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# ELIMINATION OF FINITE EIGENVALUES OF THE 2D ROESSER MODEL BY STATE FEEDBACKS

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A new problem of decreasing the degree of the closed-loop characteristic polynomial of the 2D Roesser model by a suitable choice of state feedbacks is formulated. Sufficient conditions are established under which it is possible to choose state feedbacks such that the non-zero closed-loop characteristic polynomial has degree zero. A procedure for computation of the feedback gain matrices is presented and illustrated by a numerical example.

**Keywords:** elimination, finite eigenvalue, state feedback, 2D Roesser model

## 1. Introduction

The most popular models of two-dimensional (2D) linear systems are those introduced by Roesser (1975), Fornasini and Marchesini (1976; 1978), and Kurek (1985). The models were then generalised to singular linear systems (Kaczorek, 1988; 1993). Dai showed (1988; 1989) that for singular (descriptor) linear systems $E\dot{x} = Ax + Bu$, $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\det E = 0$, it is possible to choose a matrix $K \in \mathbb{R}^{m\times n}$ of the state feedback $u = Kx$ such that the non-zero closed-loop characteristic polynomial $\det[Es - (A + BK)]$ has degree zero. It is easy to show that for standard systems $(E = I)$ such state feedbacks do not exist.

The main subject of this note is to establish conditions for the standard 2D Roesser model under which it is possible to choose state feedbacks such that the non-zero closed-loop characteristic polynomial has degree zero. This procedure of decreasing the degree of the closed-loop characteristic polynomial by state feedbacks will be called the elimination of finite eigenvalues of the 2D Roesser model, since the closed loop has no finite eigenvalues (poles).

This type of problem arises, e.g., while designing perfect observers for linear 2D systems (2001). To the best of the author's knowledge, this elimination of finite eigenvalues of the 2D Roesser model by state feedbacks has not been considered yet.

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## 2. Problem Formulation

Let $\mathbb{R}^{n\times m}$ be the set of $n\times m$ real matrices and $\mathbb{R}^n := \mathbb{R}^{n\times 1}$. The set of non-negative integers will be denoted by $\mathbb{Z}_+$. Consider the 2D Roesser model

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = A \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Bu_{ij}, \quad i,j \in \mathbb{Z}_+, \tag{1}$$

where $x^h_{ij} \in \mathbb{R}^{n_1}$ and $x^v_{ij} \in \mathbb{R}^{n_2}$ are the horizontal and vertical state vectors, and $u_{ij} \in \mathbb{R}^m$ is the input vector,

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & , A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$A_1 \in \mathbb{R}^{n_1\times n_1}, \quad A_4 \in \mathbb{R}^{n_2\times n_2}, \quad B_1 \in \mathbb{R}^{n_1}, \quad B_2 \in \mathbb{R}^{n_2}.$$

The state feedback of the model is given by

$$u_{ij} = v_{ij} + K \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} - F \begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix}, \tag{2}$$

where $K = [K_1 \ \ K_2] \in \mathbb{R}^{m\times n}$, $F \in \mathbb{R}^{m\times n}$, $n = n_1 + n_2$, and $v_{ij} \in \mathbb{R}^m$ is the new input vector. From (1) and (2) we have

$$E \begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = (A + BK) \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Bv_{ij}, \tag{3}$$

where $E = I_n + BF$ and $I_n$ is the $n \times n$ identity matrix. The problem under consideration can be stated as follows: Given $A$ and $B$, find $F$ and $K$ such that

$$\det[EZ - (A + BK)] = \alpha \neq 0, \tag{4}$$

where

$$Z = \begin{bmatrix} I_{n_1} z_1 & 0 \\ 0 & I_{n_2} z_2 \end{bmatrix}$$

and $\alpha$ is a scalar independent of $z_1$ and $z_2$.

## 3. Problem Solution

The problem will be decomposed into the following two subproblems.

**Subproblem 1.** Given $B$, find $F$ such that $E \neq 0$ and

$$\det E = 0. \tag{5}$$

**Subproblem 2.** Given $E\,(E \neq 0, \det E = 0)$, $A$ and $B$, find $K$ such that (4) holds. Solution of Subproblem 1 is based on the following theorem.

