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MOTION PLANNING, EQUIVALENCE, INFINITE DIMENSIONAL SYSTEMS

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Motion planning, i.e., steering a system from one state to another, is a basic question in automatic control. For a certain class of systems described by ordinary differential equations and called flat systems (Fliess *et al.*, 1995; 1999a), motion planning admits simple and explicit solutions. This stems from an explicit description of the trajectories by an arbitrary time function y, the flat output, and a finite number of its time derivatives. Such explicit descriptions are related to old problems on Monge equations and equivalence investigated by Hilbert and Cartan. The study of several examples (the car with n-trailers and the non-holonomic snake, pendulums in series and the heavy chain, the heat equation and the Euler-Bernoulli flexible beam) indicates that the notion of flatness and its underlying explicit description can be extended to infinite-dimensional systems. As in the finite-dimensional case, this property yields simple motion planning algorithms via operators of compact support. For the non-holonomic snake, such operators involve non-linear delays. For the heavy chain, they are defined via distributed delays. For heat and Euler-Bernoulli systems, their supports are reduced to a point and their definition domain coincides with the set of Gevrey functions of order 2.

Keywords: infinite dimensional control systems, motion planning, flatness, absolute equivalence, Pfaffian systems, delay systems, Gevrey functions

1. Introduction

The idea underlying equivalence and flatness (Fliess *et al.*, 1999a)—a one-to-one correspondence between trajectories of systems—is not restricted to control systems described by *ordinary* differential equations. It can be adapted to delay differential systems and to partial differential equations with boundary control. Of course, there are many more technicalities and the picture is far from clear. Nevertheless, this new point of view seems promising for the design of control laws. In this paper, we sketch some recent developments in this direction.

We consider three kinds of systems: nonholonomic, pendulum, and diffusion systems. Each of them admits two families of models: finite-dimensional ones and infinite-dimensional ones. The flat output, well defined in the finite-dimensional case, admits

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also a natural equivalent in the infinite-dimensional case. In a certain sense, the finite and infinite descriptions are thus equivalent, and similar motion planning algorithms can be constructed.

Thus flatness admits an infinite-dimensional extension. The systems examined here in detail suggest to us the following setting in terms of operators admitting compact supports with respect to the time t. Take $\Omega \ni x \mapsto X(x,t) \in \mathbb{R}^q$, defined on an open and smooth domain Ω of $\mathbb{R}^n$. Assume that X is a solution of a square partial differential system $P(X) = 0$ on Ω satisfying on $\partial\Omega$ the boundary condition $L(X,u) = 0$, with $t \mapsto u(t) \in \mathbb{R}^m$ the control. Roughly speaking, an explicit parameterization consists in defining compact support operators $\mathcal{A}_x$, $x \in \Omega$, and $\mathcal{B}$ with a common domain of definition $\mathcal{D}$ including enough time functions $t \mapsto y(t) \in \mathbb{R}^m$ (density, partition of unity, stability by addition and multiplication, etc.) such that $\Xi : (x,t) \mapsto \mathcal{A}_x y|_t$ and $\upsilon : t \mapsto \mathcal{B}y|_t$ with $t \mapsto y(t) \in \mathbb{R}^m$ belonging to $\mathcal{D}$ satisfy automatically $P(\Xi) = 0$ and $L(\Xi,\upsilon) = 0$. Finding $[0,T] \ni t \mapsto u(t)$ $(T > 0)$, steering X from $X(x,0) = X_0(x)$ to $X(x,T) = X_1(x)$ reduces then to finding $t \mapsto y(t)$ when $y(t)$ is prescribed by the initial state X_0 (resp. final state X_1) for t small enough (resp. large enough). Similarly, equivalence between two systems $(P(X) = 0, L(X,u) = 0)$ and $(Q(Z) = 0, M(Z,v) = 0)$ could be defined via compact support operators exchanging solutions.

Such a setting requires precise definitions. Using module theory and the notion of π-freeness (Mounier, 1995) is a first possibility. Notice that parameterizations developed in (Pommaret, 1978; 1995) deal with under-determined systems of PDE's, whereas here $P(X) = 0$ admits the same number of unknowns and equations. We shall concentrate here on three typical kinds of systems where flatness admits a direct infinite-dimensional extension.

2. Non-Holonomic Systems

Many mobile robots such as those considered in (Campion *et al.*, 1996; Murray and Sastry, 1993; Tilbury, 1994) admit the same structure. They are flat, and the flat output corresponds to the Cartesian coordinates of a special point. Starting from the classical n-trailer systems (Fliess *et al.*, 1995; 1997; Rouchon *et al.*, 1993a; 1993b), we show that, when n, the number of trailers, tends to infinity, the system tends to a trivial delay system, the non-holonomic snake. Invariance with respect to rotations and translations makes very natural the use of Frénet formulae and curve parameterization with respect to the arc length instead of time (see (Martin *et al.*, 1997; Rouchon and Rudolph, 1999) for relations between flatness and physical symmetries). The study of such systems gives us the opportunity to recall links with an old problem, stated by Hilbert (1912) and investigated by Cartan (1914), on Pfaffian systems, Goursat normal forms and (absolute) equivalence.

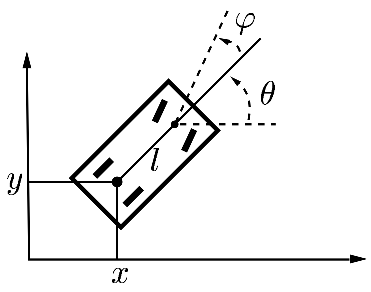

Fig. 1. Kinematics of a car.

2.1. Car

The conditions of rolling without slipping yield (see Fig. 1 for the notation)

$$\begin{cases} \dot{x} = v\cos\theta, \\ \dot{y} = v\sin\theta, \\ \dot{\theta} = \dfrac{v}{l}\tan\varphi, \end{cases} \tag{1}$$

where v, the velocity, and φ, the steering angle, are the two controls. Geometrically, these equations mean that the angle θ gives the direction of the tangent to the curve followed by P, the point of coordinates (x, y), and that $\tan\varphi/l$ corresponds to the curvature of this curve:

$$v = \pm\|\dot{P}\|, \quad \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \dot{P}/v, \quad \tan\varphi = l\det(\ddot{P}\dot{P})/v\sqrt{|v|}.$$

Thus there is a one-to-one correspondence between arbitrary smooth curves and the solutions of (1). As shown in (Fliess *et al.*, 1995), it provides a very simple algorithm to steer the car from one configuration to another.

2.2. Car with n-Trailers (Fliess *et al.*, 1995)

Take a single car above and hitch n trailers to its back (cf. Fig. 2). The resulting system still admits two control variables: the velocity of the car v and the steering angle ϕ. As for the single car, modeling is based on the assumption of rolling without slipping.

There is a one-to-one correspondence between smooth curves of arbitrary shapes and the system trajectories. It suffices to consider the curve followed by P_n, the Cartesian position of the last trailer. It is not necessary to write down explicitly the system equations in the state-space form (1). Just remember that the kinematic constraints indicate that the velocity of each trailer (more precisely, of the middle of its wheel axle) is parallel to the direction of its hitch.

Set $n = 1$ and have a look at Fig. 3. Assume that the curve $\mathcal{C}$ followed by P_1 is smooth and denote by $s \to P(s)$ an arc length parameterization. Then $P_1 = P(s)$

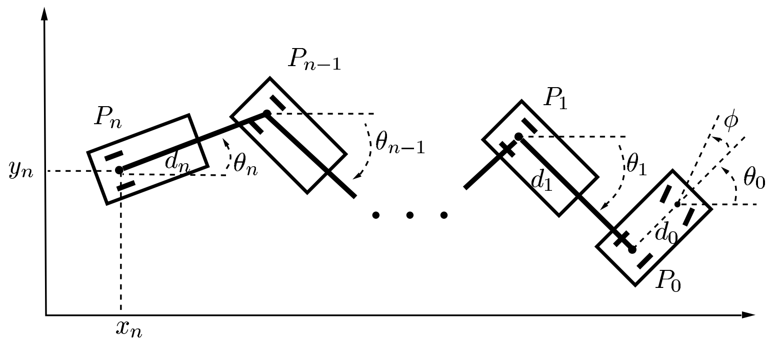

Fig. 2. Car with n trailers.

