Int. J. Appl. Math. Comput. Sci., 2002, Vol.12, No.1, 81–90

NON-SMOOTHNESS IN THE ASYMPTOTICS OF THIN SHELLS AND PROPAGATION OF SINGULARITIES. HYPERBOLIC CASE

PHILIPPE KARAMIAN*, JACQUELINE SANCHEZ-HUBERT*, ÉVARISTE SANCHEZ PALENCIA**

* Laboratoire de Mécanique, Université de Caen,
boulevard Maréchal Juin, 14032 Caen, France,
e-mail: {karen,sanchez}@meca.unicaen.fr

** Laboratoire de Modélisation en Mécanique, Université Paris VI,
8 rue du capitaine Scott, 75015 Paris, France,
e-mail: sanchez@lmm.jussieu.fr

We consider the limit behaviour of elastic shells when the relative thickness tends to zero. We address the case when the middle surface has principal curvatures of opposite signs and the boundary conditions ensure the geometrical rigidity. The limit problem is hyperbolic, but enjoys peculiarities which imply singularities of unusual intensity. We study these singularities and their propagation for several cases of loading, giving a somewhat complete description of the solution.

Keywords: hyperbolic systems, propagation of singularities, shells

1. Introduction

In this paper we study the propagation of singularities for the membrane system of shells in the hyperbolic case, i.e. when the middle surface has principal curvatures of opposite signs. The structure of the system is essentially hyperbolic, but presents certain peculiarities which imply singularities stronger than in ordinary hyperbolic systems. For instance, discontinuities of the first kind (i.e. Heaviside singularities) of the normal loading may imply δ'-like singularities of the normal displacement. As a consequence, the knowledge of the singularity gives most of the structure of the solutions, and often furnishes their good description, both from the qualitative and quantitative viewpoints. The motivation to study this problem is as follows: We are interested in a singular perturbation of the variational problems of the form

For given $f \in V'$, find $u^\varepsilon \in V$ satisfying

$$a_m\left(u^\varepsilon, v\right) + \varepsilon^2 a_f\left(u^\varepsilon, v\right) = (f, v), \quad \forall\, v \in V \tag{1}$$

involving two positive and symmetric energy forms $a_m(u,v)$ and $\varepsilon^2 a_f(u,v)$, which are called the membrane and the flexion forms, respectively, because of the mechanical application to shell theory, as we shall see in Section 2. The factor ε^2 in the second form is a small parameter. For $\varepsilon > 0$, the energy space V is such that $a_m + \varepsilon^2 a_f$ is continuous and coercive on it, whereas the limit problem for $\varepsilon = 0$ involves a new energy space V_m (membrane energy space) for which the bilinear form a_m is continuous and coercive. In fact, V_m is the completion of V equipped with the norm $\sqrt{a_m(\cdot,\cdot)}$. Clearly, the above considerations only make sense in the case when a_m is the square of a norm, i.e. under the hypothesis that

$$v \in V \ \text{ and } \ a_m(v,v) = 0 \Rightarrow v = 0. \tag{2}$$

The order of differentiation in a_f is higher than in a_m, so that as $\varepsilon \searrow 0$, a singular perturbation phenomenon appears.

Obviously, V_m contains functions less smooth than those of V. As a consequence, the solutions u^ε of the variational problem belong to V but their limit as $\varepsilon \searrow 0$ is a less smooth function (i.e. containing some kind of singularities). In fact, there is another important reason for the presence of singularities. Indeed, as $V \subset V_m$, the dual spaces satisfy $V'_m \subset V'$, so that the data f which are in V' are admissible for the variational problem with $\varepsilon > 0$, but it may happen, and often does happen in applications (see Section 2), that $f \notin V'_m$. As a consequence, the limit problem does not make sense as a variational one in V_m. The corresponding solution of the limit problem, if it exists, is out of V_m. In the sequel, we shall consider the case when the limit problem is hyperbolic and such that there is a unique solution satisfying the boundary conditions even when $f \notin V'_m$.

The case of $f \in V'_m$ will be called classical. In that situation, a well-known theorem, see, e.g., (Lions, 1973), asserts that u^ε converges to u^0 in the strong topology of

V_m, where u^ε and u^0 are the solutions of the variational problems for $\varepsilon > 0$ and $\varepsilon = 0$, respectively.

In the case of $f \notin V_m'$, even if the solution u^0 of the limit problem exists out of V_m, to our knowledge there is no theorem regarding the convergence of u^ε to u^0. Assuming that this convergence holds true, the corresponding topology is weaker than the one of V_m. Moreover, for the energy of the solution u^ε, we have

$$a_m\left(u^\varepsilon, u^\varepsilon\right) + \varepsilon^2 a_f\left(u^\varepsilon, u^\varepsilon\right) \to +\infty \text{ as } \varepsilon \searrow 0 \quad (3)$$

(see, e.g., (Gérard and Sanchez Palencia, 2000)). There is some evidence that such a convergence actually holds at least for certain examples. This evidence follows from formal asymptotic expansions and numerical experiments. The formal asymptotic expansions are concerned with boundary layer theory for either thin shell problems or their simplified models (Karamian *et al.*, 2000; Karamian and Sanchez-Hubert, 2002; Leguillon *et al.*, 1999). Moreover, the convergence for the model problem addressed in (Karamian *et al.*, 2000) was proven in (Sanchez Palencia, 2000). The numerical computations for small ε are not very reliable because of the clearly non-smooth character of the solutions; nevertheless, they seem to confirm the above-mentioned convergence.

