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ROUGH RELATION PROPERTIES

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Rough Set Theory (RST) is a mathematical formalism for representing uncertainty that can be considered an extension of the classical set theory. It has been used in many different research areas, including those related to inductive machine learning and reduction of knowledge in knowledge-based systems. One important concept related to RST is that of a rough relation. This paper rewrites some properties of rough relations found in the literature, proving their validity.

Keywords: rough set theory, rough relation, knowledge representation

1. Introduction

Rough Set Theory (RST) was proposed by Pawlak (1982) as an extension of the classical set theory, for use when representing incomplete knowledge. Rough sets can be considered sets with fuzzy boundaries—sets that cannot be precisely characterized using the available set of attributes. During the last few years RST has been approached as a formal tool used in connection with many different areas of research. There have been investigations of the relations between RST and the Dempster-Shafer Theory (Skowron and Grzymala-Busse, 1994; Wong and Lingras, 1989), and between rough sets and fuzzy sets (Pawlak, 1994; Pawlak and Skowron, 1994; Wygralak, 1989). RST has also provided the necessary formalism and ideas for the development of some propositional machine learning systems (Grzymala-Busse, 1992; Mrózek, 1992; Pawlak, 1984; 1985; Wong *et al.*, 1986). It has also been used for, among many other things, knowledge representation (Orlowska and Pawlak, 1984; Ziarko, 1991), data mining (Aasheim and Solheim, 1996; Deogun *et al.*, 1997), dealing with imperfect data (Grzymala-Busse, 1988; Szladow and Ziarko, 1993), reducing the knowledge representation (Grzymala-Busse, 1986; Jelonek *et al.*, 1994; Pawlak *at el.*, 1988), helping to solve control problems (Ohrn, 1993; Pawlak, 1997; Słowiński, 1995), and analysing attribute dependencies (Grzymala-Busse and Mithal, 1991; Mrózek, 1989).

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The notions of rough relations and rough functions are based on RST and, as discussed in (Pawlak, 1997, p.139), 'are needed in many applications, where experimental data are processes, in particular as a theoretical basis for rough controllers'. This paper presents the main concepts related to rough relations, rewrites some of its properties and proves them to be valid. It is organized as follows: Section 2 is a selection of mathematical results that constitute essential background knowledge to what follows. Section 3 presents the basic concepts and notations related to Rough Set Theory (extracted from the various sources listed in the References) as well as some results that are necessary to understand Section 4, where the basic concepts and notations related to rough relations are presented. In Section 5 the main properties of rough relations are established and proved to be valid.

2. Mathematical Prerequisites

Some of the results presented in this section have been extracted from (Berztiss, 1975).

Definition 1. A *binary relation* from set A to set B is a subset of $A \times B$. If R is a relation, we write $(x, y) \in R$ and xRy interchangeably.

Definition 2. A subset of $A \times A$ is a binary relation in the set A. In particular, the set $A \times A$ is the universal relation in A.

Definition 3. Let A be a set and R a relation in A. The set of *R-relatives* of the elements of A is $R[A] = \{y \mid \text{for some } x \text{ in } A, xRy\}$.

Definition 4. If R is a relation from A to B, the *reversed relation* of R, written as R^{-1}, is a relation from B to A such that $yR^{-1}x$ if and only if xRy.

Definition 5. A relation R in a set A is

1. *reflexive* if xRx for all $x \in A$,
2. *nonreflexive* if $\exists x \in A$ such that $x \not R x$,
3. an *identity* if it is reflexive and if xRy for $x, y \in A$ yields $x = y$,
4. *symmetric* if xRy for $x, y \in A$ yields yRx,
5. *nonsymmetric* if $\exists x, y \in A$ such that xRy and $y \not R x$,
6. *antisymmetric* if xRy and yRx for $x, y \in A$ yields $x = y$,
7. *transitive* if xRy and yRz for $x, y, z \in A$ yields xRz.

Definition 6. A reflexive, antisymmetric and transitive relation in a set is a partial order relation or a *partial ordering* in that set. If R is a partial ordering in A, the ordered pair (A, R) is a partially ordered set.

Definition 7. A relation in a set A is an *equivalence relation* in A if it is reflexive, symmetric and transitive.

Definition 8. Let R be an equivalence relation on a set A. Consider an element a of A. The set of R-relatives of a in A, $R[\{a\}]$ is called the *R-equivalence class* generated by a. When there is no danger of confusion, the symbol $R[\{a\}]$ can be abbreviated to $[a]$.

Proposition 1. *Let R be an equivalence relation on A and let $a, b \in A$. Then*
1. $a \in [a]$, and
2. if aRb then $[a] = [b]$.

Proposition 2. *If Q is an equivalence relation in A, then $Q = \cup_{1 \leq i \leq n} Z_i \times Z_i$, where Z_i, $1 \leq i \leq n$ are equivalence classes in A induced by Q.*

Proof. We have $(a, b) \in Q \Leftrightarrow a, b \in Z_i$, for some $i \in \{1, 2, \ldots, n\}$, where Z_i is an equivalence class in A induced by $Q \Leftrightarrow (a, b) \in \cup_{1 \leq i \leq n} Z_i \times Z_i$ for $i \in \{1, 2, \ldots, n\}$. ■

Definition 9. If V and W are relations in A, then $W \bullet V$ is a relation in A defined as $W \bullet V = \{(a, c) \in A \times A$ such that $(a, b) \in V$ and $(b, c) \in W$ for some $b \in A\}$.