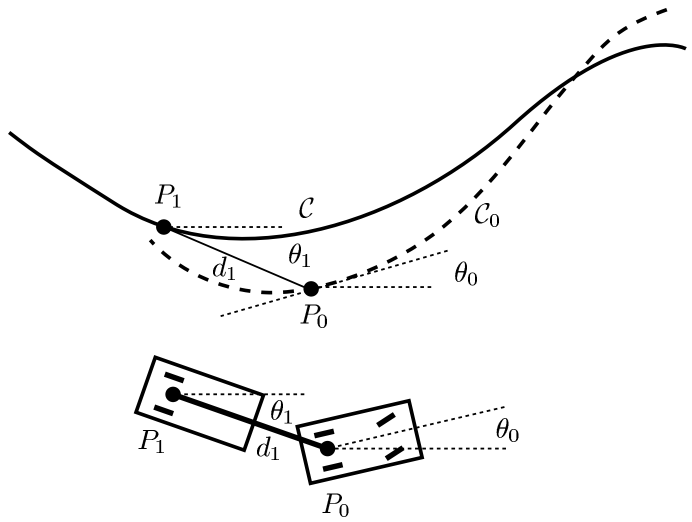

Fig. 3. Case of $n = 1$.

and θ_1 is the angle of $\vec{\tau}$, the unitary tangent vector to $\mathcal{C}$. Since $P_0 = P + d_1\vec{\tau}$, derivation with respect to s yields

$$\frac{\mathrm{d}}{\mathrm{d}s}P_0 = \vec{\tau} + d_1\kappa\vec{\nu}$$

with $(\vec{\tau}, \vec{\nu})$ the Frénet frame of $\mathcal{C}$ and κ its curvature. Thus $\mathrm{d}P_0/\mathrm{d}s \neq 0$ is tangent to $\mathcal{C}_0$, the curve followed by P_0. This curve is smooth and

$$\tan(\theta_0 - \theta_1) = d_1\kappa, \quad \vec{\tau}_0 = \frac{1}{\sqrt{1 + (d_1\kappa)^2}}\,(\vec{\tau} + d_1\kappa\vec{\nu}).$$

Derivation with respect to s_0, ($\mathrm{d}s_0 = \sqrt{1 + (d_1\kappa)^2}\ \mathrm{d}s$) yields the steering angle ϕ:

$$\tan\phi = d_0\kappa_0 = d_0\frac{1}{\sqrt{1 + (d_1\kappa)^2}}\left(\kappa + \frac{d_1}{1 + (d_1\kappa)^2}\frac{\mathrm{d}\kappa}{\mathrm{d}s}\right).$$

The car velocity v is then given by

$$v(t) = \sqrt{1 + d_1^2 \kappa^2(s(t))}\ \dot{s}(t)$$

for any C^1 time function, $t \to s(t)$. Notice that ϕ and $\theta_0 - \theta_1$ always remain in $]-\pi/2, \pi/2[$. These computations prove the one-to-one correspondence between the system trajectories determined by ϕ and $\theta_0 - \theta_1$ in $]-\pi/2, \pi/2[$, and regular planar curves of arbitrary shape with an arbitrary C^1 time parameterization.

The case $n > 1$ is just a direct generalization. The correspondence between arbitrary smooth curves $s \mapsto P(s)$ (tangent $\vec{\tau}$, curvature κ) with a C^1 time parameterization $t \mapsto s(t)$ is then defined by a smooth invertible map

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}^{n+2} & \mapsto & \mathbb{R}^2 \times \mathbb{S}^1 \times]-\pi/2, \pi/2[^{n+1} \times \mathbb{R} \\ \left(P, \vec{\tau}, \kappa, \dfrac{\mathrm{d}\kappa}{\mathrm{d}s}, \dots, \dfrac{\mathrm{d}^n\kappa}{\mathrm{d}s^n}, \dot{s}\right) & \mapsto & (P_n, \theta_n, \theta_n - \theta_{n-1}, \dots, \theta_1 - \theta_0, \phi, v)\,, \end{array}$$

where v is the car velocity. More details are given in (Fliess *et al.*, 1995).

With such a correspondence, motion planning reduces to a trivial problem: Find a smooth curve with prescribed initial and final positions, tangents, curvatures κ and curvature derivatives, $\mathrm{d}^i\kappa/\mathrm{d}s^i$, $i = 1, \dots, n$.

2.3. General One-Trailer System (Rouchon *et al.*, 1993b)

This non-holonomic system is shown in Fig. 4. Here the trailer is not directly hitched to the car at the centre of the rear axle, but more realistically at a distance a from this point. The equations take the form

$$\left\{ \begin{aligned} \dot{x} &= v \cos\alpha, \\ \dot{y} &= v \sin\alpha, \\ \dot{\alpha} &= \frac{v}{l} \tan\varphi, \\ \dot{\beta} &= \frac{v}{b}\left(\frac{a}{l} \tan\varphi \cos(\alpha - \beta) - \sin(\alpha - \beta)\right). \end{aligned} \right. \tag{2}$$

Controls are the car velocity v and the steering angle φ.

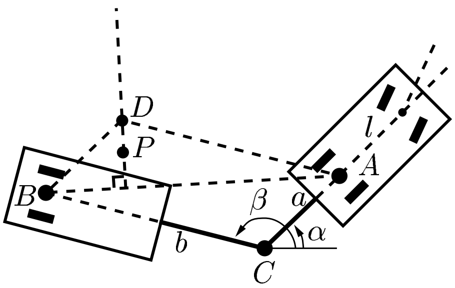

Fig. 4. Car with one trailer.

There still exists a one-to-one correspondence between the trajectories of (2) and arbitrary smooth curves with a C^1 time parameterization. Such curves are followed by the point P (see Fig. 4) of coordinates

$$\begin{aligned} X &= x + b\cos\beta + L(\alpha-\beta)\frac{b\sin\beta - a\sin\alpha}{\sqrt{a^2+b^2-2ab\cos(\alpha-\beta)}}, \\ Y &= y + b\sin\beta + L(\alpha-\beta)\frac{a\cos\alpha - b\cos\beta}{\sqrt{a^2+b^2-2ab\cos(\alpha-\beta)}}, \end{aligned} \tag{3}$$

where L is defined by the elliptic integral

$$L(\alpha-\beta) = ab\int_{\pi}^{2\pi+\alpha-\beta} \frac{\cos\sigma}{\sqrt{a^2+b^2-2ab\cos\sigma}}\,\mathrm{d}\sigma. \tag{4}$$

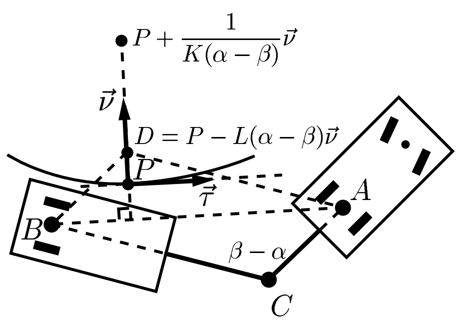

Fig. 5. Geometric construction with the Frénet frame.

