A1) is satisfied and the estimate above holds. For $A_\varepsilon(\alpha)$ we have

$$A_\varepsilon(\alpha) \subset [\alpha - c_1\varepsilon, \alpha + c_1\varepsilon] \text{ and } c_2\varepsilon \geq |x(\rho,\alpha) - y(\tau,\phi)| \geq c_3|\alpha-\phi|,$$

which implies (for a sufficiently small ε and $\phi \in A_\varepsilon(\alpha)$)

$$|\ln|x(\rho,\alpha)-y(\tau,\phi)| - \ln|x(\rho,\alpha)-\xi(\phi)|| \leq 2|\ln c_3|\alpha-\phi||.$$

Hence for all $\tau \in (0,\varepsilon)$ $(0 < \lambda < 1)$ we obtain

$$\int_{A_\varepsilon(\alpha)} |\ln|x(\rho,\alpha)-y(\tau,\phi)| - \ln|x(\rho,\alpha)-\xi_0(\phi)||\,\mathrm{d}\phi$$

$$\leq 4\int_0^{c\varepsilon} |\ln\phi|\,\mathrm{d}\phi + O(\varepsilon) = o(\varepsilon^\lambda).$$

Consequently, we get

$$\int_0^{2\pi}\int_0^{\varepsilon} \left|\ln|x(\rho,\alpha)-y(\tau,\phi)| - \ln|x(\rho,\alpha)-\xi(\phi)|\right| \,\mathrm{d}\tau\,\mathrm{d}\phi \le c\varepsilon^{1+\lambda},$$

where the constant c can be chosen independently of x and ε.

For the proof of (iib), we split the double-layer potential part as follows:

$$W_\varepsilon(\mu_\varepsilon;x) - W(\mu;x) - \varepsilon\{W(\mathrm{d}\mu[r_1];x) + \mathrm{d}W[r_1](\mu;x)\}$$
$$= e_2(\varepsilon,x) + e_3(\varepsilon,x) + e_4(\varepsilon,x),$$

where

$$e_2(\varepsilon,x) = -\frac{1}{2\pi}\int_0^{2\pi} \left[\bar{K}_\varepsilon(x,\phi) - \bar{K}(x,\phi)\right]\left[\mu_\varepsilon(\phi) - \mu(\phi)\right] \mathrm{d}\phi,$$

$$e_3(\varepsilon,x) = -\frac{1}{2\pi}\int_0^{2\pi} \left[\bar{K}_\varepsilon(x,\phi) - \bar{K}(x,\phi) - \varepsilon d\bar{K}[r_1](x,\phi)\right]\mu(\phi)\,\mathrm{d}\phi,$$

$$e_4(\varepsilon,x) = -\frac{1}{2\pi}\int_0^{2\pi} \bar{K}(x,\phi)\left[\mu_\varepsilon(\phi) - \mu(\phi) - \varepsilon\,\mathrm{d}\mu[r_1](\phi)\right] \mathrm{d}\phi.$$

For the last part, we have $|e_4(\varepsilon,x)| = o(\varepsilon)$ *uniformly* for all $x \in \Omega$ by the results of Section 4.1 and the mapping properties of the double-layer potential. For $e_2(\varepsilon,x)$, we have

$$\mu_\varepsilon(\phi) - \mu(\phi) = \varepsilon\,\mathrm{d}\mu(\phi) + o(\varepsilon) \quad \textit{uniformly} \text{ for all } \phi \in [0,2\pi].$$

