A PROBLEM OF ROBUST CONTROL OF A SYSTEM WITH TIME DELAY

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A problem of guaranteed control is under discussion. This problem consists in the attainment of a given target set by a phase trajectory of a system described by an equation with time delay. An uncontrolled disturbance (along with a control) is assumed to act upon the system. An algorithm for solving the problem in the case when information on a phase trajectory is incomplete (measurements of a 'part' of coordinates) is designed. The algorithm is stable with respect to informational noises and computational errors.

Keywords: robust control, system with time delay

1. Introduction

Consider the problem of robust control of a system with time delay of the form

$$\begin{aligned} \dot{x}(t) &= F\big(t, x_t(s), u, v\big), \quad t \in T = [t_0, \vartheta], \\ x_t(s) &= x(t+s), \quad s \in [-\tau, 0], \quad \tau = \text{const} > 0. \end{aligned} \tag{1}$$

It is supposed in many applications that the system under consideration is subjected to the law of causality, i.e. a future state of the system does not depend on past states and is determined only by a current state. If it is additionally assumed that the system is described by an equation containing varying states and velocities of their changing, then we come, as a rule, to ordinary differential equations or to partial differential equations. However, it is rather often evident that the law of causality is only a first-order approximation to some real situation, and a more realistic model should take into account several previous states of the system. Besides, many problems lose their sense if a dependence on the past is not considered. Obviously, it was known earlier, but the theory of systems with time delay has come under the scrutiny of science only in recent years. Most of investigations have been devoted to qualitative questions of differential-functional equations, as well as to numerical methods of their solution. A large number of works have been connected with problems of

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controllability, observability and stabilization of systems with time delay. In addition, the theory of program (optimal) control of differential-functional systems has been actively developed in recent years.

The essence of the problem discussed in the present paper is as follows. We have the system (1), and some control $u(t)$ and unobserved disturbance $v(t)$ act upon it simultaneously. The class of admissible disturbances $v(t)$ is wide enough and is described *a priori*. Simultaneously with the functioning of the system, a 'part' of its current phase states $x(t) = \{y(t), z(t)\}$ (namely, states $y(t)$, $t \in T$) is inaccurately measured at sufficiently frequent time moments. The problem consists in construction of a law of forming control (i.e. a rule of choosing $u = u(t)$, $t \in T$) according to the feedback principle. This law should guarantee a desired behaviour of the trajectory $x(t) = \{y(t), z(t)\}$, $t \in T$, irrespective of the disturbance $v(t)$ acting upon the system.

2. Problem Statement

Let us make the statement of the problem more precise. Consider a system of equations with time delay (1) in the form

$$\begin{aligned} \dot{y}(t) &= L_1\big(y_t(s)\big) + Cz(t) + f_0(t), \\ \dot{z}(t) &= L_2\big(z_t(s)\big) + E\big(y(t)\big) + f_1\big(t, u(t), v(t)\big), \quad t \in T, \end{aligned} \tag{2}$$

$$L_j\big(y_t(s)\big) = \sum_{i=0}^{l_j} A_i^{(j)} y(t - \tau_i^{(j)}) + \int_{\tau_{l_j}^{(j)}}^{0} A_*^{(j)}(s) y(t+s)\,\mathrm{d}s, \quad j = 1, 2,$$

with initial conditions

$$\begin{aligned} y(0) &= y^0, \quad y(s) = y^1(s) \quad \text{for} \quad s \in [-\tau_{l_1}^{(1)}, 0], \\ z(0) &= z^0, \quad z(s) = z^1(s) \quad \text{for} \quad s \in [-\tau_{l_2}^{(2)}, 0]. \end{aligned} \tag{3}$$

Here $x(t) = \{y(t), z(t)\}$ is a phase trajectory of the system, $y(t) \in \mathbb{R}^N$, $z(t) \in \mathbb{R}^n$, $y^0 \in \mathbb{R}^N$, $z^0 \in \mathbb{R}^n$, $y^1(s) \in L_2([-\tau_{l_1}^{(1)}, 0]; \mathbb{R}^N)$, $z^1(s) \in L_2([-\tau_{l_2}^{(2)}, 0]; \mathbb{R}^n)$, $0 = \tau_0^{(j)} < \tau_1^{(j)} < \cdots < \tau_{l_j}^{(j)}$, $y_t(s) : s \to y(t+s)$, $s \in [-\tau_{l_1}^{(1)}, 0]$, $z_t(s) : s \to z(t+s)$, $s \in [-\tau_{l_2}^{(2)}, 0]$, $A_i^{(j)}$ and C are constant matrices of dimensions $N \times N$ (for $j = 1$), $n \times n$ (for $j = 2$) and $N \times n$, respectively. Furthermore, the elements of matrix functions $s \to A_*^{(j)}(s)$, $s \in [-\tau_{l_j}^{(j)}, 0]$, $j = 1, 2$ are essentially bounded, $u \in \mathbb{R}^m$ is a control, $v \in \mathbb{R}^g$ is a disturbance, $E(\cdot) : \mathbb{R}^N \to \mathbb{R}^n$ denotes a matrix function satisfying the global Lipschitz condition, the function $f_1 : T \times \mathbb{R}^m \times \mathbb{R}^g \to \mathbb{R}^n$ is continuous with respect to all variables, and the function $f_0(\cdot)$ is square integrable.