The context of this paper (which will be more explicitly explained in Section 2) is the following. We consider problems for thin elastic shells the middle surface of which is hyperbolic (i.e. the principal curvatures are everywhere different from zero and of opposite sign). Taking a special parametrization (y^1, y^2), where the coordinate lines are the asymptotic curves of the middle surface, the limit problem for $\varepsilon = 0$ (the so-called membrane problem) may be written as

$$\begin{cases} -D_1 T^{11} - D_2 T^{12} = f^1, \\ -D_1 T^{12} - D_2 T^{22} = f^2, \\ -2b_{12} T^{12} = f_3, \end{cases} \quad (4)$$

$$\begin{cases} D_1 u_1 = C_{11\alpha\beta} T^{\alpha\beta}, \\ D_2 u_2 = C_{22\alpha\beta} T^{\alpha\beta}, \\ \dfrac{1}{2}\left(D_2 u_1 + D_1 u_2\right) - b_{12} u_3 = C_{12\alpha\beta} T^{\alpha\beta} \end{cases} \quad (5)$$

in a domain Ω of the plane (y^1, y^2). The unknowns are the symmetric membrane stresses $T^{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ and the displacements u_i $(i = 1, 2, 3)$. The symbols D_α are the covariant derivatives with respect to the variables y^1, y^2. The coefficients $C_{\alpha\beta\lambda\mu}$ are the compliance ones, given smooth functions. The coefficient b_{12} (coefficient of the second fundamental form) is a given smooth function everywhere different from zero. Finally, $f = (f^1, f^2, f_3)$ is a datum such that in general $f \notin V_m'$.

Obviously, the system (4)–(5) has six equations and six unknowns. Nevertheless, T^{12} is immediately given by $(4)_3$ and u_3 only appears in the last equation (5), which can be considered as a definition of u_3. Then the unknowns are essentially T^{11}, T^{22}, u_1, u_2; the first two equations of (4) only involve T^{11} and T^{22} and constitute a first-order hyperbolic system for them with the simple characteristics $y^1 = \mathrm{Const}$ and $y^2 = \mathrm{Const}$. Assuming that the boundary conditions allow us to determine T^{11} and T^{22}, the right-hand side of (5) is known and the first two equations of (5) form again a first-order hyperbolic system for u_1 and u_2 with the same simple characteristics. At this point, the high order of singularity of the solutions is easy to understand. We see that the first two equations of (4) for T^{11} and T^{22} involve as 'data' the first-order derivatives of f_3. Moreover, the unknown u_3 in the third equation of (5) inherits singularities from the first-order derivatives of u_1 and u_2. If, as usual, we focus our attention on normal forces f_3 and the normal displacement u_3, we see that the singularities are by two orders stronger than in the genuine hyperbolic system.

We are mainly concerned with the propagation of the singularities of this system. We consider the classical sequence of distributions on $\mathbb{R}$ with increasing singularities

$$\ldots xY(x), Y(x), \delta(x), \delta'(x), \ldots, \quad (6)$$

where Y and δ denote the Heaviside function and the Dirac mass, respectively. More precisely, these distributions are considered as singularities at $x = 0$ whereas their values for $x \neq 0$ are discarded; for instance, $Y(x)$ is considered merely as the unit jump at $x = 0$. In order to describe the singularity, for example, along $y^2 = 0$, we consider expansions of the form (for instance)

$$w \simeq \delta'\left(y^2\right) W^0\left(y^1\right) + \delta\left(y^2\right) W^1\left(y^1\right) + \cdots, \quad (7)$$

where it is understood that the terms denoted by dots are less singular than the previous ones at $y^2 = 0$. Such a kind of expansion is in the framework of discontinuous solutions, see, e.g., (Egorov and Shubin, 1992, Sec. 4.11; Gérard, 1988; Sanchez Palencia, 2001). We always assume that the geometric data and the coefficients are smooth, so that the sequence (7) is consistent with the singularities of the solutions provided that the singularities of the loadings are in that sequence, which covers most of the usual examples.

The very description of the singularities is given in Section 3. Precisions on the mechanical problem and the specific data will be given in Section 2. Numerical experiments exhibiting such a kind of behaviour are given in Section 4.

2. Description of the Mechanical Problem

We give here the elements of shell theory which are necessary for understanding the sequel of the paper. More explicit descriptions of shells can be found in shell treatises (Bernadou, 1994; Ciarlet ,2000; Goldenveizer, 1962; Sanchez-Hubert and Sanchez Palencia, 1997).

Let us denote by Ω a bounded and connected domain of the (y^1, y^2)-plane (the parameter plane). The middle surface $\mathcal{S}$ of the shell is defined by a smooth function $\vec{r}$, i.e.

$$\Omega \ni \left(y^1, y^2\right) \longmapsto \vec{r}\left(y^1, y^2\right) \in \mathbb{R}^3. \tag{8}$$

At any point of $\mathcal{S}$ we define the tangent vectors

$$\vec{a}_\alpha = \partial_\alpha \vec{r}, \tag{9}$$

$(\vec{a}_1, \vec{a}_2)$ being the local covariant basis of the tangent plane.

The first fundamental form which defines the distances on the surface is given by

$$\mathrm{d}s^2 = a_{\alpha\beta}\, \mathrm{d}y^\alpha\, \mathrm{d}y^\beta, \tag{10}$$

where $a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta$. The corresponding contravariant basis $\vec{a}^\alpha$ is defined by $\vec{a}^\alpha \cdot \vec{a}_\beta = \delta^\alpha_\beta$. We also consider the unit normal vector $\vec{a}^3 = \vec{a}_3$. We note that, when changing the parametrization, $\vec{a}_3$ is invariant up to the orientation, so that normal components behave essentially as scalars.