Proposition 3. *If V, W, V_1, W_1 are relations in A, $V_1 \subseteq V$ and $W_1 \subseteq W$, then $W_1 \bullet V_1 \subseteq W \bullet V$.*

Proof. Let $(a, c) \in W_1 \bullet V_1 \Rightarrow \exists b \in A$ such that $(a, b) \in V_1$ and $(b, c) \in W_1$. Then $(a, b) \in V$ and $(b, c) \in W$ so that $(a, c) \in W \bullet V$. ■

3. Rough Set Theory

3.1. Basic Concepts

The basic concept of Rough Set Theory is the notion of an approximation space, which is an ordered pair $A = (U, R)$, where U is a nonempty set of objects, called the *universe*, and R stands for the equivalence relation on U, called the *indiscernibility relation*. If $x, y \in U$ and xRy, then x and y are *indistinguishable* in A.

Each equivalence class induced by R, i.e. each element of the quotient set $\tilde{R} = U/R$, is called an *elementary set* in A. An approximation space can be alternatively denoted by $A = (U, \tilde{R})$. It is assumed that the empty set is also elementary for every approximation space A. A *definable set* in A is any finite union of elementary sets in A. For $x \in U$, let $[x]_R$ denote the equivalence class of R, containing x. For each subset X in A, X is characterized by a pair of sets—its *lower* and *upper approximations* in A, defined respectively as

$$A_{A\text{-low}}(X) = \big\{x \in U \mid [x]_R \subseteq X\big\},$$

$$A_{A\text{-upp}}(X) = \big\{x \in U \mid [x]_R \cap X \neq \emptyset\big\}.$$

When there is no risk of misunderstanding and for the sake of simplicity, we prefer to use A_{low} and A_{upp} instead of $A_{A\text{-low}}$ and $A_{A\text{-upp}}$, respectively. The lower approximation of X in A is the greatest definable set in A contained in X, and the upper approximation of X in A is the smallest definable set in A containing X, with relation to the set inclusion. A set $X \subseteq U$ is definable in A iff $A_{\mathrm{low}}(X) = A_{\mathrm{upp}}(X)$. A rough set in A is the family of all subsets of U having the same lower and upper approximations. Another definition found in (Klir and Yuan, 1995) states that 'a rough set is a representation of a given set X, by two subsets of the quotient set U/R which approach X as closely as possible from inside and outside, respectively. That is, $\langle A_{\mathrm{low}}(X), A_{\mathrm{upp}}(X)\rangle$'. Both the definitions are shown to be equivalent in (Nicoletti and Uchôa, 1997).

3.2. Some Basic RST Results

The results presented in this section are relevant to the proofs that follow. They are stated as propositions and their proofs can be found in related literature. Let $A = (U, R)$ be an approximation space and $X \subseteq U$.

Proposition 4. *The following assertions hold:*
1. $A_{\mathrm{low}}(X) = \cup Y$ *such that* Y *is definable and* $Y \subseteq X$*, and*
2. $A_{A\text{-upp}}(X) = \cap_Y$ *such that* Y *is definable and* $X \subseteq Y$*.*

Proposition 5. $A_{\mathrm{low}}(X) \subseteq X \subseteq A_{\mathrm{upp}}(X)$.

Proposition 6. $A_{\mathrm{low}}(X) = A_{\mathrm{upp}}(X) \Leftrightarrow X$ *is definable.*

Proposition 7. $A_{\mathrm{upp}}(X \cup Y) = A_{\mathrm{upp}}(X) \cup A_{\mathrm{upp}}(Y)$.

4. Rough Relations

Let $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ be two approximation spaces. The product of A_1 by A_2 is the approximation space denoted by $A = (U, S)$, where $U = U_1 \times U_2$ and the indiscernibility relation $S \subseteq (U_1 \times U_2)^2$ is defined by $((x_1, y_1), (x_2, y_2)) \in S \Leftrightarrow (x_1, x_2) \in R_1$ and $(y_1, y_2) \in R_2$, $x_1, x_2 \in U_1$ and $y_1, y_2 \in U_2$. It can be easily proven that S is an equivalence relation.

The elements (x_1, y_1) and (x_2, y_2) are indiscernible in S if and only if the elements x_1 and x_2 are indiscernible in R_1 and so are the elements y_1 and y_2 in R_2. This implies that the equivalence class of S containing (x, y), denoted by $[(x, y)]_S$, should be equal to the Cartesian product of $[x]_{R_1}$ by $[y]_{R_2}$, according to Proposition 8. Example 1 shows the approximation space resulting from the product of two approximation spaces.

Example 1. Let $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ be two approximation spaces, where $U_1 = \{x_1, x_2, x_3, x_4\}$, $R_1 = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_1, x_2), (x_2, x_1), (x_3, x_4), (x_4, x_3)\}$, $U_2 = \{a, b, c\}$ and $R_2 = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$.

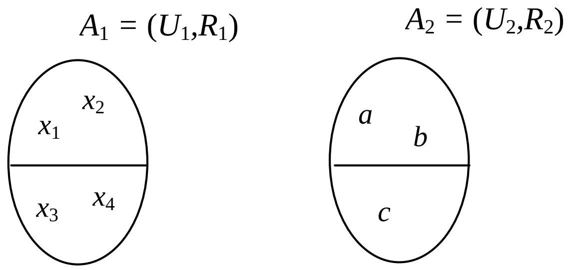

Fig. 1. Approximation spaces $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$, where $U_1 = \{x_1, x_2, x_3, x_4\}$ and $U_2 = \{a, b, c\}$.