We also have a geometrical construction (see Fig. 5): the tangent vector $\vec{\tau}$ is parallel to AB. Its curvature κ depends on $\delta = \alpha - \beta$:

$$\kappa = K(\delta) = \frac{\sin\delta}{\cos\delta\,\sqrt{a^2+b^2-2ab\cos\delta} - L(\delta\ \sin\delta}. \tag{5}$$

The function K is an increasing bijection between $]\gamma, 2\pi-\gamma[$ and $\mathbb{R}$. The constant $\gamma \in [0, \pi/2]$ is defined by the implicit equation

$$\cos\gamma\,\sqrt{a^2+b^2-2ab\cos\gamma} = ab\sin\gamma\int_{\pi}^{\gamma}\frac{\cos\sigma}{\sqrt{a^2+b^2-2ab\cos\sigma}}\,\mathrm{d}\sigma.$$

For $a = 0$, $\gamma = \pi/2$ and P coincides with B. Then D is given by $D = P - L(\delta)\vec{\nu}$ with $\vec{\nu}$ the unitary normal vector. Thus (x, y, α, β) depends on $(P, \vec{\tau}, \kappa)$. The steering angle φ depends on κ and $\mathrm{d}\kappa/\mathrm{d}s$, where s is the arc length. The car velocity v is then computed from κ, $\mathrm{d}\kappa/\mathrm{d}s$ and $\dot{s}$, the velocity of P.

2.4. Contact Structure, Equivalence and Pfaffian Systems

As opposed to the standard n-trailer systems, formulae for the general one-trailer system are not obvious. They cannot be found by some physical intuition. In fact, they rely on old questions and results relative to Pfaffian systems.

Nonholonomic systems with two controls, as the above trailer systems, are driftless systems of the form

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

defined by two vector fields f_1 and f_2. Elimination of u_1 and u_2 yields a linear system in $\dot{x}$ with coefficients depending on x. If n is the dimension of x, we have then $n-2$ equations corresponding to a Pfaffian system of codimension 2, say

$$\omega_i \equiv \sum_{j=1}^{n} a_i^j(x)\ \mathrm{d}x_j = 0, \quad i = 1, \dots, n-2.$$

The $n-2$ differential forms ω_i are independent and such that $(\omega_i, f_1) = (\omega_i, f_2) = 0$. Equivalence of Pfaffian systems via transformations of x-coordinates is an old question firstly stated by Pfaff. Weber (1898), Goursat (1923), and Cartan (1915) gave the conditions of equivalence (around a generic x) to the contact system

$$\mathrm{d}x_2 - x_3\,\mathrm{d}x_1 = 0, \quad \mathrm{d}x_3 - x_4\,\mathrm{d}x_1 = 0, \quad \dots, \quad \mathrm{d}x_{n-1} - x_n\,\mathrm{d}x_1 = 0.$$

The interest in such systems is mainly due to the fact that their general solution is given by an arbitrary function of one variable $z \mapsto w(z)$ and a finite number of its derivatives:

$$x_1 = z, \quad x_2 = w(z), \quad x_3 = \frac{\mathrm{d}w}{\mathrm{d}z}, \quad \dots \quad, x_n = \frac{\mathrm{d}^{n-2}w}{\mathrm{d}z^{n-2}}.$$

This means that we can parameterize the general solution of $\dot{x} = f_1(x)u_1 + f_2u_2$ without integrating the control $t \mapsto u(t)$. Just consider the above relations with any C^1 time function $t \mapsto z(t)$ and any C^{n-2} function of z, $z \mapsto w(z)$. The quantities $x_1 = z(t)$ and $x_2 = w(z(t))$ play a special role here. We call them the flat output (Fliess *et al.*, 1995). For trailer systems we have sketched here similar parameterizations: they are based on Frénet formulae and written in a special way in order to exploit the invariants with respect to planar isometries.

A coordinate-free characterization of contact systems was originally written in terms of the derived flag and differential form. In a dual way, it reads for the two vector fields f_1 and f_2 as follows: the generic rank of E_k has to be equal to $k+2$ for $k = 0, \dots, n-2$ where $E_0 := \mathrm{span}\{f_1, f_2\}$, $E_{k+1} := \mathrm{span}\{E_k, [E_k, E_k]\}$, $k \geq 0$. (See (Giaro *et al.*, 1978; Murray, 1994; Murray and Sastry, 1993; Pasillas-Lépine, 2000) for complementary results on chained normal forms and singularity classification of Goursat systems).

This characterization is much more general. Cartan (1914; 1915) (see also (Martin and Rouchon, 1994) for dynamic feedback refinements) proved that the E_k's characterize systems that can be solvable without any integration, a notion introduced few years earlier by Hilbert (1912). Hilbert considers the second-order Monge equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}^2 x} = F\left(x, y, z, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}z}{\mathrm{d}x}\right),$$

an under-determined differential system (a single differential equation relating two x-functions, $y(x)$ and $z(x)$). He wonders if its general solution can be expressed without integrals as for the first-order Monge equation. Hilbert also shows that this question is related to the classification of under-determined systems under a transformation group much more general than punctual transformations. Hilbert proposes a nice analogy with the group of birational transformations and the classification of algebraic manifolds.

Hilbert's original idea of 'integrallos Auflösung' can be extended to some infinite-dimensional control systems. Such extensions require advance and delay operators in complement to derivation and are based on several examples such as the nonholonomic snake characterized in what follows.

2.5. Nonholonomic Snake

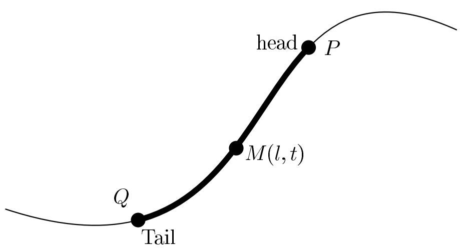

Fig. 6. The non-holonomic snake, a car with an infinite number of small trailers.

When the number of trailers is large, it is natural, as shown in Fig. 6, to introduce a continuous approximation of the 'non-holonomic snake'. The trailers are now indexed by a continuous variable $l \in [0, L]$, and their positions are given by a map $[0, L] \ni l \mapsto M(l, t) \in \mathbb{R}^2$ satisfying the following partial differential equations:

$$\left\| \frac{\partial M}{\partial l} \right\| = 1, \quad \frac{\partial M}{\partial l} \wedge \frac{\partial M}{\partial t} = 0.$$

The first equation says that $l \mapsto M(l, t)$ is an arc length parameterization. The second one amounts to the rolling-without-slipping conditions: the velocity of the trailer l is parallel to the direction of the plane of its wheels, i.e., the tangent to the curve $l \mapsto M(l, t)$. It is then obvious that the general solution to this system is

$$M(l, t) = P(s(t) - l), \quad l \in [0, L],$$

where P is the snake head and $s \mapsto P(s)$ an arc length parameterization of the curve followed by P. Similarly,

$$M(l,t) = Q(s(t)+l), \quad l \in [0,L],$$

where Q is the snake tail. It corresponds to the flat output of the finite-dimensional approximation, the n-trailer system of Fig. 2, with n large and $d_i = L/n$. The derivatives up to order n are in the infinite case replaced by advances in the arc length scale. This results from the formal relation

$$Q(s+l) = \sum_{i \geq 0} Q^{(i)}(s) l^i / i!$$

and the series truncation up to the first n terms. Nevertheless, $M(l,t) = Q(s(t)+l)$ is much simpler to use in practice. When n is large, the series experience convergence difficulties for $s \mapsto Q(s)$ smooth but not analytic.

When the number of trailers is large and the curvature radius $1/\kappa$ of $s \mapsto Q(s)$ is much larger than the length of each small trailer, such an infinite-dimensional approximation is valid. It reduces the dynamics to trivial delays. It is noteworthy that, in this case, an infinite-dimensional description yields a much better reduced model than a finite-dimensional description that gives complex nonlinear control models and algorithms[1].

3. Pendulums and Heavy Chains

3.1. Juggling Robot $2k\pi$ (Lenoir *et al.*, 1998)

The robot $2k\pi$ developed at École des Mines de Paris consists of a manipulator carrying a pendulum, see Fig. 7. There are five degrees of freedom (dof's): 3 angles for the manipulator and 2 angles for the pendulum. The 3 dof's of the manipulator are actuated by electric drives, while the 2 dof's of the pendulum are *not* actuated.