We recall that the Christoffel symbols are

$$\Gamma^\lambda_{\alpha\beta} = \partial_\beta \vec{a}_\alpha \cdot \vec{a}^\lambda$$

and that the coefficients of the second fundamental form describing the curvatures are

$$b_{\alpha\beta} = b_{\beta\alpha} = -\partial_\beta \vec{a}_3 \cdot \vec{a}_\alpha.$$

We also recall that a point of $\mathcal{S}$ is said to be elliptic, hyperbolic or parabolic when the second fundamental form is definite, indefinite or degenerate, respectively. This is equivalent to saying that the product of the principal curvatures is more than, equal to, or less than zero, respectively.

In contrast to ordinary differentiation ∂_α, the covariant differentiation is denoted by D_α. Its action on vectors and tensors is

$$\begin{cases} D_\alpha u_\beta = \partial_\alpha u_\beta - \Gamma^\lambda_{\alpha\beta} u_\lambda, \\ D_\lambda T^{\alpha\beta} = \partial_\lambda T^{\alpha\beta} + \Gamma^\alpha_{\lambda\mu} T^{\mu\beta} + \Gamma^\beta_{\lambda\mu} T^{\alpha\mu}. \end{cases} \tag{11}$$

Let $\vec{u}$ be the displacement of $\mathcal{S}$ for its deformation. Specifically, we consider that $\vec{r}$ changes into $\vec{r} + \vec{u}$ and we linearize for small $\vec{u}$. Then the strain tensor is given by the components

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(D_\alpha u_\beta + D_\beta u_\alpha\right).$$

It describes the variation produced by $\vec{u}$ on the coefficients of the first fundamental form.

Analogously, the components of the second fundamental form vary along

$$\begin{aligned} \rho_{\alpha\beta} = {} & \partial_\alpha \partial_\beta u_3 - \Gamma^\lambda_{\alpha\beta} \partial_\lambda u_3 - b^\lambda_\alpha b_{\lambda\beta} u_3 \\ & + D_\alpha\left(b^\lambda_\beta u_\lambda\right) + b^\lambda_\alpha D_\beta u_\lambda. \end{aligned}$$

Then the classical (Love Kirchhoff or Koiter) theory of thin shells is described in terms of the two bilinear forms a_m and $\varepsilon^2 a_f$ of membrane and flexion energies which are given by

$$a_m\left(\vec{u}^\varepsilon, \vec{v}\right) = \int_{\mathcal{S}} A^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu}\left(\vec{u}^\varepsilon\right) \gamma_{\alpha\beta}\left(\vec{v}\right) \mathrm{d}S, \tag{12}$$

$$a_f\left(\vec{u}^\varepsilon, \vec{v}\right) = \int_{\mathcal{S}} B^{\alpha\beta\lambda\mu} \rho_{\lambda\mu}\left(\vec{u}^\varepsilon\right) \rho_{\alpha\beta}\left(\vec{v}\right) \mathrm{d}S, \tag{13}$$

respectively, where $A^{\alpha\beta\lambda\mu}$ and $B^{\alpha\beta\lambda\mu}$ are the coefficients of membrane and flexion rigidities which satisfy usual conditions of symmetry and positivity.

Here 2ε denotes the relative thickness of the shell (equal to the ratio of the thickness to any other characteristic length of the shell). Obviously, the factor ε^2 in front of the form a_f accounts for the fact that the flexion rigidity is asymptotically small with respect to the membrane rigidity. Obviously, as the form a_f contains derivatives of higher orders than a_m, the asymptotic process $\varepsilon \searrow 0$ is a singular perturbation.

The stress membrane components $T^{\alpha\beta}$ are related to the strains by

$$T^{\alpha\beta}\left(\vec{u}^\varepsilon\right) = A^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu}\left(\vec{u}^\varepsilon\right). \tag{14}$$

Conversely, the strains can be expressed in terms of the stresses as

$$\gamma_{\lambda\mu}\left(\vec{u}^\varepsilon\right) = C_{\lambda\mu\alpha\beta} T^{\alpha\beta}\left(\vec{u}^\varepsilon\right), \tag{15}$$

where the $C_{\lambda\mu\alpha\beta}$'s are the compliance coefficients.

The energy space V of vectors $\vec{v}$ satisfying the kinematic boundary conditions (bound. cond. for brevity) is

$$\begin{aligned} V = \big\{ \vec{v} = (v_1, v_2, v_3) \in H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega);& \\ \text{bound. cond.} \big\}.& \end{aligned} \tag{16}$$

Typical kinematic boundary conditions are either fixed conditions:

$$\vec{v} = 0,$$

or clamped conditions:

$$\begin{cases} \vec{v} = 0, \\ \dfrac{\partial v_3}{\partial n} = 0 \end{cases} \tag{17}$$

on a part Γ_0 of the boundary.

Under the hypothesis that the surface is geometrically rigid or inhibited in the terminology of (Sanchez-Hubert and Sanchez Palencia, 1997), i.e. that (2) holds true, and thus has to be checked in each case, $a_m(\vec{v},\vec{v})$ is the square of a norm on V and we may construct the space V_m as the completion of V with this norm. Obviously, because of the positivity of the coefficients $A^{\alpha\beta\lambda\mu}$, this norm is equivalent to

$$\|\vec{v}\|_{V_m} = \left(\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\vec{v})\|_0^2 \right)^{\frac{1}{2}}. \tag{18}$$

From now on we make the hypothesis that the surface $\mathcal{S}$ is everywhere hyperbolic. Moreover, it is described with the special parametrization where the coordinate lines are the asymptotic curves so that $b_{11} = b_{22} = 0$, $b_{12} \neq 0$. In this context, the left-hand sides in (5) are γ_{11}, γ_{22} and γ_{12}.