Let a control $u = u(t) \in P$ being formed in the process and an unknown disturbance $v = v(t) \in Q$ act upon the system (2). Here $P \subset \mathbb{R}^m$, $Q \subset \mathbb{R}^g$ are bounded closed sets, interpreted as 'resourses' for control and disturbance, respectively. A uniform partition of the time interval T $\Delta = \{\tau_i\}_{i=0}^m$, $\tau_0 = t_0$, $\tau_m = \vartheta$, $\tau_{i+1} = \tau_i + \delta$ with a step δ is chosen. At time moments τ_i, the phase coordinate $y(\tau_i)$ is inaccurately measured. The results of the measurements (vectors $\xi_i^h \in \mathbb{R}^N$) satisfy the inequalities

$$\left|\xi_i^h - y(\tau_i)\right|_{\mathbb{R}^N} \le h, \tag{4}$$

where $h \in (0,1)$ is a value of the measurement accuracy. It is required to construct a rule of forming a feedback control for (2)

$$u(t) = u^e(t) = u_i^e(\xi_i^h) \in P, \quad t \in [\tau_i, \tau_{i+1}), \quad i = 0, \dots, m-1$$

such that, regardless of the unknown disturbance $v = v(t)$, the phase state of the system $x(t) = x(t; t_0, x_{t_0}(s), u(\cdot), v(\cdot))$ at the moment $t = \vartheta$ belong to a sufficiently small ε-neighborhood of a given set $M \subset \mathbb{R}^{N+n}$ (i.e. set M^ε). Here and below the symbol $x_{t_0}(s)$ stands for an initial state of the system (2), i.e. $x_{t_0}(s) = ((y^0, y^{(1)}(s)), (z^0, z^{(1)}(s)))$, and the symbol M^ε denotes the ε-neighborhood of the set M.

The choice of a control law, i.e. of a rule of changing parameter $u(t)$, is up to some 'player' (we use the terminology from the theory of positional differential games (Krasovskii, 1985; Krasovskii and Subbotin, 1988; Osipov, 1971a; 1971b)). The 'player' should choose this law in order to provide the above-mentioned property of the motion under any possible realization of disturbance $v = v(t)$. Note that the nature of disturbance v is insignificant from our point of view. This disturbance may be a program control or a positional feedback control. It is only necessary that two conditions be fulfilled: first, the realization $v(t)$ should be a measurable (in the Lebesgue sense) function on T; second, it should satisfy the inclusion $v(t) \in Q$ for a.a. $t \in T$.

In the present paper an algorithm for solving the problem under consideration is suggested. This algorithm is based on the method of dynamic inversion (the method of dynamic control approximation), developed in (Osipov and Kryazhimskii, 1995; Osipov *et al.*, 1991), as well as on the method of stable tracks, well-known in the theory of positional control (Krasovskii and Subbotin, 1988). In connection with incomplete information (namely, with the possibility of measuring only part $y(\tau_i)$ of the whole phase state of the system $\{y(\tau_i), z(\tau_i)\}$), in the control loop we introduce an additional block of dynamical reconstruction (approximation) of the unknown coordinate $z(t)$ (an 'identification' block). This block plays the role of a provider of information about the whole phase state of the system. The information is immediately fed into the control block functioning according to the given feedback.

It should be noted that the foundations of the theory of positional control for systems with time delay were laid in (Maksimov, 1978; Osipov, 1971a; 1971b). However, in these papers the problems of guaranteed control in the case of inaccurate measurements of the whole phase state (i.e. under 'complete' information on phase trajectories) were discussed. In contrast, in the present paper the problem of a guaranteed attainment of a given set by the phase trajectory of a system with time delay under measurements of a 'part' of the phase state is investigated in the context of the approach set forth in (Krasovskii, 1985; Krasovskii and Subbotin, 1988; Maksimov, 1978; Osipov, 1971a; 1971b; Osipov and Kryazhimskii, 1995; Osipov *et al.*, 1991). Other problems of feedback control under the conditions of the lack of information for systems with time delay were studied in (Krasovskii, 1998; Krasovskii and Krasovskii, 1995; Krasovskii and Lukoyanov, 1996).

3. Auxiliary Constructions

Introduce the notations

$$F_u(t,v) = \mathrm{co}\big[f : f = f_1(t,u,v),\ u \in P\big],$$

$$H(t) = \bigcap_{v\in Q} F_u(t,v), \quad H(\cdot) = \big\{u(\cdot) \in L_2(T;\mathbb{R}^m) : u(t) \in H(t) \quad \text{for a.a. } t \in T\big\}.$$