This system is typical of underactuated, nonlinear and unstable mechanical systems such as the PVTOL (Martin *et al.*, 1996), Caltech's ducted fan (Martin *et al.*, 1997; Murray, 1995), the gantry crane (Fliess *et al.*, 1995), Champagne flyer (Lemon and Hause, 1994). As shown in (Fliess *et al.*, 1995; Martin, 1992; Martin *et al.*, 1997) the robot $2k\pi$ is flat, with the centre of oscillation of the pendulum as a flat output. Let us recall some elementary facts.

The Cartesian coordinates of the suspension point S of the pendulum can be considered here as control variables: they are related to the angles of the manipulator θ_1, θ_2, θ_3 via static relations. Let us concentrate on the pendulum dynamics. This dynamics is similar to that of a punctual pendulum with the same mass m located at point H, the oscillation centre (the Huygens theorem). Denoting by $l = \|SH\|$ the length of the isochronous punctual pendulum, the Newton equation and geometric

[1] The finite-dimensional system does not require to be flat (trailer i can be hitched to trailer $i-1$ not directly at the centre of its wheel axle, but more realistically at a positive distance from this point (Martin and Rouchon, 1994)).

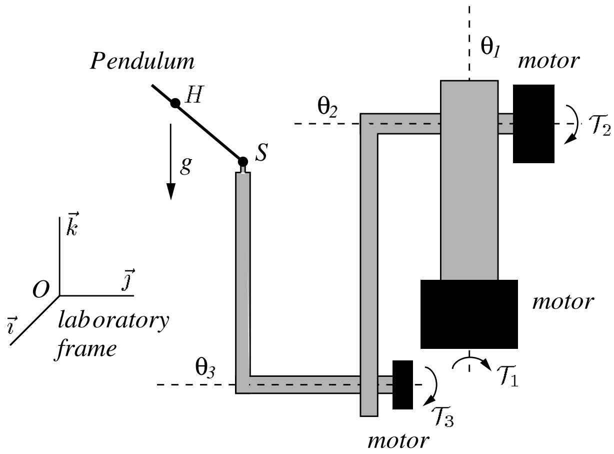

Fig. 7. The robot $2k\pi$.

constraints yield the following differential-algebraic system ($\vec{T}$ is the tension, see Fig. 8):

$$m\ddot{H} = \vec{T} + m\vec{g}, \quad \vec{SH} \wedge \vec{T} = 0, \quad \|SH\| = l.$$

If, instead of setting $t \mapsto S(t)$, we set $t \mapsto H(t)$, then $\vec{T} = m(\ddot{H} - \vec{g})$. S is located at the intersection of the sphere with centre H and radius l with the line passing through H of direction $\ddot{H} - \vec{g}$:

$$S = H \pm \frac{1}{\|\ddot{H} - \vec{g}\|}\,(\ddot{H} - \vec{g}).$$

These formulae are crucial for designing a control law steering the pendulum from the lower equilibrium to the upper equilibrium, and also for stabilizing the pendulum while the manipulator is moving around (Lenoir *et al.*, 1998).

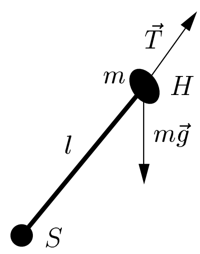

Fig. 8. The isochronous pendulum.

3.2. Towed Cable Systems (Martin *et al.*, 1997; Murray, 1996)

This system consists of an aircraft flying in a circular pattern while towing a cable with a tow body (drogue) attached at the bottom. Under suitable conditions, the

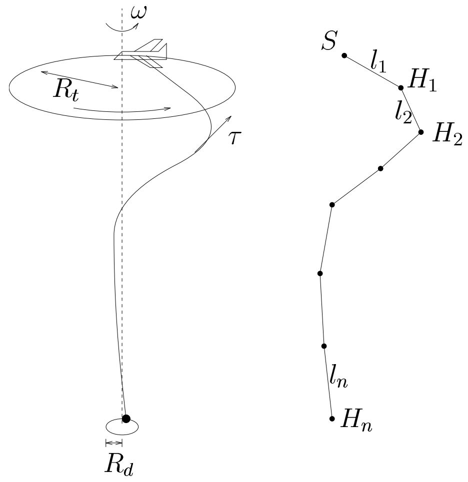

Fig. 9. The towed cable system and a finite-link approximate model.

cable reaches a relative equilibrium in which it maintains its shape as it rotates. By choosing the parameters of the system appropriately, it is possible to make the radius at the bottom of the cable much smaller than the radius at the top of the cable. This is illustrated in Fig. 9.

The motion of the towed cable system can be approximately represented using a finite-element model in which the segments of the cable are replaced by rigid links connected by spherical joints. The forces acting on the segment (tension, aerodynamic drag and gravity) are lumped and applied at the end of each rigid link. In addition to the forces on the cable, we must also consider the forces on the drogue and the towplane. The drogue is modelled as a sphere and essentially acts as a mass attached to the last link of the cable, so that the forces acting on it are included in the cable dynamics. The external forces on the drogue again consist of gravity and aerodynamic drag. The towplane is attached to the top of the cable and is subject to drag, gravity, and the force of the attached cable. For simplicity, we model the towplane as a pure force applied at the top of the cable. Our goal is to generate trajectories for this system that allow operation away from relative equilibria, as well as transition between one equilibrium point and another. Due to the high dimension of the model for the system (128 states are typical), traditional approaches to solving this problem, such as optimal control theory, cannot be easily applied. However, it can be shown that this system is differentially flat using the position of the bottom of the cable H_n as the differentially flat output. See (Murray, 1996) for a more complete description and additional references.

Assume that there is no friction and consider only gravity. Then, as for the pendulum of $2k\pi$, we have

$$H_{n-1} = H_n + \frac{1}{\|m_n \ddot{H}_n - m_n \vec{g}\|} \, (m_n \ddot{H}_n - m_n \vec{g}),$$

where m_n is the mass of link n. The Newton equation for link $n-1$ yields (with obvious notation)

$$H_{n-2} \; = \; H_{n-1} + \frac{m_n \ddot{H}_n + m_{n-1}\ddot{H}_{n-1} - (m_n + m_{n-1})\vec{g}}{\|m_n \ddot{H}_n + m_{n-1}\ddot{H}_{n-1} - (m_n + m_{n-1})\vec{g}\|}.$$

More generally, at link i we have

$$H_{i-1} = H_i + \frac{\sum_i^n m_k(\ddot{H}_k - \vec{g})}{\|\sum_i^n m_k(\ddot{H}_k - \vec{g})\|}.$$

These relations imply that S is a function of H_n and its time derivatives up to order $2n$. Thus H_n is the flat output. Let us consider now an infinite-dimensional description. It could provide simpler computations, as for the nonholonomic snake and the car with n trailers.