The limit problem for $\varepsilon = 0$ is as follows: For given $\vec{f} \in V_m'$, find $\vec{u}^0 \in V_m$ satisfying

$$\begin{aligned} a_m\left(\vec{u}^0,\vec{v}\right) &\equiv \int_{\mathcal{S}} T^{\alpha\beta}\left(\vec{u}^0\right)\gamma_{\alpha\beta}(\vec{v})\,\mathrm{d}S \\ &= \left(\vec{f},\vec{v}\right), \quad \forall\, \vec{v} \in V_m. \end{aligned} \tag{19}$$

Classical integration by parts shows that the problem (19) is equivalent to the system (4), (5) with the boundary conditions

$$u_1 = u_2 = 0 \ \text{ on } \Gamma_0 \tag{20}$$

and

$$T^{\alpha\beta} n_\beta = 0 \ \text{ on } \Gamma_1, \tag{21}$$

where $\Gamma_1 = \partial\Omega\backslash\Gamma_0$ is the free part of the boundary and $\vec{n}$ denotes the unit vector tangent to $\mathcal{S}$ and normal to the boundary. Kinematic boundary conditions (20) amount to (17) for the tangent components but the conditions for u_3 disappear because they obviously do not make sense in V_m, cf. (18). Moreover, (20) holds true under the hypothesis that Γ_0 is nowhere parallel to the characteristic curves, i.e. nowhere parallel to axes $y^1 = 0$, $y^2 = 0$. For all these questions, see (Sanchez-Hubert and Sanchez Palencia, 1997, Sec. VII.2).

Obviously, the problem for $\varepsilon > 0$ makes sense for any $\vec{f} \in V'$ which is a product of duals of standard Sobolev spaces. In contrast, the space V_m is not classical. Let us say that V_m is “large” so that its dual is “small”. As a result, quite “usual” loadings do not belong to V_m' and will be in the non classical case mentioned in the Introduction. Let us state this in a more precise form as follows:

Theorem 1. *A necessary and sufficient condition for $\vec{f}$ to be in V_m' is that there exist $T^{\alpha\beta} = T^{\beta\alpha}$ in $L^2(\Omega)$ satisfying*

$$\begin{cases} -D_\beta T^{\alpha\beta} = f^\alpha, \\ -2b_{12}T^{12} = f_3 \end{cases} \tag{22}$$

in Ω and

$$T^{\alpha\beta} n_\beta = 0 \tag{23}$$

on the free part Γ_1 of the boundary.

The proof is analogous to that of Theorem 2.3 in (Karamian *et al.*, 2000). In fact, the property that if there is no $T \in L^2(\Omega)$ satisfying (22) and (23) then $\vec{f} \notin V_m'$ follows directly from the previous considerations. Indeed, if $\vec{f} \in V_m'$, then the solution to the limit problem exists so that the corresponding $T^{\alpha\beta}(\vec{u}^0)$ exist and belong to $L^2(\Omega)$ and, by virtue of (4) and (21), may be taken as $T^{\alpha\beta}$.

Example 1. Let us take $f^\alpha = 0$ and $f_3 = Y(y^2 - c)f(y^1)$, where f is a smooth function and Y denotes the Heaviside function. System (22) gives

$$\begin{cases} -\partial_1 T^{11} = \delta\left(y^2 - c\right)\Phi\left(y^1\right) + \text{element} \in L^2(\Omega), \\ -\partial_2 T^{22} = \text{element} \in L^2(\Omega), \end{cases}$$

which is impossible with $T^{11} \in L^2(\Omega)$. Consequently, $\vec{f} \notin V_m'$. ♦

Example 2. It is even easier to prove that $f^\alpha = 0$ and $f_3 = \delta(\mathcal{C})$, where $\mathcal{C}$ denotes a curve of the surface, does not belong to V_m'. Indeed, this follows immediately from the fact that the trace of v_3 is not defined for $\vec{v} \in V_m$, cf. (18). ♦

In the two previous examples, obviously $\vec{f} \in V'$.

3. Propagation of the Singularities

For the sake of conciseness, let us consider a specific example of geometry, as well as boundary conditions. Let Ω be the domain shown in Figs. 1 or 2. The surface $\mathcal{S}$ is assumed to be smooth and uniformly hyperbolic. The parametrization is chosen such that the asymptotic curves coincide with the coordinate ones $y^1\text{const}$ and $y^1\text{const}$, so that

$$b_{11} = b_{22} = 0, \quad b_{12} \neq 0. \tag{24}$$

The boundary is fixed along $\Gamma_0 \equiv AB$, which is not a characteristic curve, so that the boundary conditions are (20). The rest of the boundary $\partial\Omega\backslash\Gamma_0$ is free. Two cases of loading will be considered, and a wide variety of examples may be handled in an analogous way.

3.1. First Example of Loading

In this subsection, the loading is defined as follows:

$$\vec{f} = \left(0, 0, \delta\left(y^2 - c^2\right)\theta_{[a^1,b^1]}\left(y^1\right)\right) F\left(y^1\right), \tag{25}$$

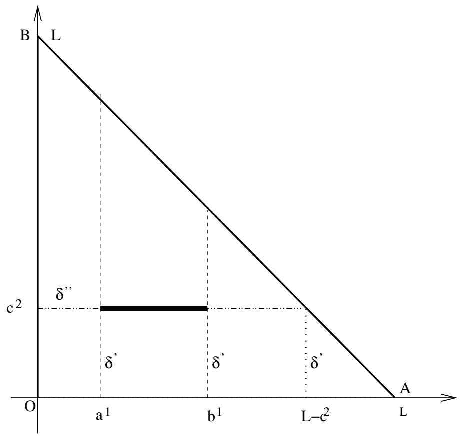

Fig. 1. Domain Ω in the first case of loading (25) (δ' and δ'' indicate the type of the singularity of u_3).