Before the description of the algorithm for solving the problem, we give auxiliary constructions which are necessary in what follows. Let $s_j(\cdot)$ be the unique solution on T of the functional-differential matrix equation

$$\frac{\mathrm{d}s_j(t)}{\mathrm{d}t} = A_0^{(j)} s_j(t) + \sum_{i=1}^{l_j} A_i^{(j)} s_j\left(t + \tau_i^{(j)}\right) + \int_{-\tau_{l_j}^{(j)}}^{0} A_*^{(j)}(s) s_j(t+s)\,\mathrm{d}s \quad \text{for a.a. } t \in T$$

with the initial state $s_j(t) = I,\ t \le 0$. Here I is the $q \times q$-identity matrix, and the operator $B^{(j)} : L_2([-\tau_{l_j}^{(j)}, 0]; \mathbb{R}^q) \to L_2([-\tau_{l_j}^{(j)}, 0]; \mathbb{R}^q)$ is of the form

$$\big(B^{(j)}\varphi\big)(\tau) = \sum_{i=1}^{l_j} A_i^{(j)} \chi_{[\tau_i^{(j)},0]}(\tau)\varphi\big(-\tau_i^{(j)} - \tau\big) + \int_{-\tau_{l_j}^{(j)}}^{0} A_*^{(j)}(\xi)\varphi(\xi - \tau)\,\mathrm{d}\xi$$

for a.a. $\tau \in [-\tau_{l_j}^{(j)}, 0]$ $(q = N$ if $j = 1,\ q = n$ if $j = 2)$.

As is well-known, the equation

$$\dot{x}_j(t) = L_j\big(x_{jt}(s)\big), \quad j = 1, 2$$

generates a C_0-semigroup of bounded linear operators $\mathcal{X}_j(t)(t \ge 0) : X_j \to X_j$ which are defined as follows (see Bernier and Manitius, 1978): We denote by $X_1 = \mathbb{R}^N \times L_2([-\tau_{l_1}^{(1)}, 0]; \mathbb{R}^N)$ the Hilbert space of all pairs $x = (x^0, x^1(s))$, with the scalar product

$$(x, y)_{X_1} = (x^0, y^0)_{\mathbb{R}^N} + \int_{\tau_{l_1}^{(1)}}^{0} \big(x^1(s), y^1(s)\big)_{\mathbb{R}^N}\,\mathrm{d}s$$

and the norm $|\cdot|_{X_1}$. In a similar manner, we define the space $X_2 = \mathbb{R}^n \times L_2([-\tau_{l_2}^{(2)}; 0]; \mathbb{R}^n)$.

Let the operator $F_j : X_j \to X_j$ be given by

$$(F_j\varphi)^0 = \varphi^0, \quad (F_j\varphi)^1 = B^{(j)}\varphi^1 \quad (\varphi = \big(\varphi^0, \varphi^1(s)\big) \in X_j).$$

Then the following equality holds (Bernier and Manitius, 1978, p.903):

$$\mathcal{X}_j(t)\varphi = G_j^t F_j \varphi + S_j(t)\varphi, \tag{5}$$

where $G_j^t : X_j \to X_j$,

$$(G_j^t\varphi)^1(\tau) = s_j(t+\tau)\varphi^0 + \int_{-\tau_{l_j}^{(j)}}^{0} s_j(t+\tau+\xi)\varphi^1(\xi)\,\mathrm{d}\xi, \quad \tau \in [-\tau_{l_j}^{(j)}, 0],$$

$$(G_j^t\varphi)^0 = (G_j^t\varphi)^1(0), \qquad (S_j(t)\varphi)^0 = 0,$$

$$\big(S_j(t)\varphi\big)^1(\tau) = \varphi(t+\tau)\chi_{[-\tau_{l_j}^{(j)}, -t]}(\tau),$$

$\chi_{[a,b]}(\cdot)$ is the characteristic function of the interval $[a, b]$.

Let $x^{(j)}(t; t_0, x_{t_0}^{(j)}(s), p^{(j)}(\cdot))$ denote the solution (in the Caratheodory sense) of the following equation with time delay:

$$\dot{x}^{(j)}(t) = L_j\big(x_t^{(j)}(s)\big) + p^{(j)}(t), \quad t \in T,$$

$$x_{t_0}^{(j)}(s) = \big(x^{(j)}, x_0^{(j)}(s)\big) \in X_j, \quad p^{(j)}(\cdot) \in L_2(T; \mathbb{R}^q),$$

and let $X^{(j)}(t; t_0, x_{t_0}^{(j)}(s), p^{(j)}(\cdot))$ stand for the weak solution of the equation in the Hilbert space X_j of the form

$$\dot{X}^{(j)}(t) = A_j X^{(j)}(t) + P^{(j)}(t), \quad t \in T,$$

$$X^{(j)}(t_0) = \big(x^{(j)}, x_0^{(j)}(s)\big) \in X_j,$$

i.e.

$$X^{(j)}(t) = \mathcal{X}_j(t-t_0)X^{(j)}(t) + \int_{t_0}^{t} \mathcal{X}_j(t-\tau)P^{(j)}(\tau)\,\mathrm{d}\tau,$$