The approximation spaces A_1 and A_2 (and their elementary sets) are shown in Fig. 1.

Let $A = (U, R) = (U_1 \times U_2, R)$ be the approximation space resulting from the product of A_1 by A_2, where $U = \{(x_1, a), (x_1, b), (x_1, c), (x_2, a), (x_2, b), (x_2, c), (x_3, a), (x_3, b), (x_3, c), (x_4, a), (x_4, b), (x_4, c)\}$ and R is defined by pairs $((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow (x_1, x_2) \in R_1$ and $(y_1, y_2) \in R_2$. Figure 2 shows the approximation space A given by its elementary sets. ⧫

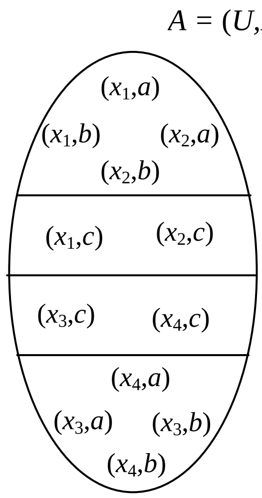

Fig. 2. $A = (U, R)$ is an approximation space resulting from the product of A_1 by A_2 shown in Fig. 1.

The concepts of RST can be easily extended to a relation, mainly due to the fact that a relation is also a set, i.e. a subset of a Cartesian product. So, let $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ be two approximation spaces and $A = (U, R) = (U_1 \times U_2, R)$ the approximation space obtained by the product of A_1 by A_2. Given a relation (or a set) $X \subseteq U_1 \times U_2$, the lower and upper approximations of X in the approximation space A can respectively be defined as:

$$A_{\mathrm{low}}(X) = \big\{(x,y) \in U_1 \times U_2 \mid [(x,y)]_R \subseteq X\big\},$$

$$A_{\mathrm{upp}}(X) = \big\{(x,y) \in U_1 \times U_2 \mid [(x,y)]_R \cap X \neq \emptyset\big\}.$$

Example 2. Let $A = (U,R) = (U_1 \times U_2, R)$ be the approximation space as defined in Example 1. Consider the relations X, Y, $Z \subseteq U_1 \times U_2$ such that $X = \{(x_1,a),(x_1,b)\}$, $Y = \{(x_1,c),(x_2,c),(x_3,c),(x_4,c)\}$, and $Z = \{(x_1,a),(x_1,c),(x_3,a),(x_3,c),(x_4,c)\}$. Consequently,

$$A_{\mathrm{low}}(X) = \emptyset, \quad A_{\mathrm{upp}}(X) = \big[(x_1,a)\big]_R = \big[(x_1,b)\big]_R = \big\{(x_1,a),(x_1,b),(x_2,a),(x_2,b)\big\},$$

$$A_{\mathrm{low}}(Y) = A_{\mathrm{upp}}(Y) = \big[(x_1,c)\big]_R \cup \big[(x_3,c)\big]_R = \big\{(x_1,c),(x_2,c),(x_3,c),(x_4,c)\big\} = Y,$$

$$A_{\mathrm{low}}(Z) = \big[(x_3,c)\big]_R = \big\{(x_3,c),(x_4,c)\big\},$$

$$A_{\mathrm{upp}}(Z) = \big[(x_1,a)\big]_R \cup \big[(x_1,c)\big]_R \cup \big[(x_3,a)\big]_R \cup \big[(x_3,c)\big]_R = U.$$

♦

Proposition 8. *Let $A = (U,R)$ be an approximation space and $B = (U^2,S)$ the approximation product space of A by A. Then:*
1. $\big[(x,y)\big]_S = [x]_R \times [y]_R$, and
2. $\big[(y,z)\big]_S \bullet \big[(x,y)\big]_S = \big[(x,z)\big]_S$.

Proof. 1. It trivially follows from the definition of the relation S.
2. Let $(a,c) \in [(y,z)]_S \bullet [(x,y)]_S$. Then there exists a $b \in U$ such that $(a,b) \in [(x,y)]_S$ and $(b,c) \in [(y,z)]_S$. It follows that $(a,b)S(x,y)$ and $(b,c)S(y,z)$. Hence aRx, bRy, bRy and cRz. Consequently, $(a,c) \in [(x,z)]_S$.

On the other hand, let $(a,c) \in [(x,z)]_S$. This gives $(a,c)S(x,z)$. We thus get aRx and cRz. Since R is an equivalence relation, aRx, yRy and cRz. This clearly forces $(a,y)S(x,y)$ and $(y,c)S(y,z)$. Hence $(a,y) \in [(x,y)]_S$ and $(y,c) \in [(y,z)]_S$, and therefore $(a,c) \in [(y,z)]_S \bullet [(x,y)]_S$. ■

5. Rough Relation Properties

The reference (Pawlak, 1981, pp.9–10) lists twelve properties of approximations of binary relations in a product space and assumes that they are all true. However, when evaluating these properties, we found that some of them do not exactly prove their validity as stated in the reference. In the following we rewrite those properties and prove those that are valid. In order to do that, we will consider an approximation space $A = (U,R)$ and $B = A \times A = (U^2,S)$ as the approximation product space, where $S \subseteq U^2$. We will also consider a relation $Q \subseteq U^2$.