**Theorem 1.** *Let $E = I_n + BF$. There exists a matrix $F = [f_{ij}]$ such that (5) holds if and only if $B \neq 0$.*

*Proof.* (Necessity) If $B = 0$, then $\det E = 1$ for any $F$.

(Sufficiency) If $B = [b_{ij}] \neq 0$, then for at least one pair $(k, l)$ $b_{kl} \neq 0$ for $k \in [1, \dots, n]$, $l \in [1, \dots, m]$, and we can choose

$$f_{ij} = \begin{cases} -1/b_{kl} & \text{for } i = l,\ j = k, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Then

$$I_n + BF = \begin{bmatrix} 1 & 0 & \cdots & 0 & b_{1l}f_{lk} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & b_{k-,1l}f_{lk} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & b_{k+1l}f_{lk} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & b_{nl}f_{lk} & 0 & \cdots & 1 \end{bmatrix}$$

and $\det E = 0$. ■

In the sequel, the following elementary operations will be used:

1. multiplication of any row (column) by a non-zero number,
2. addition to any row (column) of any other row (column) multiplied by any number,
3. interchange of any rows (columns).

A non-singular matrix $P$ obtained from $I_n$ by performing a number of elementary row operation will be called the elementary row operation matrix. Similarly, an elementary column operation matrix can be defined. Solution of Subproblem 2 is based on the following theorem.

**Theorem 2.** *Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $E \neq 0$, and $\det E = 0$. There exists a matrix $K \in \mathbb{R}^{m\times n}$ such that (4) holds if*

$$\operatorname{rank}\big[EZ - A, B\big] = n \tag{7a}$$

*for all finite $z_1, z_2 \in \mathbb{C}$ (the field of complex numbers), and*

$$\operatorname{rank}\big[E, B\big] = n. \tag{7b}$$

*The condition (7a) is necessary for the existence of $K \in \mathbb{R}^{m\times n}$ satisfying (4).*

*Proof.* To simplify the notation, it is assumed that $m = 1$. If (7b) holds and the non-zero matrix $E$ is singular, then there exists a non-singular elementary row operation matrix $P_1$ and a non-singular elementary column operation matrix $Q_1$ such that (Kaczorek, 2001)

$$\begin{aligned}\left[E', \bar{B}\right] &= P_1\,[EZ, B]\begin{bmatrix} Q_1 & 0 \\ 0 & I_m \end{bmatrix}\\ &= \left[\begin{array}{cccccc:c} e'_{11} & 0 & 0 & \cdots & 0 & 0 & 0 \\ e'_{21} & e'_{22} & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e'_{n-1,1} & e'_{n-1,2} & e'_{n-1,3} & \cdots & e'_{n-1,n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right].\end{aligned}$$

(Note that $e'_{ij}$ may depend on $z_1$ or $z_2$.) If (7a) is satisfied, then there exist non-singular elementary row and column operation matrices $P_2$, $Q_2$ and $P = P_2P_1$, $Q = Q_2Q_1$ such that

$$\begin{aligned}\left[\bar{E} - \bar{A}, \bar{B}\right] &= P\,[EZ - A, B]\begin{bmatrix} Q & 0 \\ 0 & I_m \end{bmatrix}\\ &= \left[\begin{array}{ccccc:c} \bar{e}_{11} - \bar{a}_{11} & -\bar{a}_{12} & 0 & \cdots & 0 & 0 \\ \bar{e}_{21} - \bar{a}_{21} & \bar{e}_{22} - \bar{a}_{22} & -\bar{a}_{23} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{e}_{n-1,1} - \bar{a}_{n-1,1} & \bar{e}_{n-1,2} - \bar{a}_{n-1,2} & \bar{e}_{n-1,3} - \bar{a}_{n-1,3} & \cdots & -\bar{a}_{n-1,n} & 0 \\ -\bar{a}_{n1} & -\bar{a}_{n2} & -\bar{a}_{n3} & \cdots & -\bar{a}_{nn} & 0 \end{array}\right], \end{aligned}\tag{8}$$

where $\bar{a}_{i,j+1} \neq 0$ for $i = 1, \dots, n-1$.