From (10) we additionally get the splitting $e_2(\varepsilon,x) = e_2^1(\varepsilon,x) + e_2^2(\varepsilon,x)$ by

$$\bar{K}_\varepsilon\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\phi\right) - \bar{K}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\phi\right)$$
$$= \frac{(\xi_\varepsilon(\phi)-x)\cdot\vec{a}_\varepsilon(\phi)}{|x-\xi_\varepsilon(\phi)|^2} - \frac{(\xi(\phi)-x)\cdot\vec{a}(\phi)}{|x-\xi(\phi)|^2}$$
$$= \left\langle \frac{(\xi(\phi)-x)}{|x-\xi(\phi)|^2}, \vec{a}_\varepsilon(\phi) - \vec{a}(\phi)\right\rangle$$
$$+ \left\langle \frac{(\xi_\varepsilon(\phi)-x)}{|x-\xi_\varepsilon(\phi)|^2} - \frac{(\xi(\phi)-x)}{|x-\xi(\phi)|^2}, \vec{a}_\varepsilon(\phi) - \vec{a}(\phi) + \vec{a}(\phi)\right\rangle.$$

Moreover, from $r_\varepsilon(\phi) = r(\phi) + \varepsilon r_1(\phi)$ we have

$$\vec{a}_\varepsilon(\phi) - \vec{a}(\phi) = \varepsilon \begin{pmatrix} r_1(\phi)\cos\phi + r_1'(\phi)\sin\phi \\ r_1(\phi)\sin\phi - r_1'(\phi)\cos\phi \end{pmatrix}.$$

The L_1-norm of the first part $e_2^1(\varepsilon, x)$ of $e_2(\varepsilon, x)$ can be estimated as follows:

$$e_2^1 = \int\limits_{\Omega_\varepsilon \cap \Omega} |e_2^1(\varepsilon, x)|\,\mathrm{d}x$$

$$= \int\limits_{\Omega_\varepsilon \cap \Omega} \Big| \int\limits_0^{2\pi} \left\langle \frac{(\xi(\phi) - x)}{|x - \xi(\phi)|^2}, \vec{a}_\varepsilon(\phi) - \vec{a}(\phi) \right\rangle \frac{\mu_\varepsilon(\phi) - \mu(\phi)}{2\pi}\,\mathrm{d}\phi \Big|\,\mathrm{d}x$$

$$= \int\limits_{\Omega_\varepsilon \cap \Omega} \Big| \frac{\varepsilon}{2\pi} \int\limits_0^{2\pi} \left\langle \frac{(\xi(\phi) - x)}{|x - \xi(\phi)|^2}, \begin{pmatrix} r_1(\phi)\cos\phi + r_1'(\phi)\sin\phi \\ r_1(\phi)\sin\phi - r_1'(\phi)\cos\phi \end{pmatrix} \right\rangle$$

$$\times \left[\varepsilon\,\mathrm{d}\mu[r_1](\phi) + o(\varepsilon)\right]\mathrm{d}\phi \Big|\,\mathrm{d}x$$

$$\leq \left[\frac{\varepsilon^2}{2\pi} + o(\varepsilon^2)\right] \int\limits_0^{2\pi} \int\limits_\Omega \frac{c\,\mathrm{d}x}{|x - \xi(\phi)|}\,\mathrm{d}\phi \leq \tilde{c}\varepsilon^2.$$

For $e_2^2(\varepsilon, x)$ we discuss only $\langle (\xi_\varepsilon(\phi) - x)/|x - \xi_\varepsilon(\phi)|^2 - (\xi(\phi) - x)/|x - \xi(\phi)|^2, \vec{a}(\phi)\rangle$. We have

$$\left| \frac{(\xi_\varepsilon(\phi) - x)}{|x - \xi_\varepsilon(\phi)|^2} - \frac{(\xi(\phi) - x)}{|x - \xi(\phi)|^2} \right| = \left| \int\limits_0^1 \langle F_\xi[\xi + t(\xi_\varepsilon - \xi)], \xi_\varepsilon - \xi \rangle\,\mathrm{d}t \right|$$

$$\leq 2|\xi_\varepsilon - \xi| \int\limits_0^1 \frac{\mathrm{d}t}{|x - \xi_t|^2} \leq 2 \frac{|\xi_\varepsilon - \xi|^\lambda |\xi_\varepsilon - \xi|^{1-\lambda}}{\min|x - \xi_t|^2}.$$