Property 1. *If Q is an identity relation in U, then*
1. $A_{\text{upp}}(Q)$ is an identity relation in $U \Leftrightarrow A_{\text{upp}}(Q) = Q$,
2. $A_{\text{low}}(Q)$ is an identity relation in $U \Leftrightarrow A_{\text{low}}(Q) = Q$, and
3. if $A_{\text{low}}(Q) \neq \emptyset$ and $A_{\text{low}}(Q) \neq Q$, then $A_{\text{low}}(Q)$ is an identity relation in a proper subset of U.

Proof. 1. From the fact that Q is an identity relation in U, knowing from Proposition 5 that $Q \subseteq A_{\text{upp}}(Q)$, it follows that $A_{\text{upp}}(Q)$ is an identity relation in $U \Leftrightarrow A_{\text{upp}}(Q) = Q$.
2. This results from the facts that $A_{\text{low}}(Q) \subseteq Q$ (by Proposition 5) and that Q is an identity relation in U.
3. It trivially follows from the fact that a subset of an identity relation defined in U will be an identity relation in a proper subset of U. ■

As a consequence of Property 1, it can be said that *if Q is an identity relation in U, $A_{\text{low}}(Q)$ and $A_{\text{upp}}(Q)$ are both identity relations in U iff Q is definable.*

Property 2. *If Q is a reflexive relation in U, then*
1. $A_{\text{upp}}(Q)$ is a reflexive relation in U, and
2. if $A_{\text{low}}(Q) \neq Q$, nothing can be said about the reflexivity of $A_{\text{low}}(Q)$.

Proof. 1. Since Q is reflexive in U and $Q \subseteq A_{\text{upp}}(Q)$, we conclude that $A_{\text{upp}}(Q)$ is reflexive in U.
2. This inconclusive assertion can be evidenced in Example 3. ■

Example 3. Let $A = (U, R)$ be an approximation space such that $U = \{a, b, c, d\}$ and $R = \{(a, a), (b, b), (a, b), (b, a), (c, c), (d, d), (c, d), (d, c)\}$, i.e. $U/R = \{\{a, b\}, \{c, d\}\}$. Consider the approximation space B given by $A \times A$, i.e. $B = (U^2, S)$, such that $U^2/S = \{\{(a, a), (b, b), (a, b), (b, a)\}, \{(a, c), (a, d), (b, c), (b, d)\}, \{(c, c), (d, d), (c, d), (d, c)\}, \{(c, a), (d, a), (c, b), (d, b)\}\}$. This situation is depicted in Figs. 3 and 4.

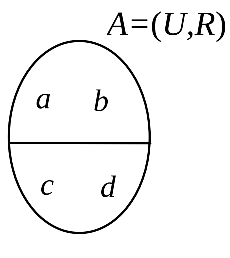

Fig. 3. Approximation space $A = (U, R)$ where $U = \{a, b, c, d\}$.

We may have, for example, the following settings:

- Let $Q = \{(a, a), (b, b), (c, c), (d, d)\}$ be a reflexive relation in U. Then we get $A_{\text{low}}(Q) = \emptyset$, i.e. we have a nonreflexive relation.

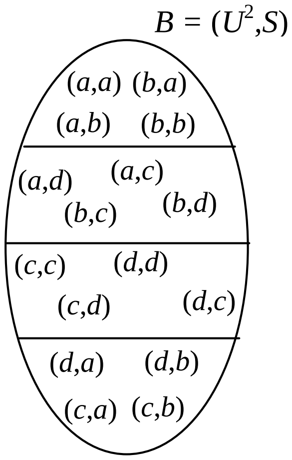

Fig. 4. $B = (U^2, S)$ is the approximation product space $A \times A$.

- Let $Q = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(c,d),(d,c),(d,a)\}$ be a reflexive relation in U. Then $A_{\mathrm{low}}(Q) = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(c,d),(d,c)\}$ is a reflexive relation in U.
- Let $Q = \{(a,a),(b,b),(c,c),(d,d),(a,c),(a,d),(b,c),(b,d)\}$ be a reflexive relation in U. Then $A_{\mathrm{low}}(Q) = \{(a,c),(a,d),(b,c),(b,d)\}$ is a nonreflexive relation in any subset of U.
- Let $Q = \{(a,a),(a,b),(b,b),(b,a),(c,c),(d,d)\}$ be a reflexive relation in U. Then $A_{\mathrm{low}}(Q) = \{(a,a),(b,b),(a,b),(b,a)\}$ constitutes a reflexive relation in a proper subset of U. ♦

Property 3. *If Q is a symmetric relation in U, then*
1. $A_{\mathrm{upp}}(Q)$ is symmetric, and
2. $A_{\mathrm{low}}(Q)$ is symmetric provided that $A_{\mathrm{low}}(Q) \neq \emptyset$.