Let (for $m = 1$)

$$\bar{K} = KQ = \left[1 - \bar{a}_{n1},\ -\bar{a}_{n2},\ \dots,\ -\bar{a}_{nn}\right].\tag{9}$$

Then

$$\begin{aligned}P\left[EZ - (A + BK)\right]Q &= \left[\bar{E} - (\bar{A} + \bar{B}\bar{K})\right]\\ &= \begin{bmatrix} \bar{e}_{11} - \bar{a}_{11} & -\bar{a}_{12} & 0 & \cdots & 0 \\ \bar{e}_{21} - \bar{a}_{21} & \bar{e}_{22} - \bar{a}_{22} & -\bar{a}_{23} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{e}_{n-1,1} - \bar{a}_{n-1,1} & \bar{e}_{n-1,2} - \bar{a}_{n-1,2} & \bar{e}_{n-1,3} - \bar{a}_{n-1,3} & \cdots & -\bar{a}_{n-1,n} \\ -1 & 0 & 0 & \cdots & 0 \end{bmatrix}\end{aligned}$$

and

$$\det\left[EZ-(A+BK)\right] = -\det P^{-1}\det Q^{-1}a_{12}a_{23}\cdots a_{n-1,n} \neq 0.$$

From the equality

$$\left[EZ-(A+BK)\right] = [EZ-A, B]\begin{bmatrix} I_n \\ -K \end{bmatrix}$$

it follows that (4) implies (7a). ■

**Example 1.** For given

$$E = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right], \quad A = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ 1 & 2 & 1 \\ \hdashline 1 & 0 & 1 \end{array}\right], \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{10}$$

we wish to find $K = [k_1 \;\; k_2 \;\; k_3]$ such that (4) holds.

It is easy to check that the matrices (10) satisfy the assumptions of Theorem 2 since

$$\operatorname{rank}\left[EZ-A, B\right] = \operatorname{rank}\left[\begin{array}{ccc:c} z_1 & -1 & 0 & 0 \\ -1 & -2 & -1 & 1 \\ -1 & 0 & z_2-1 & 0 \end{array}\right] = 3$$

for all finite $z_1,\ z_2 \in \mathbb{C}$, and

$$\operatorname{rank}\left[E, B\right] = \operatorname{rank}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] = 3.$$

Using elementary operations, the matrix

$$[EZ-A, B] = \left[\begin{array}{ccc:c} z_1 & -1 & 0 & 0 \\ -1 & -2 & -1 & 1 \\ -1 & 0 & z_2-1 & 0 \end{array}\right]$$

can be reduced to the form

$$\left[\begin{array}{ccc:c} z_2-1 & -1 & 0 & 0 \\ 0 & -z_1 & -1 & 0 \\ -1 & -1 & -2 & 1 \end{array}\right] \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

From (9) we obtain

$$K = \bar{K}Q^{-1} = [0 \ \ -1 \ \ -2] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [-1 \ \ -2 \ \ 0].$$

♦

**Theorem 3.** *Let* $B \neq 0$ *and* $F$ *be chosen so that* $E \neq 0$ *and* $\det E = 0$. *Then there exists* $K \in \mathbb{R}^{m\times n}$ *such that (4) holds if*

$$\operatorname{rank}\big[Z - A, B\big] = n \ \ \textit{for all finite } z_1,\ z_2 \in \mathbb{C}. \tag{11}$$

*Proof.* By Theorem 2 there exists $K$ such that (4) holds if the conditions (7) are satisfied. The condition (7a) is satisfied if and only if (11) holds, since

$$\begin{aligned} \operatorname{rank}\big[EZ - A, B\big] &= \operatorname{rank}\big[(I_n + BF)Z - A, B\big] \\ &= \operatorname{rank}\left([Z - A, B]\begin{bmatrix} I_n & 0 \\ FZ & I_m \end{bmatrix}\right) = \operatorname{rank}\big[Z - A, B\big]. \end{aligned}$$

The condition (7b) is always satisfied, since

$$\begin{aligned} \operatorname{rank}\big[E, B\big] &= \operatorname{rank}\big[I_n + BF, B\big] \\ &= \operatorname{rank}\left([I_n, B]\begin{bmatrix} I_n & 0 \\ F & I_m \end{bmatrix}\right) = \operatorname{rank}\big[I_n, B\big] = n. \end{aligned}$$

■

From Theorems 1 and 3 we immediately have the following result.