$j = 1, 2$. Here $P^{(j)}(t) = (p^{(j)}(t), 0) \in X_j$ for a.a. $t \in T$, (i.e. $p^{(j)}(t) \in \mathbb{R}^q$ for a.a. $t \in T$, $0 \in L_2([-\tau_{l_j}^{(j)}; 0]; \mathbb{R}^q)$). The operator A_j is given by (cf. Bernier and Manitius, 1978, Proposition 2.1):

$$D(A_j) = \Big\{\varphi = \big(\varphi^0, \varphi^1(s)\big) \in X_j :$$

$$\varphi^1(s) \in W^{1,2}\big([-\tau_{l_j}^{(j)}, 0]; \mathbb{R}^q\big), \quad \varphi^1(0) = \varphi^0\Big\}, \tag{6}$$

$$A_j\varphi = \big(L_j(\varphi^1), \dot{\varphi}^1(s)\big), \quad \varphi = \big(\varphi^0, \varphi^1(s)\big) \in D(A_j).$$

Let

$$P(\cdot) = \{u(\cdot) \in L_2(T; \mathbb{R}^m) : u(t) \in P \quad \text{for a.a. } t \in T\},$$

$$Q(\cdot) = \{v(\cdot) \in L_2(T; \mathbb{R}^q) : v(t) \in Q \quad \text{for a.a. } t \in T\}.$$

Lemma 1. (Bernier and Manitius, 1978) *The following equality is true:*

$$X^{(j)}\big(t; t_0, x_0^{(j)}(s), P^{(j)}(\cdot)\big)$$
$$= \Big\{x^{(j)}\big(t; t_0, x_0^{(j)}(s), p^{(j)}(\cdot)\big), x_t^{(j)}\big(s; t_0, x_0^{(j)}(s), p^{(j)}(\cdot)\big)\Big\}, \quad t \in T.$$

Lemma 2. (Kappel and Maksimov, 2000) *The set of all solutions to (2)*

$$X\big(x_{t_0}(s)\big) = \big\{x\big(\cdot; t_0, x_{t_0}(s), u(\cdot), v(\cdot)\big) : u(\cdot) \in P(\cdot), \quad v(\cdot) \in Q(\cdot)\big\}$$

is bounded in $W^{1,2}(T; \mathbb{R}^{N+n}) = \{x(\cdot) \in L_2(T; \mathbb{R}^{N+n}) : x_t(\cdot) \in L_2(T; \mathbb{R}^{N+n})\}$.

4. Algorithm for Solving the Main Problem

Let the following condition be fulfilled:

Condition 1. *(a) Sets* $H(t)$ *are nonempty for all* $t \in T$*, (b) there exists a control* $u_*(\cdot) \in H(\cdot)$ *which makes the phase trajectory of the system*

$$\begin{aligned}
\dot{y}^0(t) &= L_1\big(y_t^0(s)\big) + Cz^0(t) + f_0(t),\\
\dot{z}^0(t) &= L_2\big(z_t^0(s)\big) + E\big(y^0(t)\big) + u_*(t), \quad t \in T,\\
x_{t_0}^0(s) &= \big(y_{t_0}^0(s), z_{t_0}^0(s)\big) = x_{t_0}(s)
\end{aligned} \tag{7}$$

attain the set M *at the moment* ϑ*, (c) the existence condition for a saddle point in a 'small game' is valid:*

$$\min_{u \in P} \max_{v \in Q} s' f_1(t, u, v) = \max_{v \in Q} \min_{u \in P} s' f_1(t, u, v), \quad \forall s \in \mathbb{R}^n, \quad t \in T.$$

To solve the problem, along with the system (7) we introduce an auxiliary system (an 'identification' block) described by the vector equation with time delay

$$\dot{w}^h(t) = L_1\big(w_t^h(s)\big) + Cv^h(t) + f_0(t), \quad t \in T \tag{8}$$

with the initial condition

$$w^h(0) = y^0, \quad w^h(s) = y^1(s) \quad \text{for} \quad s \in [-\tau_{l_1}^{(1)}, 0].$$

Here the control $v^h(t)$ belongs to $\mathbb{R}^n$. We denote by $w^h(t; t_0, w_{t_0}^h(s), v^h(\cdot)) \in \mathbb{R}^N$ the Caratheodory solution of this system on the interval T.

The auxiliary system (7) (to be more precise, the 'stable track', cf. (Krasovskii and Subbotin, 1988)) is analogous to a special model which is known in the theory of positional differential games (Krasovskii, 1985). In essence, it represents some virtual construction which helps the 'player' to form a necessary control in the real system. In turn, the system (8) (the model, see (Osipov and Kryazhimskii, 1995)) constitutes an instrument for dynamic reconstruction of the unknown coordinate $z(t)$. The process of synchronous control of the system (2), track (7) and model (8) is realized simultaneously with the real motion $x(t) = \{y(t), z(t)\},\ t \in T$ of the system (2).