Proof. 1. Assume that Q is symmetric. Let $(x,y) \in A_{\mathrm{upp}}(Q)$. Thus, $[(x,y)]_S \cap Q \neq \emptyset$, i.e. $\exists (x_1,y_1) \in Q$ such that xRx_1 and yRy_1. This clearly forces yRy_1 and xRx_1. Hence $(y,x)S(y_1,x_1)$. Since $(x_1,y_1) \in Q$ and Q is symmetric, we have $(y_1,x_1) \in Q$. So, $[(y,x)]_S \cap Q = [(y_1,x_1)]_S \cap Q \neq \emptyset$, and therefore $(y,x) \in A_{\mathrm{upp}}(Q)$, i.e. $A_{\mathrm{upp}}(Q)$ is symmetric.
2. Assume that Q is symmetric and $A_{\mathrm{low}}(Q) \neq \emptyset$. Let $(x,y) \in A_{\mathrm{low}}(Q)$. We see that $[(x,y)]_S \subseteq Q$ and, since Q is symmetric, this implies that $[(y,x)]_S \subseteq Q$, i.e. $(y,x) \in A_{\mathrm{low}}(Q)$, which means that $A_{\mathrm{low}}(Q)$ is symmetric. ■

Property 4. *If Q is an antisymmetric relation in U, then*
1. $A_{\mathrm{low}}(Q)$ is antisymmetric in U, provided that $A_{\mathrm{low}}(Q) \neq \emptyset$, and
2. nothing can be said about the antisymmetry of $A_{\mathrm{upp}}(Q)$ if $Q \neq A_{\mathrm{upp}}(Q)$.

Proof. 1. If $A_{\mathrm{low}}(Q) \neq \emptyset$, then $\exists (x,y) \in A_{\mathrm{low}}(Q)$. But if $(x,y) \in A_{\mathrm{low}}(Q)$ and $(y,x) \in A_{\mathrm{low}}(Q)$, since $A_{\mathrm{low}}(Q) \subseteq Q$ and Q is antisymmetric, we have $(x,y) = (y,x)$.

2. Consider again Example 3 and an antisymmetric relation given by:

- $Q = \{(a,a),(b,b),(c,c),(d,d)\}$; $A_{\text{upp}}(Q) = \{(a,a),(b,b),(a,b),(b,a),(c,c),(d,d),(c,d),(d,c)\}$ is not an antisymmetric relation since $(a,b),(b,a) \in A_{\text{upp}}(Q)$ and $a \neq b$;
- $Q = \{(a,c),(a,d),(b,c)\}$; $A_{\text{upp}}(Q) = \{(a,c),(a,d),(b,c),(b,d)\}$ is an antisymmetric relation. ■

Property 5. *If Q is a nonsymmetric relation in U, then nothing can be said about $A_{\text{low}}(Q)$ and $A_{\text{upp}}(Q)$ being or not a nonsymmetric relation.*

In the settings of Example 3 consider the following nonsymmetric relations:

- $Q = \{(a,a),(b,b),(a,b),(b,a),(c,d)\}$; thus $A_{\text{low}}(Q) = \{(a,a),(a,b),(b,a),(b,b)\}$ and $A_{\text{upp}}(Q) = \{(a,a),(b,b),(a,b),(b,a),(c,c),(d,d),(c,d),(d,c)\}$ are symmetric relations;
- $Q = \{(a,c),(a,d),(b,c),(b,d),(c,d)\}$; thus $A_{\text{low}}(Q) = \{(a,c),(a,d),(b,c),(b,d)\}$ and $A_{\text{upp}}(Q) = \{(a,c),(a,d),(b,c),(b,d),(c,c),(d,d),(c,d),(d,c)\}$ are nonsymmetric relations;
- $Q = \{(a,c),(a,d),(b,c),(b,d),(d,a)\}$; thus $A_{\text{low}}(Q) = \{(a,c),(a,d),(b,c),(b,d)\}$ is a nonsymetric relation and $A_{\text{upp}}(Q) = \{(a,c),\ (a,d),\ (b,c),\ (b,d),\ (d,a),\ (d,b),\ (c,a),\ (c,b)\}$ constitutes a symmetric relation;
- $Q = \{(a,a),(a,b),(b,b),(b,a),(a,c)\}$; thus $A_{\text{low}}(Q) = \{(a,a),(a,b),(b,b),(b,a)\}$ is a symmetric relation and $A_{\text{upp}}(Q) = \{(a,a),(a,b),(b,b),(b,a),(a,c),(a,d),(b,c),(b,d)\}$ is a nonsymmetric relation.

Property 6. *If Q is a transitive relation in U, then*
1. $A_{\text{low}}(Q)$ is a transitive relation in U provided that $A_{\text{low}}(Q) \neq \emptyset$, and
2. nothing can be said about the transitivity of $A_{\text{upp}}(Q)$.

Proof. 1. Assume that Q is transitive. Let $(x,y) \in A_{\text{low}}(Q)$ and $(y,z) \in A_{\text{low}}(Q)$. Consequently, we have

- $(x,y) \in A_{\text{low}}(Q) \Rightarrow [(x,y)]_S \subseteq Q$,
- $(y,z) \in A_{\text{low}}(Q) \Rightarrow [(y,z)]_S \subseteq Q$,
- $(x,y) \in Q$ and $(y,z) \in Q \Rightarrow (x,z) \in Q$, i.e. $[(x,z)]_S \cap Q \neq \emptyset$.