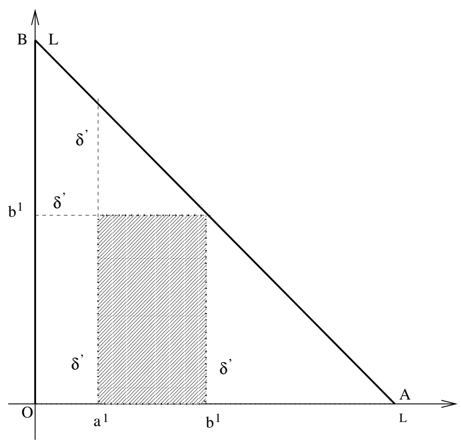

Fig. 2. Domain Ω in the second case of loading (49) (δ' indicates the type of the singularity along the characteristics).

where $\theta_{[a^1,b^1]}$ is the characteristic function of the interval $[a^1,b^1]$ and F is assumed to be a smooth function.

Let us now address the propagation of the singularities along the characteristic curve which supports the loading.

3.1.1. Propagation of the Singularities Along $y^2 = c^2$

We first study the singularities of the components $T^{\alpha\beta}$ in the system (4), which is of the form

$$\left\{\begin{aligned} &-\partial_1 T^{11} - \left(2\Gamma^1_{11} + \Gamma^2_{12}\right) T^{11} - \Gamma^1_{22} T^{22} \\ &\qquad = \partial_2 T^{12} + \left(3\Gamma^1_{12} + \Gamma^2_{22}\right) T^{12}, \\ &-\partial_2 T^{22} - \left(2\Gamma^2_{22} + \Gamma^1_{12}\right) T^{22} - \Gamma^2_{11} T^{11} \\ &\qquad = \partial_1 T^{12} + \left(3\Gamma^2_{12} + \Gamma^1_{11}\right) T^{12}, \\ &-2b_{12} T^{12} = \delta\left(y^2 - c^2\right) \theta_{[a^1,b^1]}\left(y^1\right) f\left(y^1\right). \end{aligned}\right. \tag{26}$$

By substituting the expression for T^{12} in $(26)_3$ into $(26)_1$ and $(26)_2$, we obtain for the leading order of singularity

$$\left\{\begin{aligned} &-\partial_1 T^{11} - \left(2\Gamma^1_{11} + \Gamma^2_{12}\right) T^{11} - \Gamma^1_{22} T^{22} \\ &\qquad \simeq - \delta'\left(y^2 - c^2\right) \Phi_1\left(y^1\right) + \cdots, \\ &-\partial_2 T^{22} - \left(2\Gamma^2_{22} + \Gamma^1_{22}\right) T^{22} - \Gamma^2_{11} T^{11} \\ &\qquad \simeq - \delta\left(y^2 - c^2\right) \Phi_2\left(y^1\right) + \cdots, \\ &T^{12} = - \delta\left(y^2 - c^2\right) \Phi_1\left(y^1\right), \end{aligned}\right. \tag{27}$$

where

$$\Phi_1\left(y^1\right) = \frac{\theta_{[a^1,b^1]}\left(y^1\right) f\left(y^1\right)}{2b_{12}\left(y^1, c^2\right)} \tag{28}$$

and

$$\Phi_2\left(y^1\right) = \partial_1 \Phi_1\left(y^1\right). \tag{29}$$

We note that Φ_2 contains terms in $\delta(y^1 - a^1)$ and $\delta(y^1 - b^1)$.

We see that the appropriate singularity expansions for $T^{\alpha\beta}$ in the framework of (7) are

$$\left\{\begin{aligned} &T^{11} \simeq \delta'\left(y^2 - c^2\right) \mathcal{T}^{11}\left(y^1\right) + \cdots, \\ &T^{22} \simeq \delta\left(y^2 - c^2\right) \mathcal{T}^{22}\left(y^1\right) + \cdots, \\ &T^{12} = -\delta\left(y^2 - c^2\right) \Phi_1\left(y^1\right). \end{aligned}\right. \tag{30}$$

The system (27) then gives the system satisfied by $\mathcal{T}^{11}$ and $\mathcal{T}^{22}$:

$$\left\{\begin{aligned} &\frac{\mathrm{d}\mathcal{T}^{11}}{\mathrm{d}y^1} + \left(2\Gamma^1_{11} + \Gamma^2_{12}\right) \mathcal{T}^{11} = \Phi_1\left(y^1\right), \\ &\qquad \mathcal{T}^{22} + \Gamma^2_{11} \mathcal{T}^{11} = 0. \end{aligned}\right. \tag{31}$$

This system is of total order one. Let us look for the corresponding boundary condition. At the leading order (23) gives $\mathcal{T}^{11} n_1 = 0$, where $n_1 \neq 0$. Indeed, we have $\vec{n} = n_1 \vec{a}^1 + n_2 \vec{a}^2$, where $\vec{n}$ is normal to $\vec{a}_2$, i.e. parallel to $\vec{a}^1$. The boundary condition for $\mathcal{T}^{11}$ is then

$$\mathcal{T}^{11}(0) = 0. \tag{32}$$

We then have

$$\left\{\begin{aligned} &\mathcal{T}^{11}\left(y^1\right) = \int_{a^1}^{y^1} \Phi_1(\eta) \exp\Bigg[\int_{y^1}^{\eta} \left(2\Gamma^1_{11}\left(\xi, c^2\right)\right. \\ &\qquad\qquad \left. + \Gamma^2_{12}\left(\xi, c^2\right)\right) \mathrm{d}\xi\Bigg] \mathrm{d}\eta, \\ &\mathcal{T}^{22} = -\Gamma^2_{11}\left(y^1, c^2\right) \mathcal{T}^{11}\left(y^1\right), \end{aligned}\right. \tag{33}$$

and the leading order of the singularity is completely determined.