In order to estimate e_2^2, we change the ϕ- and x-integration and remember that the estimate (A1) holds on $B_\varepsilon(\phi)$ for all $t \in [0, 1]$. Therefore, we arrive at

$$e_2^2 \leq \frac{[2\varepsilon]^{1+\lambda}}{\pi} \int\limits_0^{2\pi} |a(\phi)||r_1(\phi)|^\lambda [\mathrm{d}\mu[r_1](\phi) + \delta(\varepsilon)] \int\limits_{B_\varepsilon(\phi)} \frac{\mathrm{d}x}{|x - \bar{\xi}(\phi)|^{1+\lambda}}\,\mathrm{d}\phi$$

$$+ \frac{\varepsilon}{2\pi} \int\limits_0^{2\pi} [\mathrm{d}\mu(\phi) + \delta(\varepsilon)] \int\limits_{R_\varepsilon(\phi)} \left[|\bar{K}_\varepsilon(x, \phi)| + |\bar{K}(x, \phi)|\right]\mathrm{d}x\,\mathrm{d}\phi$$

$$\leq c_1(a, r_1, \mathrm{d}\mu)\varepsilon^{1+\lambda} \int\limits_0^{2\pi} \frac{\mathrm{d}x}{|x - \bar{\xi}(\phi)|^{1+\lambda}}\,\mathrm{d}\phi + c_2(a, r_1)\varepsilon^2 \int\limits_0^{2\pi} [\mathrm{d}\mu(\phi) + \delta(\varepsilon)]\,\mathrm{d}\phi,$$

where $R_\varepsilon(\phi) := \Omega_\varepsilon \cap \Omega \setminus B_\varepsilon(\phi)$ and the last estimate results from

$$\int\limits_{R_\varepsilon(\phi)} \left[|\bar{K}_\varepsilon(x,\phi)| + |\bar{K}(x,\phi)|\right] \, \mathrm{d}x \leq 4 \int\limits_0^{2\pi} \int\limits_0^{\varepsilon c(\phi)} \frac{c_2(a,\dots)}{\rho} \rho \, \mathrm{d}\rho \, \mathrm{d}\alpha \leq \tilde{c}\varepsilon.$$

For the discussion of $e_3(\varepsilon, x)$, we need the 'singularity reduction' (see the proof of Theorem 3)

$$e_3(\varepsilon, x) = -\frac{1}{2\pi} \int\limits_0^{2\pi} \left[\bar{K}_\varepsilon(x,\phi) - \bar{K}(x,\phi) - \varepsilon \, \mathrm{d}\bar{K}[r_1](x,\phi)\right] [\mu(\phi) - \mu(\alpha)] \, \mathrm{d}\phi,$$

where we used (19) and

$$-\int\limits_0^{2\pi} \bar{K}_\varepsilon(x,\phi) \, \mathrm{d}\phi = -\int\limits_0^{2\pi} \bar{K}(x,\phi) \, \mathrm{d}\phi = 2\pi.$$

The Taylor expansion of $[\bar{K}_\varepsilon(x,\phi) - \bar{K}(x,\phi) - \varepsilon \, \mathrm{d}\bar{K}[r_1](x,\phi)]$ now leads to 'worst-case terms' of the type

$$\left|\bar{K}_\varepsilon(x,\phi) - \bar{K}(x,\phi) - \varepsilon \, \mathrm{d}\bar{K}[r_1](x,\phi)\right| \leq c(\phi) \left\{ \frac{|\xi_\varepsilon - \xi|^2}{\min |x - \xi_t|^3} + \cdots \right\}.$$