3.3. Nonlinear Heavy-Chain Systems

The nonlinear conservative model of a homogenous heavy chain with an end mass is the following:

$$\begin{cases} \rho \dfrac{\partial^2 M}{\partial t^2} = \dfrac{\partial}{\partial s}\left(T \dfrac{\partial M}{\partial s}\right) + \rho \vec{g}, \\ \left\| \dfrac{\partial M}{\partial s} \right\| = 1, \\ M(L,t) = u(t), \\ T(0,t)\dfrac{\partial M}{\partial s}(0,t) = m \dfrac{\partial^2 M}{\partial t^2}(0,t) - m\vec{g}, \end{cases} \tag{6}$$

where $[0, L] \ni s \mapsto M(s,t) \in \mathbb{R}^3$ is an arc length parameterization of the chain and $T(s,t) > 0$ is the tension. The control u is the position of the suspension point. If we use

$$N(s,t) = \int_0^s M(\sigma, t) \, \mathrm{d}\sigma$$

instead of $M(s,t)$ (Bäcklund transformation), we get

$$\begin{aligned}
\rho\frac{\partial^2 N}{\partial t^2} &= T(s,t)\frac{\partial^2 N}{\partial s^2}(s,t) - T(0,t)\frac{\partial^2 N}{\partial s^2}(0,t) + \rho s\vec{g}, \\
\left\|\frac{\partial^2 N}{\partial s^2}\right\| &= 1, \\
\frac{\partial N}{\partial s}(L,t) &= u(t), \\
T(0,t)\frac{\partial^2 N}{\partial s^2}(0,t) &= m\frac{\partial^3 N}{\partial t^2 \partial s}(0,t) - m\vec{g}, \\
N(0,t) &= 0.
\end{aligned}$$

Assume that the load trajectory is given by

$$t \mapsto y(t) = \frac{\partial N}{\partial s}(0,t).$$

Then (we take the positive branch)

$$T(s,t) = \left\|\rho\frac{\partial^2 N}{\partial t^2}(s,t) - (\rho s + m)\vec{g} + m\ddot{y}(t)\right\|$$

and we have the Cauchy-Kovalevskaya problem

$$\begin{aligned}
\frac{\partial^2 N}{\partial s^2}(s,t) &= \frac{1}{T(s,t)}\left(\rho\frac{\partial^2 N}{\partial t^2}(s,t) - (\rho s + m)\vec{g} + m\ddot{y}(t)\right), \\
N(0,t) &= 0, \\
\frac{\partial N}{\partial s}(0,t) &= y(t).
\end{aligned}$$

Formally, its series solution is expressed in terms of y and its derivatives of an infinite order. This could be problematic since y must be analytic and the series converges for $s \geq 0$ small enough.

We will see below that the solution of the tangent linearization of this Cauchy-Kovalevskaya system around the stable vertical steady-state can be expressed via advances and delays of y. Such a formulation avoids series with y derivatives of arbitrary orders. For the nonlinear system above, we conjecture a solution involving nonlinear delays and advances of y.

3.4. Linear Heavy-Chain Systems (Petit and Rouchon, 2001)

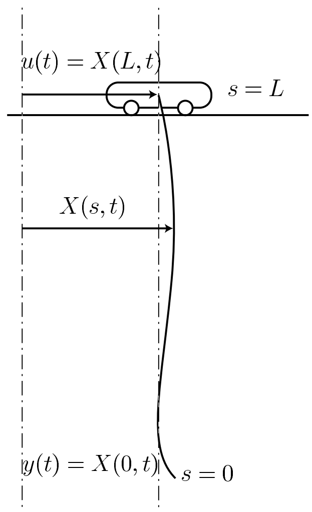

Fig. 10. The homogeneous chain without any load.

A small-angle approximation of (6) with $m = 0$ yields the following dynamics around the stable vertical steady-state:

$$\begin{cases} \dfrac{\partial}{\partial s}\left(gs\dfrac{\partial X}{\partial s}\right) - \dfrac{\partial^2 X}{\partial t^2} = 0, \\ \\ X(L,t) = u(t), \end{cases} \tag{7}$$

where $X(s,t)$ is the horizontal coordinate of M. In this case, the vertical and horizontal dynamics are decoupled. The two horizontal dynamics are also decoupled. The tension T equals $g\rho s$. The control u is the trolley horizontal position.

We prove in (Petit and Rouchon, 2001) that the general solution of (7) is given by the following formulae where y is the free-end position $X(0,t)$:

$$X(s,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} y\big(t + 2\sqrt{s/g}\sin\theta\big)\ \mathrm{d}\theta. \tag{8}$$

Simple computations show that (8) corresponds to the series solution of the (singular) Cauchy-Kovalevskaya problem:

$$\begin{cases} \dfrac{\partial}{\partial s}\left(gs\dfrac{\partial X}{\partial s}\right) = \dfrac{\partial^2 X}{\partial t^2}, \\ \\ X(0,t) = y(t). \end{cases}$$

The relation (8) means that there is a one-to-one correspondence between the (smooth) solutions of (7) and the (smooth) functions $t \mapsto y(t)$. For each solution of (7), set $y(t) = X(0,t)$. For each function $t \mapsto y(t)$, set X via (8) and u via

$$u(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(t + 2\sqrt{L/g}\sin\theta)\ \mathrm{d}\theta \tag{9}$$

to obtain a solution of (7).

Finding $t \mapsto u(t)$ steering the system from a steady state $X \equiv 0$ to another $X \equiv D$ becomes obvious. It just consists in finding $t \mapsto y(t)$ that is equal to 0 for $t \leq 0$ and to D for t large enough (at least for $t > 4\sqrt{L/g}$), and in computing u via (9).

For example, take

$$y(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ \dfrac{3L}{2}\left(\dfrac{t}{T}\right)^2\left(3 - 2\left(\dfrac{t}{T}\right)\right) & \text{if} \quad 0 \leq t \leq T, \\ \dfrac{3L}{2} & \text{if} \quad t > T, \end{cases}$$

where the chosen transfer time T equals 2Δ with $\Delta = 2\sqrt{L/g}$, the travelling time of a wave between $x = L$ and $x = 0$. For $t \leq 0$ the chain is vertical at position 0. For $t \geq T$ the chain is vertical at position $D = 3L/2$.

When $m > 0$, a small-angle approximation of (6) gives

$$\begin{cases} \dfrac{\partial}{\partial s}\left(g(s+a)\dfrac{\partial X}{\partial s}\right) - \dfrac{\partial^2 X}{\partial t^2} = 0, \\ \dfrac{\partial^2 X}{\partial t^2}(0,t) = g\dfrac{\partial X}{\partial s}(0,t), \\ X(L,t) = u(t), \end{cases}$$

where $a = m/\rho$ is homogenous with respect to the length. We also prove in (Petit and Rouchon, 2001) that its general solution depends on advances and delays of y and its first derivative.

4. Diffusion and Gevrey Functions

4.1. Finite-Dimensional Models

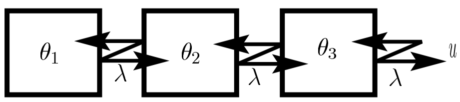

Fig. 11. A finite volume model of a heating system.

Consider the three-compartment model illustrated in Fig. 11. Its dynamics is based on the following energy balance equations (m, ρ, C_p and λ are physical constants):

$$\begin{cases} m\rho C_p\ \dot{\theta}_1 = \lambda(\theta_2 - \theta_1), \\ m\rho C_p\ \dot{\theta}_2 = \lambda(\theta_1 - \theta_2) + \lambda(\theta_3 - \theta_2), \\ m\rho C_p\ \dot{\theta}_3 = \lambda(\theta_2 - \theta_3) + \lambda(u - \theta_3). \end{cases} \tag{10}$$

It is obvious that this linear system is controllable with $y = \theta_1$ as the the Brunovsky output: it can be transformed via a linear transformation of coordinates and a linear static feedback into $y^{(3)} = v$.

Taking an arbitrary number n of compartments yields

$$\begin{cases} m\rho C_p\ \dot{\theta}_1 = \lambda(\theta_2 - \theta_1), \\ m\rho C_p\ \dot{\theta}_2 = \lambda(\theta_1 - \theta_2) + \lambda(\theta_3 - \theta_2), \\ \qquad\vdots \\ m\rho C_p\ \dot{\theta}_i = \lambda(\theta_{i-1} - \theta_i) + \lambda(\theta_{i+1} - \theta_i), \\ \qquad\vdots \\ m\rho C_p\ \dot{\theta}_{n-1} = \lambda(\theta_{n-2} - \theta_{n-1}) + \lambda(\theta_n - \theta_{n-1}), \\ m\rho C_p\ \dot{\theta}_n = \lambda(\theta_{n-1} - \theta_n) + \lambda(u - \theta_n). \end{cases} \tag{11}$$

Here $y = \theta_1$ remains the Brunovsky output: via a linear transformation of coordinates and a linear static feedback we have $y^{(n)} = v$.