Let us proceed with the description of the algorithm. Fix a function $\alpha(h) : (0,1) \to \mathbb{R}^+ = \{r \in \mathbb{R}^1 : r \geq 0\}$ (a regularizer) and a family of partitions Δ_h, $h \in (0,1)$ of the interval T with the following properties:

$$\Delta_h = \{\tau_i\}_{i=0}^m, \quad \tau_i = \tau_{h,i}, \quad m = m_h, \quad \tau_{i+1} = \tau_i + \delta, \quad \delta = \delta(h), \tag{9}$$

$$\alpha(h) \to 0, \quad \delta(h) \to 0, \quad \big(h + \delta^{1/2}(h)\big)\alpha^{-1}(h) \to 0 \quad \text{as} \quad h \to 0.$$

Then we organize the process of the synchronous feedback control of the track (7), model (8) and real system (2) in such a way that for sufficiently small h and δ the motion of the system (2) belongs at the moment ϑ to a sufficiently small neighborhood M (i.e. to a set M^ε) for all possible realizations $v(\cdot) \in Q(\cdot)$. For this purpose we fix h, $\alpha(h)$ and Δ_h prior to the beginning of the algorithm. This work is decomposed into $m-1$ $(m = m_h)$ identical steps. At the i-th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, the following operations are performed: First, the 'identification' block calculates the vector

$$v_i^h = v_i^h\big(\xi_i, w^h(\tau_i)\big) = \arg\min\big\{L(\alpha, v, s_i) : v \in S(d)\big\} \tag{10}$$

using the measurement ξ_i and model state $w^h(\tau_i)$. Here

$$L(\alpha, v, s_i) = \alpha(h)|v|^2_{\mathbb{R}^n} + 2(s_i, Cv)_{\mathbb{R}^N}, \quad s_i = \big(w^h(\tau_i) - \xi_i\big)\exp(-2\omega_1\tau_{i+1}),$$

$$d = \sup\Big\{\big|z\big(t; t_0, x_{t_0}(s), u(\cdot), v(\cdot)\big)\big|_{\mathbb{R}^n} : u(\cdot) \in P(\cdot),\ v(\cdot) \in Q(\cdot),\ \ t \in T\Big\},$$

$$S(d) = \big\{v \in \mathbb{R}^n : |v|_{\mathbb{R}^n} \leq d\big\}.$$

Then the 'control' block determines the vector

$$u_i^e = u_i\big(v_i^h, z^0(\tau_i)\big) = \arg\min\Big\{\max_{v \in Q}\big(v_i^h - z^0(\tau_i), f_1(\tau_i, u, v)\big)_n : u \in P\Big\}.$$

After that, during the time interval δ_i, the constant control

$$u^e(t) = u_i^e = u_i^e(v_i^h, z^0\big(\tau_i)\big) \tag{11}$$

is fed onto the input of the system (1), and the control

$$v^h(t) = v_i^h = v_i^h\big(\xi_i, w^h(\tau_i)\big)$$

is fed onto the input of the model. As a result of the action of these two controls and some unknown disturbance $v(t)$, $t \in \delta_i$ (below the latter is denoted by $v_{\tau_i,\tau_{i+1}}(\cdot)$), the system (2) passes from state $\{y_{\tau_i}(s), z_{\tau_i}(s)\}$ to state $\{y_{\tau_{i+1}}(s), z_{\tau_{i+1}}(s)\}$:

$$y_{\tau_{i+1}}(s) = y_{\tau_{i+1}}\big(s; \tau_i, y_{\tau_i}(s), z_{\tau_i}(s), u_i^e, v_{\tau_i,\tau_{i+1}}(\cdot)\big),$$

$$z_{\tau_{i+1}}(s) = z_{\tau_{i+1}}\big(s; \tau_i, y_{\tau_i}(s), z_{\tau_i}(s), u_i^e, v_{\tau_i,\tau_{i+1}}(\cdot)\big),$$

and the model (8) passes from state $w_{\tau_i}^h(s)$ to state $w_{\tau_{i+1}}^h(s)$:

$$w_{\tau_{i+1}}^h(s) = w_{\tau_{i+1}}^h\big(s; \tau_i, w_{\tau_i}^h(s), v_i^h\big).$$

In the next, $(i+1)$-th step, analogous actions are repeated. The procedure stops at the moment $t = \vartheta$.

Note that in the relation (10) one can take any number $d > d_1$ instead of d_1.

Assume that the following relations between the parameters are valid:

$$\alpha(h) \to 0, \quad \Big\{\big(h + \delta^{1/2}(h) + \alpha(h)\big)^{1/2} + \big(h + \delta^{1/2}(h)\big)\alpha^{-1}(h)\Big\} \to 0 \text{ as } h \to 0. \quad (12)$$

For example, we can set $\delta = h^2$, $\alpha = h^{1/2}$, $\beta = h^\nu$, $\nu \in (0, 1/4)$.

Condition 2. *Let $n \le N$ and the following conditions be fulfilled:*

(a) *there exists a number $d_1 > 0$ such that for all $x \in \mathbb{R}^N$ we have*

$$\inf_{t \in T} |s_1^{-1}(t)x|_{\mathbb{R}^N} \ge d_1 |x|_{\mathbb{R}^N},$$

(b) *there exist a number $d_2 > 0$ and an n-th order minor of matrix $s_1(t)C_1$ such that the $n \times n$-matrix $\overline{s_1(t)C_1}$ corresponding to this minor satisfies the inequality*

$$\inf_{t \in T} |\overline{s_1(t)C_1} v|_{\mathbb{R}^n} \ge d_2 |v|_{\mathbb{R}^n}$$

for all $v \in \mathbb{R}^n$,

(c) *for any solution $z(\cdot)$ of (2), the inclusion $(\overline{s_1(\vartheta - t)C_1})^{-1} z(t) \in V(T; \mathbb{R}^n)$ is true.*

Here the symbol $V(T; \mathbb{R}^n)$ denotes the space of all functions $t \to x(t) \in \mathbb{R}^n$ with bounded variation $\mathrm{var}_X(T; x(\cdot))$.