One order of the singularity can be reduced by the $[\mu(\phi) - \mu(\alpha)]$ term and, additionally, a procedure similar to the discussion of $e_2(\varepsilon, x)$ can be applied. More precisely, we have

$$\begin{aligned} &\bar{K}_\varepsilon(x,\phi) - \bar{K}(x,\phi) - \varepsilon \, \mathrm{d}\bar{K}[r_1](x,\phi) \\ &\quad = \left\langle \frac{(\xi_\varepsilon(\phi) - x)}{|x - \xi_\varepsilon(\phi)|^2} - \frac{(\xi(\phi) - x)}{|x - \xi(\phi)|^2}, \vec{a}_\varepsilon - \vec{a} \right\rangle \\ &\quad\quad + \left\langle \frac{(\xi_\varepsilon - x)}{|x - \xi_\varepsilon|^2} - \frac{(\xi - x)}{|x - \xi|^2} - \left[\frac{(\xi_\varepsilon - \xi)}{|x - \xi|^2} - 2\frac{(\xi - x)(\xi - x)^T(\xi_\varepsilon - \xi)}{|x - \xi|^4} \right], \vec{a} \right\rangle, \end{aligned}$$

where

$$\left\langle \frac{(\xi_\varepsilon(\phi) - x)}{|x - \xi_\varepsilon(\phi)|^2} - \frac{(\xi(\phi) - x)}{|x - \xi(\phi)|^2}, \vec{a}_\varepsilon - \vec{a} \right\rangle [\mu(\phi) - \mu(\alpha)]$$

can be estimated just as e_2^1 and the second part similarly to e_2^2:

$$\left\langle \frac{(\xi_\varepsilon - x)}{|x - \xi_\varepsilon|^2} - \frac{(\xi - x)}{|x - \xi|^2} - \left[\frac{(\xi_\varepsilon - \xi)}{|x - \xi|^2} - 2\frac{(\xi - x)(\xi - x)^T(\xi_\varepsilon - \xi)}{|x - \xi|^4} \right], \vec{a} \right\rangle [\mu(\phi) - \mu(\alpha)].$$

This completes the proof. ∎

A more detailed investigation of the proof shows that the uniform L_1-estimate can be improved at least to an estimate with respect to L_p for $p < 2$. Furthermore, by a more careful investigation of Cauchy principal values and their limiting behaviour, it seems to us that it is possible to show (according to $\nabla u \in L_p$, cf. Remark 1) that

(i) $\mathrm{d}u[r_1](\cdot) \in L_p$, and

(ii) $\sum_{k=1}^{4} e_k(\varepsilon;\cdot) \in L_p$

for every $p \in [1,\infty)$, if we use our standard assumptions $f \in C$ and $g \in C^1$. More regularity can be expected for more regular data (cf. Section 4).

B. Regularity of $\mathrm{d}u[r_1]$ for $g \in C^{1,\gamma}(D)$

Lemma 4. *For $f \in C(D)$ and $g \in C^{1,\gamma}(D)$, the directional derivative $\mathrm{d}u[r_1]$ is Hölder continuous on $\bar{\Omega}$, i.e. $\mathrm{d}u[r_1] \in C^{0,\alpha}(\bar{\Omega})$.*

Proof. Owing to $\Gamma \in C^2$ and the regularity of the data, we have $\mu \in C^{1,\gamma}$ and $\mathrm{d}\mu \in C^{0,\gamma}$ from the regularity of the right-hand sides of the related boundary integral equations. More precisely, on $[0,2\pi]$ we have $g|_\Gamma$, $V(f)|_\Gamma \in C^{1,\alpha}$, $\mathrm{d}g[r_1]$, $\mathrm{d}_\Gamma V[r_1] \in C^{0,\gamma}$ and $\mathrm{d}\mathbf{K}\mu \in C^{1,\gamma}$ (cf. Remark 7). As an immediate consequence, for the first two parts we get

$$\mathrm{d}V[r_1](f;\cdot) \in C^{0,\gamma}(\bar{\Omega}) \quad \text{and} \quad W(\mathrm{d}\mu[r_1];\cdot) \in C^{0,\gamma}(\bar{\Omega}).$$