When n tends to infinity, m and λ tend to zeros as $1/n$, and (11) tends to the classical heat equation (12) considered below. We will see that the temperature on the side opposite to u, i.e., $y = \theta(0, t)$, still plays a special role.

4.2. Heat Equation (Laroche *et al.*, 1998)

Consider the linear heat equation

$$\begin{cases} \partial_t\theta(x,t) = \partial_x^2\theta(x,t), \quad x \in [0,1], \\ \partial_x\theta(0,t) = 0, \\ \theta(1,t) = u(t), \end{cases} \tag{12}$$

where $\theta(x, t)$ is the temperature and $u(t)$ is the control input. We claim that

$$y(t) := \theta(0, t)$$

is a 'flat' output. Exchange the role of time t and space x, and consider the following Cauchy-Kovalevskaya system

$$\begin{cases} \dfrac{\partial^2\theta}{\partial x^2} = \dfrac{\partial\theta}{\partial t}, \\ \theta(0,t) = 0, \\ \dfrac{\partial\theta}{\partial x}(0,t) = 0. \end{cases} \tag{13}$$

Its series solution is represented by

$$\begin{cases} \theta(x,t) = \displaystyle\sum_{i=1}^{\infty} \frac{y^{(i)}(t)}{(2i)!}\ x^{2i}, \\ u(t) = \displaystyle\sum_{i=1}^{\infty} \frac{y^{(i)}(t)}{(2i)!}. \end{cases} \tag{14}$$

Whenever $t \mapsto y(t)$ is an arbitrary function (i.e., a trajectory of the trivial system $y = v$), $t \mapsto \big(\theta(x,t), u(t)\big)$ defined by (14) is a (formal) trajectory of (12) and vice versa. This is exactly the idea underlying our definition of flatness in (Fliess *et al.*, 1999a). Notice that these calculations have been known for a long time, see (Valiron, 1950, pp.588 and 594).

To make the statement precise, we now turn to convergence issues. On the one hand, $t \mapsto y(t)$ must be a smooth function such that

$$\exists\ K, M > 0, \quad \forall i \geq 0, \forall t \in [t_0, t_1], \quad |y^{(i)}(t)| \leq M(Ki)^{2i}$$

to ensure convergence. On the other hand, $t \mapsto y(t)$ cannot in general be analytic. Indeed, if the system is to be steered from an initial temperature profile $\theta(x,t_0) = \alpha_0(x)$ at time t_0 to a final profile $\theta(x,t_1) = \alpha_1(x)$ at time t_1, eqn. (12) implies

$$\forall t \in [0,1], \forall i \geq 0, \quad y^{(i)}(t) = \partial_t^i \theta(0,t) = \partial_x^{2i}\theta(0,t),$$

and in particular

$$\forall i \geq 0, \qquad y^{(i)}(t_0) = \partial_x^{2i}\alpha_0(0) \quad \text{and} \quad y^{(i)}(t_1) = \partial_x^{2i}\alpha_1(1).$$

If, for instance, $\alpha_0(x) = c$ for all $x \in [0,1]$ (i.e., a uniform temperature profile), then $y(t_0) = c$ and $y^{(i)}(t_0) = 0$ for all $i \geq 1$, which implies $y(t) = c$ for all t provided the function is analytic. It is thus impossible to reach any final profile but $\alpha_1(x) = c$ for all $x \in [0,1]$.

Smooth functions $t \in [t_0, t_1] \mapsto y(t)$ that satisfy

$$\exists\ K, M > 0, \quad \forall i \geq 0, \qquad |y^{(i)}(t)| \leq M(Ki)^{\sigma i}$$

are known as Gevrey functions of order σ (Ramis, 1979) (they are also closely related to functions of class S (Gelfand and Shilov, 1964)). The Taylor expansion of such functions is convergent for $\sigma \leq 1$ and divergent for $\sigma > 1$ (the larger σ is, the 'more divergent' the Taylor expansion becomes). Analytic functions are thus Gevrey of order ≤ 1.

In other words, we need a Gevrey function on $[t_0, t_1]$ of order > 1 but ≤ 2, with initial and final Taylor expansions imposed by the initial and final temperature profiles. With such a function, we can then compute an open-loop control steering the system from one profile to the other by the formula (14).

For instance, we steered the system from the uniform temperature zero at $t = 0$ to the uniform unit temperature at $t = 1$ by using the function

$$\mathbb{R} \ni t \mapsto y(t) := \begin{cases} 0 & \text{if} \quad t < 0, \\ 1 & \text{if} \quad t > 1, \\ \dfrac{\int_0^t \exp\bigl(-1/(\tau(1-\tau))^\gamma\bigr)\mathrm{d}\tau}{\int_0^1 \exp\bigl(-1/(\tau(1-\tau))^\gamma\bigr)\mathrm{d}\tau} & \text{if} \quad t \in [0, 1], \end{cases}$$

with $\gamma = 1$ (Gevrey order $1 + 1/\gamma$).

More details can be found in (Laroche *et al.*, 1998) where numerical tests indicate practical interest in using Gevrey functions of order > 2 and divergent series.

4.3. Flexible (Euler-Bernoulli) Beam (Aoustin *et al.*, 1997; Fliess *et al.*, 1996)

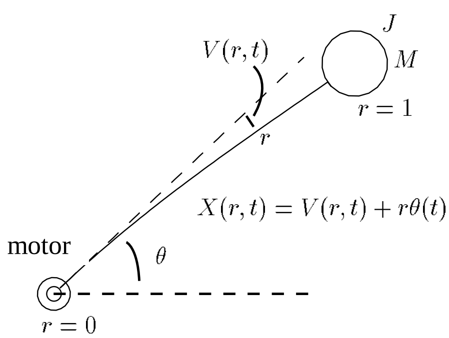

Fig. 12. A flexible beam rotating around a control axle.

Symbolic computations 'à la Heaviside' with s instead of $\partial/\partial t$ are important here. We will not develop the formal aspect with the use of the Mikusiński operational calculus as in (Fliess *et al.*, 1996). We just concentrate on the computations. We have the following $1D$ modeling:

$$\partial_{tt}X = -\partial_{xxxx}X,$$

$$X(0,t) = 0, \quad \partial_x X(0,t) = \theta(t),$$
$$\ddot{\theta}(t) = u(t) + k\partial_{xx}X(0,t),$$

$$\partial_{xx}X(1,t) = -\lambda\partial_{ttx}X(1,t),$$
$$\partial_{xxx}X(1,t) = \mu\partial_{tt}X(1,t),$$

where the control is the motor torque u, $X(r,t)$ is the deformation profile, k, λ and μ are physical parameters (t and r are on reduced scales).

We will show that the general solution can be expressed in terms of an arbitrary C^∞ function y (Gevrey order ≤ 2 for convergence):

$$X(x,t) = \sum_{n\geq 0} \frac{(-1)^n\, y^{(2n)}(t)}{(4n)!} P_n(x) + \sum_{n\geq 0} \frac{(-1)^n\, y^{(2n+2)}(t)}{(4n+4)!} Q_n(x) \tag{15}$$

with $\imath = \sqrt{-1}$,

$$P_n(x) = \frac{x^{4n+1}}{2(4n+1)} + \frac{(\Im - \Re)(1-x+\imath)^{4n+1}}{2(4n+1)} + \mu\Im(1-x+\imath)^{4n}$$

and

$$\begin{aligned} Q_n(x) =& \frac{\lambda\mu}{2}(4n+4)(4n+3)(4n+2)\left((\Im - \Re)(1-x+\imath)^{4n+1} - x^{4n+1}\right) \\ &- \lambda(4n+4)(4n+3)\Re(1-x+\imath)^{4n+2} \end{aligned}$$

($\Re$ and $\Im$ stand for the real and imaginary parts, respectively). Notice that θ and u result from (15): it suffices to derive term by term.