Theorem 1. *Let Conditions 1 and 2 be fulfilled. Then for any $\varepsilon > 0$ one can indicate a number $h_* = h_*(\varepsilon)$ such that for $h \in (0, h_*)$ the inclusion*

$$x\big(\vartheta; t_0, x_{t_0}(s), u^e(\cdot), v(\cdot)\big) \in M^\varepsilon$$

is true irrespective of the disturbance $v(\cdot) \in Q(\cdot)$.

Proof. Let us estimate the variation of the value

$$\varepsilon(t) = \big|x(t) - x^0(t)\big|_{X_1\times X_2} = \big|y(t) - y^0(t)\big|^2_{X_1} + \big|z(t) - z^0(t)\big|^2_{X_2},$$

where $x(\cdot) = x(\cdot; t_0, x_{t_0}(s), u^e(\cdot), v(\cdot)) = (y(\cdot), z(\cdot))$ is the phase trajectory of the system (2), and $x^0(\cdot) = x^0(\cdot; t_0, x_{t_0}(s), u_*(\cdot)) = (y^0(\cdot), z^0(\cdot))$ is the phase trajectory of the system (7). For $t \in [\tau_i, \tau_{i+1}]$ we have

$$\varepsilon(\tau_{i+1}) = \nu_1^{(i+1)} + \nu_2^{(i+1)} + \nu_3^{(i+1)}, \tag{13}$$

$$\nu_1^{(i+1)} = \Big|x(\tau_i) - x^0(\tau_i) + \int\limits_{\tau_i}^{\tau_{i+1}} \big\{f\big(t, x_t(s), u_i^e, v(t)\big) - f\big(t, x_t^0(s), u_*(t)\big)\big\}\,\mathrm{d}t\Big|^2_{N+n},$$

$$\nu_2^{(i+1)} = \int\limits_{-\delta}^{0} \Big|x(\tau_i) - x^0(\tau_i) + \int\limits_{\tau_i}^{\tau_{i+1}+s} \big\{f\big(t, x_t(s), u_i^e, v(t)\big) - f\big(t, x_t^0(s), u_*(t)\big)\big\}\,\mathrm{d}t\Big|^2_{N+n}\mathrm{d}s,$$

$$\nu_3^{(i+1)} = \int\limits_{-\tau_{l_1}^{(1)}}^{-\delta} \big|y(\tau_{i+1}+s) - y^0(\tau_{i+1}+s)\big|^2_N\,\mathrm{d}s + \int\limits_{-\tau_{l_2}^{(2)}}^{-\delta} \big|z(\tau_i+s) - z^0(\tau_i+s)\big|^2_n\,\mathrm{d}s.$$

Here

$$f\big(t, x_t(s), u_i^e, v(t)\big) = \begin{pmatrix} L_1\big(y_t(s)\big) + Cz(t) + f_0(t), \\ L_2\big(z_t(s)\big) + E\big(y(t)\big) + f_1\big(t, u_i^e, v(t)\big) \end{pmatrix},$$

$$f\big(t, x_t^0(s), u_*(t)\big) = \begin{pmatrix} L_1\big(y_t^0(s)\big) + Cz^0(t) + f_0(t), \\ L_2\big(z_t^0(s)\big) + E\big(y^0(t)\big) + u_*(t) \end{pmatrix}.$$

Estimating each term on the right-hand side of (13), we deduce that

$$\nu_3^{(i+1)} = \int\limits_{-\tau_{l_1}^{(1)}+\delta}^{0} \big|y(\tau_i+s) - y^0(\tau_i+s)\big|^2_N\,\mathrm{d}s + \int\limits_{-\tau_{l_2}^{(2)}+\delta}^{0} \big|z(\tau_i+s) - z^0(\tau_i+s)\big|^2_n\,\mathrm{d}s,$$

$$\begin{aligned} \nu_1^{(i+1)} &= \big|x(\tau_i) - x^0(\tau_i)\big|^2_{N+n} + \nu_4^{(i+1)} \\ &\quad + \Big|\int\limits_{\tau_i}^{\tau_{i+1}} \big\{f\big(t, x_t(s), u(t), v(t)\big) - f\big(t, x_t^0(s), u_*(t)\big)\big\}\,\mathrm{d}t\Big|^2_{N+n}, \end{aligned} \tag{14}$$

$$\nu_4^{(i+1)} = 2\Big(x(\tau_i) - x^0(\tau_i),$$

$$\int\limits_{\tau_i}^{\tau_{i+1}} \big\{f\big(t, x_t(s), u_i^e, v(t)\big) - f\big(t, x_t^0(s), u_*(t)\big)\big\}\,\mathrm{d}t\Big)_{N+n} = \mu_i^{(1)} + \mu_i^{(2)},$$