In order for $A_{\text{low}}(Q)$ to be transitive, we have to prove that $[(x,z)]_S \subseteq Q$. Let $(a,b) \in [(x,z)]_S$. This forces $(a,b)S(x,z)$. Hence aRx and bRz. Given aRx and yRy, it can be said that $(a,y)S(x,y)$. This gives $(a,y) \in [(x,y)]_S \subseteq Q$. Also, if yRy and bRz, then $(y,b)S(y,z)$ and so $(y,b) \in [(y,z)]_S \subseteq Q$. Since Q is transitive, it follows that $(a,b) \in Q$.
2. Consider Example 3 and the transitive relation given by

- $Q = \{(a,d),(c,b)\}$; thus $A_{\text{upp}}(Q) = \{(a,c),(a,d),(b,c),(b,d),(d,a),(d,b),(c,a),(c,b)\}$ is not transitive since $(a,d) \in A_{\text{upp}}(Q)$, $(d,b) \in A_{\text{upp}}(Q)$ and $(a,b) \notin A_{\text{upp}}(Q)$;
- $Q = \{(a,a),(a,b),(b,b),(b,a),(c,c)\}$; thus $A_{\text{upp}}(Q) = \{(a,a),(a,b),(b,a),(b,b),(c,c),(d,d),(c,d),(d,c)\}$ is a transitive relation. ■

Property 7. *If Q is an equivalence relation in U, then*
1. $A_{\text{upp}}(Q)$ is a tolerance relation (reflexive and symmetric), and
2. if each Z_i, $1 \le i \le n$ is an equivalence class induced by Q, then we have

$$A_{B\text{-upp}}(Q) = \bigcup_{1 \le i \le n} U_i \times U_i,$$

where $U_i = A_{A\text{-upp}}(Z_i)$, $1 \le i \le n$.

Proof. 1. If Q is an equivalence relation, then it is reflexive, symmetric and transitive. Property 2 assures that if Q is reflexive, so is $A_{\text{upp}}(Q)$ and, in turn, Property 3 assures that if Q is symmetric, so is $A_{\text{upp}}(Q)$. Consequently, if Q is an equivalence relation, $A_{\text{upp}}(Q)$ is a tolerance relation.
2. We shall prove that for Q being an equivalence relation, if each Z_i, $1 \le i \le n$ is an equivalence class induced by Q, we get

$$A_{B\text{-upp}}(Q) = \bigcup_{1 \le i \le n} U_i \times U_i,$$

where $U_i = A_{A\text{-upp}}(Z_i)$, $1 \le i \le n$.

First, let us show that $\bigcup_{1 \le i \le n} U_i \times U_i \subseteq A_{B\text{-upp}}(Q)$. If $(x, y) \in \bigcup_{1 \le i \le n} U_i \times U_i$, then $x, y \in U_i = A_{A\text{-upp}}(Z_i)$ for some i. We thus get $[x]_R \cap Z_i \neq \emptyset$ and $[y]_R \cap Z_i \neq \emptyset$. Hence $\exists a \in Z_i$ such that aRx, and $\exists b \in Z_i$ such that bRy. For this reason, aQb, aRx, bRy. This forces $(a, b) \in Q$ and $(a, b)S(x, y)$. Thus $(a, b) \in Q \cap [(x, y)]_S$, and then $Q \cap [(x, y)]_S \neq \emptyset$. Consequently, $(x, y) \in A_{B\text{-upp}}(Q)$.

Now, let us show that $A_{B\text{-upp}}(Q) \subseteq \bigcup_{1 \le i \le n} U_i \times U_i$. If $(x, y) \in A_{B\text{-upp}}(Q)$, then $[(x, z)]_S \cap Q \neq \emptyset$. Thus $\exists (a, b) \in [(x, y)]_S \cap Q$. This gives $(a, b)S(x, y)$ and $(a, b) \in Q$. We get aRx, bRy and aQb, so $a, b \in Z_i$ for some i, $1 \le i \le n$, $a \in [x]_R$ and $b \in [y]_R$. Therefore $a \in Z_i \cap [x]_R$ and $b \in Z_i \cap [y]_R$, so $x \in A_{A\text{-upp}}(Z_i) = U_i$ and $y \in A_{A\text{-upp}}(Z_i) = U_i$. It follows that $(x, y) \in U_i \times U_i$ and then $(x, y) \in \bigcup_{1 \le i \le n} U_i \times U_i$. ■

Example 4. Let $A = (U, R)$ be an approximation space, where $U = \{a, b, c, d, e, f, g\}$ and $U/R = \{\{a, b\}, \{c, d\}, \{e, f\}, \{g\}\}$, as shown in Fig. 5. Let $B = (U^2, S)$ be the approximation space product of A by A as shown in Fig. 6. Consider the equivalence relation on U given by

$$Q = \big\{(a, a), (c, c), (a, c), (c, a), (d, d), (e, e), (f, f), (d, e), (d, f), (e, d), (e, f), (f, e), (f, d), (b, b), (g, g)\big\}.$$

The equivalence classes induced by Q are $Z_1 = \{a, c\}$, $Z_2 = \{b\}$, $Z_3 = \{d, e, f\}$ and $Z_4 = \{g\}$. The upper approximations of these classes are given by

$$U_1 = A_{A\text{-upp}}(Z_1) = \{a, b, c, d\}, \quad U_2 = A_{A\text{-upp}}(Z_2) = \{a, b\},$$

$$U_3 = A_{A\text{-upp}}(Z_3) = \{c, d, e, f\}, \quad U_4 = A_{A\text{-upp}}(Z_4) = \{g\}.$$

It should be noted that in this case we have $U = U_1 \cup U_2 \cup U_3 \cup U_4$. The upper approximation of Q is given by $A_{\text{upp}}(Q) = \{(a, a), (a, b), (b, a), (b, b),$

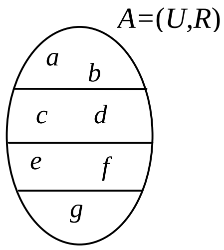

Fig. 5. Approximation space $A = (U, R)$, where $U = \{a, b, c, d, e, f, g\}$.