We just show here how to get these formulae with $\lambda = \mu = 0$ (no inertia at the free end $r = 1$, $M = J = 0$). The method remains unchanged in the general case. The problem is how to get

$$X(x,t) = \sum_{n\geq 0} \frac{y^{(2n)}(t)(-1)^n}{(4n)!} \pi_n(x) \tag{16}$$

with

$$\pi_n(x) = \frac{x^{4n+1}}{2(4n+1)} + \frac{(\Im - \Re)(1-x+\imath)^{4n+1}}{2(4n+1)}.$$

With the Laplace variable s, we have the ordinary differential system

$$X^{(4)} = -s^2 X,$$

where

$$X(0) = 0, \quad X^{(2)}(1) = 0, \quad X^{(3)}(1) = 0.$$

The derivatives are with respect to the spatial variable r and s stands here for a parameter. The general solution depends on an arbitrary constant, i.e., an arbitrary function of s, since we have three boundary conditions. With the following four elementary solutions of $X^{(4)} = -s^2 X$:

$$C_+(x) = \Big(\cosh\big((1-x)\sqrt{s}\xi\big) + \cosh\big((1-x)\sqrt{s}/\xi\big)\Big)/2,$$

$$C_-(x) = \Big(\cosh\big((1-x)\sqrt{s}\xi\big) - \cosh\big((1-x)\sqrt{s}/\xi\big)\Big)/(2\imath),$$

$$S_+(x) = \Big(\imath\sinh\big((1-x)\sqrt{s}\xi\big) + \sinh\big((1-x)\sqrt{s}/\xi\big)\Big)/(2\xi\sqrt{s}),$$

$$S_-(x) = \xi\Big(\imath\sinh\big((1-x)\sqrt{s}\xi\big) - \sinh\big((1-x)\sqrt{s}/\xi\big)\Big)/(2\sqrt{s}),$$

where $\xi = \exp(\imath\pi/4)$, X reads as

$$X(x) = aC_+(x) + bC_-(x) + cS_+(x) + dS_-(x).$$

The three boundary conditions provide three equations relating the constants a, b, c and d:

$$aC_+(0) + bC_-(0) + cS_+(0) + dS_-(0) = 0, \quad sb = 0, \quad sc = 0.$$

Thus $b = c = 0$ and we have just one relation between a and d:

$$aC_+(0) + dS_-(0) = 0.$$

Since

$$C_+(0) = \Re\big(\cosh(\xi\sqrt{s})\big), \quad S_-(0) = \Im\big(\xi\sinh(\xi\sqrt{s}/\sqrt{s})\big)$$

are entire functions of s very similar to $\cosh(\sqrt{s})$ and $\sinh\sqrt{s}/\sqrt{s}$ appearing for the heat eqn. (12), we can associate with them two operators, algebraically independent and commuting,

$$\delta_+ = C_+(0), \quad \delta_- = S_-(0).$$

They are in fact ultra-distributions belonging to the dual space of Gevrey functions of order less than ≤ 2 and with pointwise support (Gelfand and Shilov, 1964). We have thus a module generated by two elements (a, d) satisfying $\delta_+ a + \delta_- d = 0$. This is an $\mathbb{R}[\delta_+, \delta_-]$-module. This module is not free but δ_+-free (Mounier, 1995):

$$a = \delta_- y, \quad d = -\delta_+ y$$

with $y = -\delta_+^{-1} d$.

The basis y plays the role of a flat output since

$$X(x) = \big(S_-(0)C_+(x) - S_-(x)C_+(0)\big)y.$$

Simple but tedious computations using hyperbolic trigonometric formulae yield then

$$X(x) = -\frac{1}{2}\Big[S_{-}(x) + \Im\big(S_{-}(1 - x + \imath)\big)\Big] y.$$

The series of the entire function S_{-} provides (16). We conjecture that the quantity y possesses a physical sense as an explicit expression with integrals of X over $r \in [0, 1]$ (centre of flexion).

5. Conclusion

The above infinite-dimensional examples can be completed by several others. For advance/delay parameterizations we have:

- water-tanks systems (Dubois *et al.*, 1999);
- telegraph equation (Fliess *et al.*, 1999b; Mounier *et al.*, 1996);
- flexible beams (Fliess *et al.*, 1998b; Mounier *et al.*, 1995);
- Burger equation and nonlinear delays (Petit *et al.*, 1998) (see also (Mounier and Rudolph, 1998) for flatness-based control of non-linear delay systems).

For series parameterization with Gevrey functions, we have

- tubular chemical reactors (a multi-input case) (Fliess *et al.*, 1998a; Rouchon and Rudolph, 2000);
- heat equation with variable coefficients (Laroche and Martin, 2000; Rothfuß *et al.*, 2000).

All of the above examples are 1D systems. Such explicit parameterizations also exist for higher space dimensions. Take, e.g., the 2D wave equation corresponding, in the linear approximation, to the surface wave generated by the horizontal motions of a cylindric tank containing a fluid (linearized Saint-Venant equations—a shallow water approximation):

$$\begin{cases} \dfrac{\partial^2 \xi}{\partial t^2} = g\bar{h}\ \Delta\xi & \text{on } \Omega, \\ g\nabla\xi \cdot n = -\ddot{D} \cdot n & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where Ω is the interior of a circle with radius R and centre $D(t) \in \mathbb{R}^2$, the control (n is the normal to the boundary $\partial\Omega$), $\bar{h} + \xi$ is the height of the liquid, and g denotes the gravity. A family of solutions to (17) is given by the following formulae ((r, θ) are the polar coordinates):

$$\xi(r, \theta, t) = \frac{1}{\pi}\sqrt{\frac{\bar{h}}{g}}\ \left(\int_0^{2\pi} \cos\alpha \left[\dot{a}\left(t - \frac{r\cos\alpha}{c}\right)\cos\theta + \dot{b}\left(t - \frac{r\cos\alpha}{c}\right)\sin\theta\right]\ \mathrm{d}\alpha\right)$$

and $((u, v)$ are the Cartesian coordinates of $D)$

$$u = \frac{1}{\pi} \int_0^{2\pi} a\left(t - \frac{R\cos\alpha}{c}\right) \cos^2\alpha \; \mathrm{d}\alpha,$$

$$v = \frac{1}{\pi} \int_0^{2\pi} b\left(t - \frac{R\cos\alpha}{c}\right) \cos^2\alpha \; \mathrm{d}\alpha,$$

where $t \mapsto (a(t), b(t)) \in \mathbb{R}^2$ is an arbitrary smooth function.

References

Aoustin Y., Fliess M., Mounier H., Rouchon P. and Rudolph J. (1997): *Theory and practice in the motion planning and control of a flexible robot arm using Mikusiński operators.* — Proc. 5th IFAC Symp. Robot Control, Nantes, France, pp.287–293.

Campion G., d'Andrea Novel G. and Bastin G. (1996): *Structural properties and classification of kinematic and dynamic models of wheeled mobile robots.* — IEEE Trans. Robot. Automat., Vol.12, No.1, pp.47–62.

Cartan E. (1914): *Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courves.* — Bull. Soc. Math. France, Vol.42, pp.12–48, also in: *Œuvres Complètes*, Part II, Vol.2, (1984), Paris: CNRS, pp.1133–1168.

Cartan E. (1915): *Sur l'intégration de certains systèmes indéterminés d'équations différentielles.* — J. für reine und angew. Math., Vol.145, pp.86–91, also in: *Œuvres Complètes*, Part II, Vol.2, (1984), Paris: CNRS, pp.1164–1174.

Dubois F., Petit N. and Rouchon P. (1999): *Motion planing and nonlinear simulations for a tank containing a fluid.* — Proc. European Control Conf., Karlsruhe, Germany, published on CD-ROM.

Fliess M., Lévine J., Martin Ph. and Rouchon P. (1995): *Flatness and defect of nonlinear systems: introductory theory and examples.* — Int. J. Contr., Vol.61, No.6, pp.1327–1361.