**Theorem 4.** *Let* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times m}$ *be given. The problem has a solution if* $B \neq 0$ *and (11) holds.*

If the condition (11) is satisfied and $B \neq 0$, then $F$ and $K$ can be computed by using the following procedure:

**Procedure**

**Step 1.** *Using (6), compute* $F$ *satisfying* $\det E = 0$ *and* $E = I_n + BF$.

**Step 2.** *Compute $K$ such that (4) holds using the method of elementary operations, or by assuming $a_{kl} = 0$ for $k = 1, \dots, r_1$, $l = 1, \dots, r_2$ and $a_{00} \neq 0$ of the polynomial*

$$\det\big[EZ - (A + BK)\big] = a_{r_1 r_2} z_1^{r_1} z_2^{r_2} + a_{r_{11}-1, r_2} z_1^{r_1 - 1} z_2^{r_2} + \cdots + a_{11} z_1 z_2 + a_{10} z_1 + a_{01} z_2 + a_{00}.$$

**Example 2.** For the matrices

$$A = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ 1 & 2 & 1 \\ \hdashline 1 & 0 & 1 \end{array}\right], \quad B = \begin{bmatrix} 0 \\ 1 \\ \hdashline 0 \end{bmatrix}$$

choose matrices $F = [f_1 \;\; f_2 \;\; f_3]$ and $K = [k_1 \;\; k_2 \;\; k_3]$ such that (4) is satisfied. It is easy to check that the assumptions of Theorem 4 are met, since $B \neq 0$ and

$$\operatorname{rank}\big[Z - A, B\big] = \operatorname{rank}\left[\begin{array}{ccc:c} z_1 & -1 & 0 & 0 \\ -1 & z_1 - 2 & -1 & 1 \\ -1 & 0 & z_2 - 1 & 0 \end{array}\right] = 3 \quad \text{for all finite } z_1, z_2 \in \mathbb{C}.$$

Using the foregoing procedure, we obtain:

**Step 1.** From (6) we have $F = [0 \;\; -1 \;\; 0]$ and

$$E = I_n + BF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Step 2.** Using (4), we obtain

$$\begin{aligned} \det\big[EZ - (A + BK)\big] &= \begin{vmatrix} z_1 & -1 & 0 \\ -k_1 - 1 & -k_2 - 2 & -k_3 - 1 \\ -1 & 0 & z_2 - 1 \end{vmatrix} \\ &= -(k_2 + 2) z_1 z_2 + (k_2 + 2) z_1 - (k_1 + 1) z_2 + k_1 + 1 - k_3 - 1. \end{aligned}$$

For $k_1 = -1$, $k_2 = -2$, $k_3 = -1 - \alpha$ we get (4). The same result was obtained using the elementary operation method, cf. Example 1. ♦

## 4. Concluding Remarks

A new problem of decreasing the degree of the closed-loop characteristic polynomial of the 2D Roesser model by state feedbacks was formulated and solved. Sufficient conditions were established under which it is possible to choose the state feedbacks (2) for the standard 2D Roesser model (1) such that (4) holds. It was shown that the problem has a solution if $B \neq 0$ and the condition (11) is satisfied. A procedure for computation of the gain matrices $F$ and $K$ of (2) was presented and illustrated by a numerical example. If the 2D Roesser model is singular $(\det E = 0)$, then there exists a gain matrix $K$ of (2) for $F = 0$ such that (4) holds if the condition (7) is satisfied. The considerations can be extended to 2D Fornasini-Marchesini models (1976; 1978) and the Kurek model (1985).

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