It remains to prove that $\mathrm{d}W[r_1](\mu,\cdot) \in C^{0,\gamma}(\bar{\Omega})$ for $\mu \in C^{1,\gamma}$. Obviously, the crucial region for this property is close to the boundary. Therefore, in the sequel, we assume that

$$x \in B_\delta(\Gamma) := \{x = (\rho,\beta) \in \Omega \mid d(x,\Gamma) < \delta\}, \tag{B1}$$

for a sufficiently small fixed $\delta > 0$. Moreover, the following is satisfied for $x \in \bar{B}_\delta(\Gamma)$:

(i) $\vec{d}(x) = \vec{d}(\beta(x)) := r_1(\beta)\vec{e}_r(\beta)$ is a smooth function of x;

(ii) for $\rho = \rho(x)$ we have $r(\beta) - \rho < M\delta$ for some constant $M > 0$;

(iii) from $W(\mu[r_1];\cdot) \in C^{1,\gamma}(\bar{\Omega})$ for $\mu \in C^{1,\gamma}$, cf. Kress (1989), and (i) it follows that

$$\langle \nabla_x W(\mu[r_1];\cdot), \vec{d}(\cdot)\rangle \in C^{0,\gamma}(\bar{B}_\delta(\Gamma)).$$

As will be seen below, the formal 'singularity behaviour' of the last expression is very similar to that of $\mathrm{d}W[r_1](\mu,\cdot)$. To this end, recall the structure of the kernel $\mathrm{d}\bar{K}$ of $\mathrm{d}W$:

$$\mathrm{d}\bar{K}[r_1](x,\phi) = r_1(\phi)\langle \nabla^2_{\xi\xi} \ln|x-\xi(\phi)|\vec{e}_r(\phi), \vec{a}(\phi)\rangle + \langle \nabla_\xi \ln|x-\xi(\phi)|, \mathrm{d}\vec{a}[r_1](\phi)\rangle.$$

The first part has a singularity of order 2 (for $x = x(\beta) \to \xi(\phi)$), whereas the other is only strongly singular (order 1). One order can be reduced by (19), i.e.

$$\mathrm{d}W[r_1](\mu;x) = -\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\bar{K}[r_1](x,\phi)\big[\mu(\phi)-\mu(\beta)\big]\,\mathrm{d}\phi$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} \big[\mu(\phi)-\mu(\beta)\big]\Big\{\big\langle\nabla_\xi \ln|x-\xi(\phi)|,\ \mathrm{d}\vec{a}[r_1](\phi)\big\rangle$$

$$+\, r_1(\phi)\big\langle\nabla^2_{\xi\xi}\ln|x-\xi(\phi)|\cdot\vec{e}_r(\phi),\vec{a}(\phi)\big\rangle\Big\}\,\mathrm{d}\phi.$$

Moreover, a comparison to the (spatial) derivative (with $\vec{d}(\beta) := r_1(\beta)\vec{e}_r(\beta)$)

$$\big\langle\nabla_x W(\mu;x),\vec{d}(\beta)\big\rangle = -\frac{1}{2\pi}\int_0^{2\pi} \big\langle\nabla^2_{\xi x}\ln|x-\xi(\phi)|\vec{d}(\beta),\vec{a}(\phi)\big\rangle\mu(\phi)\,\mathrm{d}\phi$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} \big\langle\nabla^2_{\xi x}\ln|x-\xi(\phi)|\vec{d}(\beta),\vec{a}(\phi)\big\rangle\big[\mu(\phi)-\mu(\beta)\big]\,\mathrm{d}\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \big\langle\nabla^2_{\xi\xi}\ln|x-\xi(\phi)|\vec{d}(\beta),\vec{a}(\phi)\big\rangle\big[\mu(\phi)-\mu(\beta)\big]\,\mathrm{d}\phi$$

due to $\nabla_x W(\mathbf{1};x)\equiv 0$ and $\nabla^2_{\xi x}E(x,\xi) = -\nabla^2_{\xi\xi}E(x,\xi)$) shows that the difference $D(\mu;x) := \mathrm{d}W[r_1](\mu;x) - \langle\nabla_x W(-\mu;x),\vec{d}(\beta)\rangle$ has the structure