Fliess M., Mounier H., Rouchon P. and Rudolph J. (1996): *Systèmes linéaires sur les opérateurs de Mikusiński et commande d'une poutre flexible.* — ESAIM Proc. Conf. Élasticité, viscolélasticité et contrôle optimal, 8ème entretiens du centre Jacques Cartier, Lyon, France, pp.157–168.

Fliess M., Lévine J., Martin P., Ollivier F. and Rouchon P. (1997): *Controlling nonlinear systems by flatness*, In: Systems and Control in the Twenty-First Century, (C.I. Byrnes, B.N. Datta, D.S. Gilliam and C.F. Martin, Eds.). — Boston: Birkhäuser.

Fliess M., Mounier H., Rouchon P. and Rudolph J. (1998a): *A distributed parameter approach to the control of a tubular reactor: A multi-variable case.* — Proc. Conf. Decision and Control, Tampa, U.S.A., pp.439–442.

Fliess M., Mounier H., Rouchon P. and Rudolph J. (1998b): *Tracking control of a vibrating string with an interior mass viewed as delay system.* — ESAIM: Contr. Optim. Calc. Var, `www.eamth.fr/cocv`, Vol.3, pp.315–321.

Fliess M., Lévine J., Martin Ph. and Rouchon P. (1999a): *A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems.* — IEEE Trans. Automat. Contr., Vol.44, No.5, pp.922–937.

Fliess M., Martin Ph., Petit N. and Rouchon P. (1999b): *Active signal restoration for the telegraph equation.* — Proc. Conf. Decision and Control, Phenix, U.S.A., pp.1107–1111.

Gelfand I.M. and Shilov G.E. (1964): *Les Distributions, T.3.* — Paris: Dunod.

Giaro A., Kumpera A. and Ruiz C. (1978): *Sur la lecture d'un résultat d'Élie Cartan.* — C.R. Acad. Sc. Paris, Série A, Vol.287, pp.241–244.

Goursat E. (1923): *Leçons sur le Problème de Pfaff.* — Paris: Hermann.

Hilbert D. (1912): *Über den Begriff der Klasse von Differentialgleichungen.* — Math. Ann., Vol.73, pp.95–108, also in: *Gesammelte Abhandlungen*, Vol.III, (1965) New York: Chelsea, pp.81–93.

Laroche B. and Martin Ph. (2000): *Motion planning for a 1-d diffusion equation using a brunovsky-like decomposition.* — Proc. *MTNS 2000*, Perpignan, France, published on CD-ROM.

Laroche B., Martin Ph. and Rouchon P. (1998): *Motion planning for a class of partial differential equations with boundary control.* — Proc. Conf. Control and Decision, Tampa, U.S.A., pp.3494–3497.

Lemon C. and Hause J.E. (1994): *Design and initial flight test of the Champagne Flyer.* — Proc. 33nd IEEE Conf. Decision and Control, Lake Buena Vista, U.S.A., pp.3852–3853.

Lenoir Y., Martin Ph. and Rouchon P. (1998): *$2k\pi$, the juggling robot.* — Proc. Conf. Control and Decision, Tampa, U.S.A., pp.1995–2000.

Martin Ph. (1992): *Contribution à l'étude des systèmes diffèrentiellement plats.* — Ph.D. Thesis, École des Mines de Paris.

Martin Ph., Devasia S. and Paden B. (1996): *A different look at output tracking: Control of a VTOL aircraft.* — Automatica, Vol.32, No.1, pp.101–107.

Martin Ph., Murray R. and Rouchon P. (1997): *Flat systems.* — Proc. 4th European Control Conf., Brussels, Belgium, Plenary Lectures and Mini-courses, pp.211–264.

Martin Ph. and Rouchon P. (1994): *Feedback linearization and driftless systems.* — Math. Contr. Signal Syst., Vol.7, No.7, pp.235–254.

Mounier H. (1995): *Propriétés structurelles des systèmes linéaires à retards: Aspects théoriques et pratiques.* — Ph.D. Thesis, Université Paris Sud, Orsay.

Mounier H. and Rudolph J. (1998): *Flatness based control of nonlinear delay systems: A chemical reactor example.* — Int. J. Contr., Vol.71, pp.871–890.

Mounier H., Rudolph J., Petitot M. and Fliess M. (1995): *A flexible rod as a linear delay system.* — Proc. 3rd European Control Conf., Rome, Italy, pp.3676–3681.

Mounier H., Rouchon P. and Rudolph J. (1996): *π-freeness of a long electric line.* — Proc. CESA '96 IMACS Multiconf., Lille, France, pp.28–29.

Murray R.M. (1994): *Nilpotent bases for a class on nonintegrable distributions with applications to trajectory generation for nonholonomic systems.* — Math. Contr. Signal Syst., Vol.7, No.1, pp.58–75.

Murray R.M. (1995): *Caltech ducted fan reference manual.* — Techn. Rep., California Institute of Technology, Division of Engineering and Applied Science.

Murray R.M. (1996): *Trajectory generation for a towed cable flight control system.* — Proc. *IFAC World Congress*, San Francisco, U.S.A., pp.395–400.

Murray R.M. and Sastry S.S. (1993): *Nonholonomic motion planning: Steering using sinusoids.* — IEEE Trans. Automat. Contr., Vol.38, No.5, pp.700–716.

Pasillas-Lépine W. (2000): *Systèmes de contact et structures de Goursat: Théorie et application au contrôle des systèmes mécaniques non holonomes.* — Ph.D. Thesis, Université de Rouen, France.

Petit N., Creff Y. and Rouchon P. (1998): *Motion planning for two classes of nonlinear systems with delays depending on the control.* — Proc. Conf. *Decision and Control*, Tampa, U.S.A., pp.1007–1011.

Petit N. and Rouchon P. (2001): *Motion planning for heavy chain systems.* — Wave 2000, (to appear).

Pommaret J.F. (1978): *Systems of Partial Differential Equations and Lie Pseudogroups.* — New York: Gordon & Breach.

Pommaret J.F. (1995): *Dualité différentielle et applications.* — C.R. Acad. Sci. Paris, Série I, Vol.320, pp.1225–1230.

Ramis J.P. (1979): *Dévissage Gevrey.* — Astérisque, Vol.59-60, pp.173–204.

Rothfuß R., Becker U. and Rudolph R. (2000): *Controlling a solenoid valve: A distributed parameter approach.* — Proc. *MTNS 2000*, Perpignan, France, published on CD-ROM.

Rouchon P., Fliess M., Lévine J. and Martin Ph. (1993a): *Flatness and motion planning: the car with n-trailers.* — Proc. *European Control Conf.*, Groningen, the Netherlands, pp.1518–1522.

Rouchon P., Fliess M., Lévine J. and Martin Ph. (1993b): *Flatness, motion planning and trailer systems.* — Proc. 32nd IEEE Conf. *Decision and Control*, San Antonio, U.S.A., pp.2700–2705.

Rouchon P. and Rudolph J. (2000): *Réacteurs chimiques différentiellement plats: Planification et suivi de trajectoires.* — Part II, Chapter III.f of the book *Automatique et procédés chimiques* (J.P. Corriou, Ed.), Paris: Hermès, (in print).

Rouchon P. and Rudolph J. (1999): *Invariant tracking and stabilization: Problem formulation and examples*, In: Stability and Stabilization of Nonlinear Systems (D. Acyels, F. Lamnabhi-Lagarrigve, A. van der Schaft, Eds.). — Berlin: Springer, pp.261–273.

Tilbury D.M. (1994): *Exterior differential systems and nonholonomic motion planning.* — Ph.D. Thesis, University of California, Berkeley, Memorandum No. UCB/ERL M94/90.

Valiron G. (1950): *Equations Fonctionnelles, 2nd Ed.* — Paris: Masson.

von Weber E. (1898): *Zur Invarianten Theorie der Systeme Pfaff'scher Gleigungen.* — Berichte Verhandlungen der Königlich Sächsichen Gesellshaft der Wissenschaften Mathematische-Physicalische Klasse, Leipzig, Vol.50, pp.207–229.