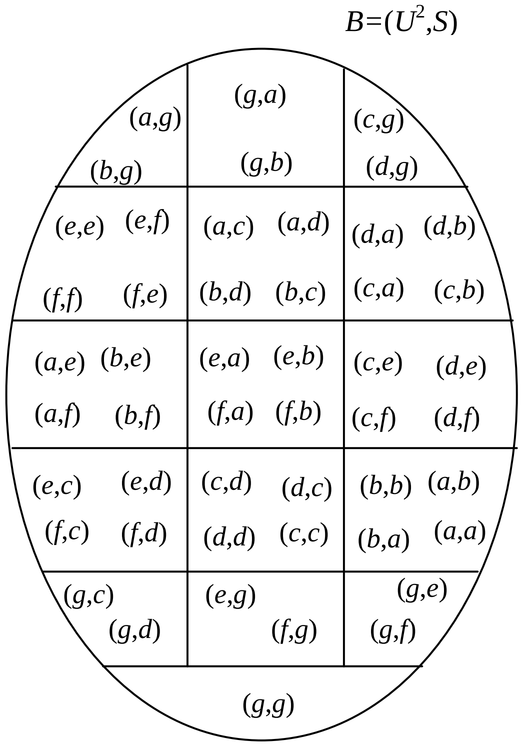

Fig. 6. Approximation product space $B = (U^2, S)$ resulting from $A \times A$.

$(c, c), (c, d), (d, c), (d, d), (a, c), (a, d), (b, c), (b, d), (c, a), (c, b), (d, a), (d, b), (e, e), (e, f), (f, e), (f, f), (c, e), (c, f), (d, e), (d, f), (e, c), (e, d), (f, c), (f, d), (g, g)\}$. It can be easily verified that $A_{\text{upp}}(Q) = (U_1 \times U_1) \cup (U_2 \times U_2) \cup (U_3 \times U_3) \cup (U_4 \times U_4)$. ♦

Property 8. *If Q is an equivalence relation in U, then*

$$A_{B\text{-low}}(Q) = \bigcup_{1 \le i \le n} V_i \times V_i,$$

where $V_i = A_{A\text{-low}}(Z_i)$, $1 \le i \le n$ and Z_i is an equivalence class induced by Q.

Proof. Let us first show that $\bigcup_{1\le i\le n} V_i \times V_i \subseteq A_{B\text{-low}}(Q)$. Indeed, for $(x,y) \in \bigcup_{1\le i\le n} V_i \times V_i$ we have $(x,y) \in V_i \times V_i$ for some i. Thus $x,y \in A_{A\text{-low}}(Z_i)$ and then $[x]_R \subseteq Z_i$, $[y]_R \subseteq Z_i$ for some i. We thus get $[x]_R \times [y]_R \subseteq Z_i \times Z_i \subseteq Q$. But since $[(x,y)]_S = [x]_R \times [y]_R$ (see Proposition 8), it follows that $[(x,y)]_S \subseteq Q$, i.e. $(x,y) \in A_{B\text{-low}}(Q)$.

Now, let us show that $A_{B\text{-low}}(Q) \subseteq \bigcup_{1\le i\le n} V_i \times V_i$. For $(x,y) \in A_{B\text{-low}}(Q)$ we have $[(x,y)]_S \subseteq Q$. From Proposition 8 it follows that $[x]_R \times [y]_R \subseteq Q$. Thus $[x]_R \subseteq Z_i$ and $[y]_R \subseteq Z_i$ for some equivalence class Z_i of U, induced by Q. Consequently, $x \in A_{A\text{-low}}(Z_i) = V_i$, $y \in A_{A\text{-low}}(Z_i) = V_i$ for some i, and so $(x,y) \in \bigcup_{1\le i\le n} V_i \times V_i$. This property is illustrated in Example 5. ■

Example 5. Let $A = (U,R)$ be an approximation space and $B = (U^2,R)$ the approximation product space $B = (U^2,R)$ as defined in Example 4. Also consider the same relation Q defined there. We then have

$$A_{\text{low}}(Q) = \{(e,e),(f,f),(e,f),(f,e),(g,g)\},$$

which is an equivalence relation in $\{e,f,g\}$, a subset of U. The lower and upper approximations of the equivalence classes $Z_1,\dots,Z_4$ induced by Q are given by

$$U_1 = A_{A\text{-low}}(Z_1) = U_2 = A_{A\text{-low}}(Z_2) = \emptyset,$$

$$U_3 = A_{A\text{-low}}(Z_3) = \{e,f\}, \quad U_4 = A_{A\text{-low}}(Z_4) = \{g\}.$$

It should be noted that U_3 and U_4 are the equivalence classes of Q. ♦

Property 9. *If Q is a partial ordering in U, then*
1. if $A_{\text{low}}(Q) \ne Q$, then nothing can be said about $A_{\text{low}}(Q)$ being a partial ordering,
2. if $A_{\text{upp}}(Q) \ne Q$ then nothing can be said about $A_{\text{upp}}(Q)$ being a partial ordering.