$$D(\mu;x) = \frac{1}{2\pi}\int_0^{2\pi} i_1(x,\phi)+i_2(x,\phi)\,\mathrm{d}\phi,$$

where

$$i_1(x,\phi) = \big[\mu(\phi)-\mu(\beta)\big]\big\langle\nabla_\xi\ln|x-\xi(\phi)|,d\vec{a}[r_1](\phi)\big\rangle,$$

$$i_2(x,\phi) = \big[\mu(\phi)-\mu(\beta)\big]\big\langle\nabla^2_{\xi\xi}\ln|x-\xi(\phi)|(\vec{d}(\phi)-\vec{d}(\beta)),\vec{a}(\phi)\big\rangle.$$

Because $i_{1/2}$ can be continuously extended to $\bar{B}_\delta(\Gamma)\times[0,2\pi]$, we have $D(\mu;\cdot)\in C(\bar{\Omega})$. To show the Hölder continuity of D, we use the bijection $\phi\leftrightarrow\xi(\phi)$ and reformulate

$$D(\mu;x) = \int_\Gamma \hat{i}_1(x,\xi)+\hat{i}_2(x,\xi)\,\mathrm{d}S_\xi,$$

where $\hat{i}_1$ and $\hat{i}_2$ are related to i_1 and i_2, respectively. Furthermore, for both the integrands we have

$$|\nabla_x\hat{i}(x,\xi)| \le \frac{M}{|x-\xi|},\quad x\in B_\delta(\Gamma),$$

from which we conclude that

$$|\hat{\imath}(x,\xi)-\hat{\imath}(y,\xi)| \le \frac{M|x-y|}{|\xi-\eta(x,y)|}, \quad \eta(x,y)\in[x,y].$$

In order to estimate (for given $x,y\in B_\delta(\Gamma)$)

$$|f(x)-f(y)| = \Big|\int_\Gamma \hat{\imath}(x,\xi)-\hat{\imath}(y,\xi)\,\mathrm{d}S_\xi\Big| \le \int_\Gamma \big|\hat{\imath}(x,\xi)-\hat{\imath}(y,\xi)\big|\,\mathrm{d}S_\xi,$$

we split Γ into

$$B := \left\{\xi\in\Gamma \Big| |\xi-\frac{x+y}{2}| \le 4|x-y|\right\}.$$

and $\Gamma\backslash B$. Obviously, we have $\operatorname{mes}(B)\le L|x-y|$ for some $L>0$ and, for $\xi\in\Gamma\backslash B$,

(i) $|\xi-\eta(x,y)|\ge 3|x-y|$, $|\xi-x|\ge 3|x-y|$, and

(ii) $\frac{4}{3}|x-\xi| \ge |\xi-\eta(x,y)| \ge \frac{2}{3}|x-\xi|$.

We obtain

$$\begin{aligned}|f(x)-f(y)| &\le \int_B |\hat{\imath}(x,\xi)-\hat{\imath}(y,\xi)|\,\mathrm{d}S_\xi + \int_{\Gamma\backslash B} |\hat{\imath}(x,\xi)-\hat{\imath}(y,\xi)|\,\mathrm{d}S_\xi \\ &\le 2ML|x-y| + |x-y|^\gamma \int_{\Gamma\backslash B} \frac{3M|\xi)-\eta(y,x)|^{1-\gamma}}{|\xi-\eta(x,y)|}\,\mathrm{d}S_\xi \\ &\le \tilde{L}|x-y| + |x-y|^\gamma \int_\Gamma \frac{\tilde{M}\,\mathrm{d}S_\xi}{|\xi-x|^\gamma},\end{aligned}$$

which ensures the Hölder continuity of $D(\mu;\cdot)$, because the last integral above is only weakly singular and hence uniformly bounded. ■

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Received: 16 November 1999
Revised: 17 August 2000