Remark 1. For $0 < y^1 < a^1$, $\mathcal{T}^{11}(y^1) = 0$ but for $b^1 < y^1 < L - c^2$, in general, $\mathcal{T}^{11}(y^1) \neq 0$ though $\theta_{[a^1,b^1]}(y^1) \equiv 0$: this manifests the *phenomenon of propagation of singularities*.

Remark 2. According to the previous results, at the leading order both boundary conditions (21) are automatically satisfied.

Let us now examine the singularities of the displacement components u_i. The system (5) gives at the leading orders

$$\begin{cases} \partial_1 u_1 - \Gamma^1_{11} u_1 - \Gamma^2_{11} u_2 \\ \qquad \simeq C_{1111}\delta'\left(y^2 - c^2\right)\mathcal{T}^{11}\left(y^1\right) + \cdots, \\ \partial_2 u_2 - \Gamma^1_{22} u_1 - \Gamma^2_{22} u_2 \\ \qquad \simeq C_{2211}\delta'\left(y^2 - c^2\right)\mathcal{T}^{11}\left(y^1\right) + \cdots, \\ \dfrac{1}{2}\left(\partial_1 u_2 + \partial_2 u_1\right) - \Gamma^1_{12} u_1 - \Gamma^2_{12} u_2 - b_{12} u_3 \\ \qquad \simeq C_{1211}\delta'\left(y^2 - c^2\right)\mathcal{T}^{11}\left(y^1\right) + \cdots. \end{cases} \tag{34}$$

Then the appropriate expansions of the components u_i are

$$\begin{cases} u_1 \simeq \delta'\left(y^2 - c^2\right) U_1\left(y^1\right) + \cdots, \\ u_2 \simeq \delta\left(y^2 - c^2\right) U_2\left(y^1\right) + \cdots, \\ u_3 \simeq \delta''\left(y^2 - c^2\right) U_3\left(y^1\right) + \cdots. \end{cases} \tag{35}$$

Substitution of (35) into (34) leads to

$$\begin{cases} \dfrac{\mathrm{d}U_1}{\mathrm{d}y^1} - \Gamma^1_{11} U_1 = C_{1111}\left(y^1, c^2\right)\mathcal{T}^{11}\left(y^1\right) \\ \qquad \equiv \Psi_1\left(y^1\right), \\ U_2 - \Gamma^1_{22} U_1 = C_{2211}\left(y^1, c^2\right)\mathcal{T}^{11}\left(y^1\right) \\ \qquad \equiv \Psi_2\left(y^1\right), \\ \dfrac{1}{2} U_1 - b_{12} U_3 = 0. \end{cases} \tag{36}$$

The components U_1 and U_2 satisfy a system of total order one with the boundary condition

$$U_1\left(L - c^2\right) = 0 \tag{37}$$

and we obtain

$$\begin{cases} U_1\left(y^1\right) = \left(\displaystyle\int_{L-c^2}^{y^1} \Psi_1(\eta)\right)\left[\exp\left(\displaystyle\int_{L-c^2}^{\eta} \Gamma^1_{11}\left(\xi, c^2\right)\mathrm{d}\xi\right)\right]\mathrm{d}\eta, \\ U_2\left(y^1\right) = \Gamma^1_{22} U_1 + \Psi_2\left(y^1\right), \\ U_3\left(y^1\right) = \dfrac{1}{2b_{12}\left(y^1, c^2\right)} U_1\left(y^1\right). \end{cases} \tag{38}$$

The leading orders of the singularities of the components u_i are completely known.

Remark 3. We observe that $U_2(L - c^2) \neq 0$. The boundary condition (20) for u_2 (which is of an order of singularity lower than u_1, see (35)) is not satisfied. This provokes a new (reflected) singularity of lower order along $y^1 = c^2$. This kind of phenomenon was considered in (Karamian, 1998b). See also Remark 6 here after.

Remark 4. The components U_1 and U_3 are different from zero on the whole interval $0 < y^1 < L - c^2$ (propagation of the singularities).

3.1.2. Propagation of the Singularities along the Characteristic $y^1 = a^1$

In the sequel, we shall study the propagation along the characteristic $y^1 = a^1$. Propagation along $y^1 = b^1$ is analogous. We now have

$$T^{12} = -Y\left(y^1 - a^1\right)\frac{f\left(a^1\right)\delta\left(y^2 - c^2\right)}{2b_{12}\left(a^1, y^2\right)}. \tag{39}$$

We note that (39) is merely the singularity of $(26)_3$ at $y^1 = a^1$. Nevertheless, the roles of $Y(y^1 - a^1)$ and $\delta(y^2 - c^2)$ are “reversed” in the study of the propagation along $y^2 = c^2$ (Subsection 3.1.1) and along $y^1 = a^1$ (now). Indeed, along $y^2 = c^2$, the “singularity” in the sense of (6) or (7) was $\delta(y^2 - c^2)$ and the “coefficient”, Φ_1, was given by (28), which contains Y terms in the tangential variable y^1. Consequently, for studying the propagation along $y^1 = a^1$, the “singularity ” is $Y(y^1 - a^1)$ and the “coefficient” is $\delta(y^2 - c^2)$; it is “more singular”, but in the tangential variable. Consequently,