A relation Q is a partial ordering if it is reflexive, antisymmetric and transitive. If Q is a partial ordering on U, based on Properties 2, 4 and 6, nothing can be said about $A_{\text{low}}(Q)$ and $A_{\text{upp}}(Q)$ being or not a partial ordering relation.

Property 10. *If Q is a relation in U, then*
1. $A_{\text{low}}(Q^{-1}) = (A_{\text{low}}(Q))^{-1}$,
2. $A_{\text{upp}}(Q^{-1}) = (A_{\text{upp}}(Q))^{-1}$.

Proof. 1. Note that $(x,y) \in A_{\text{low}}(Q^{-1}) \Leftrightarrow [(x,y)]_S \subseteq Q^{-1} \Leftrightarrow [(y,x)]_S \subseteq Q \Leftrightarrow (y,x) \in A_{\text{low}}(Q) \Leftrightarrow (x,y) \in (A_{\text{low}}(Q))^{-1}$. The equivalence $[(x,y)]_S \subseteq Q^{-1} \Leftrightarrow [(y,x)]_S \subseteq Q$ can be proved by observing that $(a,b) \in [(x,y)]_S \Leftrightarrow (b,a) \in [(y,x)]_S$.
2. This part is similar to Part 1. ■

Property 11. *If V and W are any relations in U and $Q = W \bullet V$ (composition of V and W), then*
1. $A_{\text{low}}(W) \bullet A_{\text{low}}(V) \subseteq A_{\text{low}}(Q)$,
2. $A_{\text{upp}}(Q) \subseteq A_{\text{upp}}(W) \bullet A_{\text{upp}}(V)$.

Proof. 1. For $(x,z) \in A_{\mathrm{low}}(W) \bullet A_{\mathrm{low}}(V)$ there exists a $y \in U$ such that $(x,y) \in A_{\mathrm{low}}(V)$ and $(y,z) \in A_{\mathrm{low}}(W)$. Thus $[(x,y)]_S \subseteq V$ and $[(y,z)]_S \subseteq W$. From Proposition 3 it follows that $[(y,z)]_S \bullet [(x,y)]_S \subseteq W \bullet V = Q$. Proposition 8 assures that $[(x,z)]_S \subseteq Q \Rightarrow (x,z) \in A_{\mathrm{low}}(Q)$. Example 6 illustrates this property.
2. For $(x,y) \in A_{\mathrm{upp}}(Q)$ we have $[(x,y)]_S \cap Q \neq \emptyset$, so there exists $(a,b) \in [(x,y)]_S$ such that $(a,b) \in Q = W \bullet V$. Consequently, there exists a $c \in U$ such that $(a,c) \in V$ and $(c,b) \in W$. This gives $(a,c) \in A_{\mathrm{upp}}(V)$ and $(c,b) \in A_{\mathrm{upp}}(W)$. Hence $(a,b) \in A_{\mathrm{upp}}(W) \bullet A_{\mathrm{upp}}(V)$ and so $[(a,b)]_S \cap (A_{\mathrm{upp}}(W) \bullet A_{\mathrm{upp}}(V)) \neq \emptyset$. Therefore $(x,y) \in A_{\mathrm{upp}}(W) \bullet A_{\mathrm{upp}}(V)$, since $[(a,b)]_S = [(x,y)]_S$. Example 7 illustrates this property. ■

Example 6. Let $A = (U,R)$ be an approximation space and $B = (U^2,R)$ the approximation product space $B = (U^2,R)$ as defined in Example 4. Let $Q = W \bullet V$, where

$$V = \{(a,a),(b,b),(e,d),(e,g),(f,g)\},$$

$$W = \{(a,a),(a,b),(b,a),(b,b),(e,f),(e,g),(f,g)\},$$

$$Q = W \bullet V = \{(a,a),(a,b),(b,a),(b,b)\},$$

$$A_{\mathrm{low}}(V) = \{(e,g),(f,g)\},$$

$$A_{\mathrm{low}}(W) = \{(a,a),(a,b),(b,a),(b,b),(e,g),(f,g)\},$$

$$A_{\mathrm{low}}(Q) = \{(a,a),(a,b),(b,a),(b,b)\},$$

$$A_{\mathrm{low}}(W) \bullet A_{\mathrm{low}}(V) = \emptyset \subseteq A_{\mathrm{low}}(Q).$$

♦

Example 7. Let $A = (U,R)$ be an approximation space and $B = (U^2,R)$ the approximation product space $B = (U^2,R)$ as defined in Example 4. Let $Q = V \bullet W$, where

$$V = \{(d,d)\}, \quad W = \{(c,a)\}, \quad Q = V \bullet W = \emptyset,$$

$$A_{\mathrm{upp}}(V) = \{(c,c),(d,d),(c,d),(d,c)\},$$

$$A_{\mathrm{upp}}(W) = \{(c,a),(d,a),(c,b),(d,b)\},$$

$$A_{\mathrm{upp}}(Q) = \emptyset \subseteq A_{\mathrm{upp}}(W) \bullet A_{\mathrm{upp}}(V) = \{(c,a),(c,b),(d,a),(d,b)\}.$$

♦

6. Conclusion

This work establishes main properties related to rough relations and proves their validity. It is worth mentioning that many of these properties are rewritten versions of those listed in (Pawlak, 1981). We have also shown, using examples, the properties that are not valid.

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