$$\mu_i^{(1)} = 2\Big(y(\tau_i) - y^0(\tau_i), \int\limits_{\tau_i}^{\tau_{i+1}} \big\{L_1\big(y_t(s) - y_t^0(s)\big) + C\big(z(t) - z^0(t)\big)\big\}\,\mathrm{d}t\Big)_N,$$

$$\mu_i^{(2)} = 2\Big(z(\tau_i) - z^0(\tau_i), \int\limits_{\tau_i}^{\tau_{i+1}} \big\{L_2\big(z_t(s) - z_t^0(s)\big) + E\big(y(t)\big) - E\big(y^0(t)\big)\big\}\,\mathrm{d}t$$

$$+ f_1\big(t, u_i^e, v(t)\big) - u_*(t)\Big)_n.$$

Let $w^h(t; t_0, w_{t_0}^h(s), v^h(\cdot))$ stand for the weak solution of the following equation in the Hilbert space X_1:

$$\dot{w}^h(t) = A_1 w^h(t) + V^h(t) + F^0(t), \quad t \in T,$$

$$w^h(t_0) = w_{t_0}^h(s) = \big(y^0, y^{(1)}(s)\big),$$

and $Y(t; t_0, y_{t_0}(s), Z(\cdot))$ denote the weak solution of the equation

$$\dot{Y}(t) = A_1 Y(t) + Z(t) + F^0(t),$$

$$Y(t_0) = w^h(t_0).$$

Here $F^0(t) = (f_0(t), 0) \in X_1$, $V^h(t) = (v^h(t), 0) \in X_1$, $Z(t) = (z(t), 0) \in X_1$, the symbol $z(t) = z(t; t_0, x_{t_0}(s), u^e(\cdot), v(\cdot))$ means the corresponding part of the solution $x(t; t_0, x_{t_0}(s), u^e(\cdot), v(\cdot))$ of (2) and the control $v^h(t)$ is found from (10). Based on the results of (Kappel and Maksimov, 2000), it can be easily proved that under Condition 2 the following inequality is true:

$$\int\limits_{t_0}^{\vartheta} \big|V^h(t) - Z(t)\big|_{X_\Pi}^2\,\mathrm{d}t \le \rho(h) \equiv c\big(h + \delta^{1/2}(h) + \alpha(h)\big)^{1/2} + \big(h + \delta^{1/2}(h)\big)/\alpha(h).$$

Here the symbol X_Π denotes the subspace

$$X_\Pi = \mathbb{R}^n \times \{0\} \subset X_2 \quad \Big(0 \in L_2\big([-\tau_{l_2}^{(2)}, 0]; \mathbb{R}^n\big)\Big),$$

$|\cdot|_{X_\Pi}$ is the norm on X_Π induced by the norm on the space X_2. By Lemma 1 we see that

$$\int\limits_{t_0}^{\vartheta} |v^h(t) - z(t)|_n^2\,\mathrm{d}t \le \rho(h).$$

Then we have

$$\delta \sum_{i=1}^{m-1} \left|v_i^h - z(\tau_i)\right|_n^2 \le c_0\rho(h), \quad c_0 > 1. \tag{15}$$

For $t \in [\tau_i, \tau_{i+1})$ we obtain

$$\begin{aligned}\left|L\big(y_t(s)\big) - L\big(y_{\tau_i}(s)\big)\right| \;\le\; c_1\Big(&\sum_{i=0}^{l_1} \left|y_t\big(t-\tau_i^{(1)}\big) - y_{\tau_i}\big(t-\tau^{(1)}\big)\right|_N \\ &+ \int_{-\tau}^{0} \left|y(t+s) - y(\tau_i+s)\right|_N \,\mathrm{d}s\Big).\end{aligned}$$

Thus, by Lemma 2,

$$\int_{\tau_i}^{\tau_{i+1}} \Big|L_1\big(y_t(s) - y_t^0(s)\big) - L_1\big(y_{\tau_i}(s) - y_{\tau_i}^0(s)\big)\Big|_N \,dt \le c_2\delta^{3/2}.$$

It is clear that the inequality

$$\int_{\tau_i}^{\tau_{i+1}} \Big|C\big(z(t) - z^0(t)\big) - C\big(z(\tau_i) - z^0(\tau_i)\big)\Big|_N \,\mathrm{d}t \le c_3\delta^{3/2}$$

is fulfilled. We have

$$\begin{aligned}\mu_i^{(1)} \;\le\; 2\delta c_4 &\left|y(\tau_i) - y^0(\tau_i)\right|_N \Big\{\left|z(\tau_i) - z^0(\tau_i)\right|_n \\ &+ \left|y_{\tau_i}(s) - y_{\tau_i}^0(s)\right|_{L_2([-\tau_{l_1}^{(1)},0];\mathbb{R}^N)}\Big\} + c_5\delta^{3/2}.\end{aligned} \tag{16}$$