$$\begin{cases} \partial_1 T^{12} = -\delta\left(y^1 - a^1\right)\dfrac{f\left(a^1\right)\delta\left(y^2 - c^2\right)}{2b_{12}\left(a^1, c^2\right)}, \\ \partial_2 T^{12} = -Y\left(y^1 - a^1\right)\dfrac{f\left(a^1\right)\delta'\left(y^2 - c^2\right)}{2b_{12}\left(a^1, c^2\right)}, \end{cases}$$

and the system (5) reduces to

$$\begin{cases} -\partial_1 T^{11} - \left(2\Gamma^1_{11} + \Gamma^2_{12}\right)T^{11} - \Gamma^1_{22} T^{22} \\ \qquad = -Y\left(y^1 - a^1\right)\dfrac{f\left(a^1\right)\delta'\left(y^2 - c^2\right)}{2b_{12}\left(a^1, c^2\right)}, \\ -\partial_2 T^{22} - \left(2\Gamma^2_{22} + \Gamma^1_{12}\right)T^{22} - \Gamma^2_{11} T^{11} \\ \qquad = \delta\left(y^1 - a^1\right)\dfrac{f\left(a^1\right)\delta\left(y^2 - c^2\right)}{2b_{12}\left(a^1, c^2\right)} \end{cases} \tag{40}$$

with (39). The appropriate expansion is then

$$\begin{cases} T^{11} \simeq Y\left(y^1 - a^1\right)\Upsilon^{11}\left(y^2\right) + \cdots, \\ T^{22} \simeq \delta\left(y^1 - a^1\right)\Upsilon^{22}\left(y^2\right) + \cdots, \end{cases} \tag{41}$$

where Υ^{11} and Υ^{22} satisfy

$$\begin{cases} -\Upsilon^{11} - \Gamma^1_{22}\Upsilon^{22} = 0, \\ -\dfrac{\mathrm{d}\Upsilon^{22}}{\mathrm{d}y^2} - \left(2\Gamma^1_{11} + \Gamma^2_{12}\right)\Upsilon^{22} \\ \qquad = \dfrac{f\left(a^1\right)}{2b_{12}\left(a^1, c^2\right)}\delta\left(y^2 - c^2\right). \end{cases} \tag{42}$$

The explicit solution is

$$\begin{aligned} &\Upsilon^{22}\left(y^2\right) \\ &\quad = \frac{f\left(a^1\right)}{2b_{12}\left(a^1, c^2\right)} Y\left(y^2 - c^2\right) \\ &\qquad \times \exp\left(-\int_{c^2}^{y^2}\left(2\Gamma^1_{11}\left(a^1,\eta\right) + \Gamma^2_{12}\left(a^1,\eta\right)\right)\mathrm{d}\eta\right). \end{aligned} \tag{43}$$

The corresponding system satisfied by the displacement components is

$$\begin{cases} \partial_1 u_1 - \Gamma^1_{11}u_1 - \Gamma^2_{11}u_2 \\ \quad = C_{1122}\left(a^1, y^2\right)\delta\left(y^1 - a^1\right)\Upsilon^{22}\left(y^2\right) + \cdots, \\ \partial_2 u_2 - \Gamma^1_{22}u_1 - \Gamma^2_{22}u_2 \\ \quad = C_{2222}\left(a^1, y^2\right)\delta\left(y^1 - a^1\right)\Upsilon^{22}\left(y^2\right) + \cdots, \\ \dfrac{1}{2}\left(\partial_2 u_1 + \partial_1 u_2\right) - \Gamma^1_{12}u_1 - \Gamma^2_{12}u_2 - b_{12}u_3 \\ \quad = C_{1222}\left(a^1, y^2\right)\delta\left(y^1 - a^1\right)\Upsilon^{22}\left(y^2\right) + \cdots, \end{cases} \tag{44}$$

and their expansions are of the form

$$\begin{cases} u_1 \simeq Y\left(y^1 - a^1\right)V_1\left(y^2\right) + \cdots, \\ u_2 \simeq \delta\left(y^1 - a^1\right)V_2\left(y^2\right) + \cdots, \\ u_3 \simeq \delta'\left(y^1 - a^1\right)V_3\left(y^2\right) + \cdots, \end{cases} \tag{45}$$

where the functions V_i satisfy

$$\begin{cases} V_1 - \Gamma^2_{11}V_2 = C_{1122}\Upsilon^{22}\left(y^2\right), \\ \dfrac{\mathrm{d}V_2}{\mathrm{d}y^2} - \Gamma^2_{22}V_2 = C_{2222}\Upsilon^{22}\left(y^2\right), \\ \dfrac{1}{2}V_2 - b_{12}V_3 = 0. \end{cases} \tag{46}$$

Taking account of the boundary condition $V_2(L - c^2) = 0,$ we obtain the solution

$$\begin{aligned} V_2\left(y^2\right) = &\left[-\int_{y^2}^{L-a^1}\exp\left(-\int_0^{\eta}\Gamma^2_{22}\left(a^1,\xi\right)\mathrm{d}\xi\right)\mathcal{F}\left(\eta\right)\mathrm{d}\eta\right] \\ &\times\exp\left(\int_0^{y^2}\Gamma^2_{22}\left(a^1,\eta\right)\mathrm{d}\eta\right), \end{aligned} \tag{47}$$

where

$$\mathcal{F}\left(\eta\right) = C_{2222}\left(a^1,\eta\right)\Upsilon^{22}\left(\eta\right). \tag{48}$$

Then V_1 is given by $(46)_1$ and V_3 by $(46)_3$, so that the propagation of the singularity along the characteristic $y^1 = a^1$ is completely determined at the leading order.