By analogy, we conclude that

$$\begin{aligned}\mu_i^{(2)} \;\le\; 2\delta c_6 &\left|z(\tau_i) - z^0(\tau_i)\right|_n \Big\{\left|z_{\tau_i}(s) - z_{\tau_i}^0(s)\right|_{L_2([-\tau_{l_1}^{(2)},0];\mathbb{R}^n)} \\ &+ \left|y(\tau_i) - y^0(\tau_i)\right|_N\Big\} + c_7\delta^{3/2} + \mu_i^{(3)},\end{aligned} \tag{17}$$

where

$$\mu_i^{(3)} = 2\Big(z(\tau_i) - z^0(\tau_i), \int_{\tau_i}^{\tau_{i+1}} \big(f_1\big(t, u_i^e, v(t)\big) - u_*(t)\big)\,\mathrm{d}t\Big)_n.$$

It is obvious that

$$\nu_i^{(2)} \le c_8\delta^2. \tag{18}$$

Combining (14)–(18), we obtain

$$\varepsilon(\tau_{i+1}) \le (1 + c_9\delta)\varepsilon(\tau_i) + c_{10}\delta^{3/2} + \mu_i^{(3)}. \tag{19}$$

But

$$\mu_i^{(3)} \le \mu_i^{(4)} + \mu_i^{(5)},$$

where

$$\mu_i^{(4)} = 2\Big(v_i^h - z^0(\tau_i), \int\limits_{\tau_i}^{\tau_{i+1}} \big(f_1\big(\tau_i, u_i^e, v(t)\big) - u_*(t)\big)\, \mathrm{d}t\Big)_n,$$

$$\mu_i^{(5)} = c_{10}\Big(|v_i^h - z^0(\tau_i)|_n + \omega(\delta)\Big)\delta,$$

and the symbol $\omega(\delta)$ stands for the continuity modulo of function $f_1(t,u,v)$, i.e.

$$\omega(\delta) = \sup\left\{|f_1(t_1,u,v) - f_1(t_2,u,v)|_n : \ t_1,t_2 \in T, \ |t_1 - t_2| < \delta, \ u \in P, \ v \in Q\right\}.$$

From (15) it follows that

$$\sum_{i=1}^{m-1} \big|v_i^h - z^0(\tau_i)\big|_n \delta \le c_{11}\rho^{1/2}(h). \tag{20}$$

From Condition 1(a) we deduce that, irrespective of the disturbance $v(t) \in Q$, $t \in [\tau_i, \tau_{i+1})$, acting on the system (2), there exists a control $u^0_{\tau_i,\tau_{i+1}}(\cdot; v_{\tau_i,\tau_{i+1}}(\cdot))$ such that

$$f_1\left(t, u^0\big(t; v_{\tau_i,\tau_{i+1}}(\cdot)\big), v(t)\right) = u_*(t) \quad \text{for a.a.} \quad t \in [\tau_i, \tau_{i+1}).$$

Therefore

$$\mu_i^{(4)} \ \le\ 2\int\limits_{\tau_0}^{\tau_{i+1}} \Big(v_i^h - z^0(\tau_i), f_1\big(\tau_i, u_i^e, v(t)\big) - f_1\left(\tau_i, u^0\big(t; v_{\tau_i,\tau_{i+1}}(\cdot)\big), v(t)\right)\Big)_n \mathrm{d}t$$
$$+\, c_{12}\delta\omega(\delta).$$

Condition 1(c) and the definition of u_i^e give

$$\mu_i^{(4)} \le c_{12}\delta\omega(\delta). \tag{21}$$

From (21) it follows that

$$\mu_i^{(3)} \le |v_i^h - z^0(\tau_i)|_n + c_{13}\delta\omega(\delta).$$

From this and (19), we obtain

$$\varepsilon(\tau_{i+1}) \le (1 + c_9\delta)\varepsilon(\tau_i) + c_{14}\delta\big(\omega(\delta) + \delta^{1/2}\big) + c_{10}\delta\big|v_i^h - z^0(\tau_i)\big|_n.$$

Thus (20) yields

$$\varepsilon(\tau_{i+1}) \ \le\ c_{14}\Big(\varepsilon(t_0) + \omega(\delta) + \delta^{1/2} + \sum_{i=1}^{m-1} \delta|v_i^h - z^0(\tau_i)|_n\Big)$$
$$\le\ c_{15}\Big(\omega\big(\delta(h)\big) + \delta^{1/2}(h) + \rho^{1/2}(h)\Big), \quad i = 0,\dots,m-1,$$

which completes the proof. ■

Remarks:

- The algorithm suggested in the paper can be applied to solving the problem of robust control for other classes of equations with time delay.
- In the process of solving the reconstruction problem according to the algorithm described above, an 'identification' block was introduced in the control loop. The construction of this block is based on the method of a smoothing functional (Thikhonov's method), which is well-known in the theory of ill-posed problems. The reconstruction algorithm can be also modified by introducing an 'identification' block based on the dynamical discrepancy method (Blizorukova, 2000; Kryazhimskii and Osipov, 1988; Maksimov, 1994) in the control loop.
- Instead of the method of stable tracks, one can also use a more general method of the so-called stable bridges (Krasovskii and Subbotin, 1988).

5. Conclusion

The problem of robust control of a system with time delay under measurements of a part of coordinates has been considered. An algorithm for solving this problem based on the methods of the theories of dynamical reconstruction and guaranteed control has been designed. This algorithm is stable with respect to informational noise and computational errors. It can be rather easily implemented even on low-cost personal computers.

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