3.2. Second Example of Loading

We now consider another loading which is less singular than the previous one. In order to make comparisons with Section 3.1, we keep the same surface $\mathcal{S}$ and domain Ω. The loading is

$$\begin{aligned} \vec{f} = \Big(&0, 0, Y\left(y^2 - b^1\right)Y\left(y^1 - a^1\right) \\ &\times Y\left(b^1 - y^1\right)F\left(y^1, y^2\right)\Big), \end{aligned} \tag{49}$$

where F is a smooth function. Clearly, we have discontinuities of f_3 along the characteristics $y^1 = a^1$, $y^1 = b^1$ and $y^2 = b^1$ (see Fig. 2).

As regards the singularities along $y^2 = b^1$, the loading f_3 is singular in $Y(y^2 - b^1)$ instead of $\delta(y^2 - c^2)$ as in Section 3.1, so that the singularities of the unknowns are studied exactly in the same manner as in Section 3.1, but their order is lower by one. As a result, instead of (30) and (35), we have

$$\begin{cases} T^{11} \simeq \delta\left(y^2 - b^1\right)\mathcal{T}^{11}\left(y^1\right) + \cdots, \\ T^{22} \simeq Y\left(y^2 - b^1\right)\mathcal{T}^{22}\left(y^1\right) + \cdots, \\ T^{12} = -Y\left(y^2 - b^1\right)\Phi_1\left(y^1\right) \end{cases} \tag{50}$$

and

$$\begin{cases} u_1 \simeq \delta\left(y^2 - b^1\right)U_1\left(y^1\right) + \cdots, \\ u_2 \simeq Y\left(y^2 - b^1\right)U_2\left(y^1\right) + \cdots, \\ u_3 \simeq \delta'\left(y^2 - b^1\right)U_3\left(y^1\right) + \cdots, \end{cases} \tag{51}$$

respectively, where the functions $\mathcal{T}^{11}, \dots, U_3$ can be determined in much the same way as in Section 3.1.

As for the singularities along $y^1 = a^1$ (resp. $y^1 = b^1$), the loading is singular in $Y(y^1 - a^1)$ (resp. $Y(b^1 - y^1)$), i.e. of the same order as in Section 3.1.2, so that nothing is changed in formulae (41) and (45), where $\Upsilon^{11}, \dots, V_3$ can be determined as in that section.

Fig. 2 shows the order of the singularity of u_3 along the above-mentioned characteristics.

Remark 5. For the present loading, where f_3 is distributed and does not vanish on a part of the characteristic boundary $y^2 = 0$, in addition to the previous singularities, there is a strong boundary layer along $y^2 = 0$ enjoying propagation properties (Sanchez Palencia, 2001) (see also an analogous situation for a model problem in (Karamian *et al.*, 2000)).

4. Numerical Experiments

Numerical experiments are concerned with u^ε for $\varepsilon > 0$. As we mentioned in Section 1, when $\vec{f} \notin V'_m$, to our knowledge there is no proof of the convergence of u^ε as $\varepsilon \searrow 0$ in the general case. Nevertheless, a proof in appropriate topologies after a re-scaling was given for a model problem in (Sanchez Palencia, 2000). Of course, as we shall see in the sequel, there is "numerical evidence" of such convergence. Clearly, for $\varepsilon > 0$ the singularities become internal layers with thickness $\eta(\varepsilon) \searrow 0$. We must emphasize that such numerical computations are very tricky since the finite element approximation $u_h^\varepsilon \to u^\varepsilon$ is not uniform with respect to ε with values in V_m or in any smaller space (Gérard and Sanchez Palencia, 2000, Prop. 4.1). Consequently, the smaller ε is, the smaller h must be taken to have a good approximation. This peculiarity generates a variety of difficulties when computing thin shells (Chapelle and Bathe, 1998; Karamian, 1998b; 1999; Sanchez-Hubert and Sanchez Palencia, 1998). Some of these difficulties are linked to the presence of boundary layers and the corresponding local locking phenomena (Pitkaranta *et al.* (to appear); Sanchez-Hubert and Sanchez Palencia, 2001a; 2001b).

Let us recall some elementary properties of distributions of $\mathcal{D}'(\mathbb{R})$, which will be useful for understanding the numerical experiments and, more precisely, the sections on the internal layers. It is classical that the Dirac mass is the limit of a sequence of functions

$$\frac{1}{\eta}\varphi\left(\frac{x}{\eta}\right) \to \delta(x) \quad \text{as} \quad \eta \to 0$$

provided that

$$\int_{\text{support}} \varphi(x)\,\mathrm{d}x = 1.$$

More generally (Sanchez-Hubert and Sanchez Palencia, 1989, Sec. VI.14), a sequence of functions $\varphi^\eta(x) = \varphi(x/\eta)$ can be expanded in the form

$$\varphi^\eta(x) \simeq \eta m^0(\varphi)\,\delta(x) - \frac{\eta^2}{2!}m^1(\varphi)\,\delta'(x) + \frac{\eta^3}{3!}m^2(\varphi)\,\delta''(x) + \cdots,$$

where the coefficients are the moments of φ:

$$m^k(\varphi) = \int_{\text{support}} x^k \varphi(x)\,\mathrm{d}x.$$

Consequently, if φ is such that

$$m^k(\varphi) = 0 \quad \text{for } k = 0, \dots, p, \tag{52}$$

then

$$\frac{(-1)^p\,p!}{\eta^{p+1}m^p(\varphi)}\varphi^\eta(x) \to \delta^{(p)}(x). \tag{53}$$

In this section, we present some numerical experiments concerning the cases considered in Section 3 for two different cases of loading. The numerical computations are implemented with reduced Hermite finite elements that are used for the normal displacement u_3, as well as for the tangential displacement (u_1, u_2). The numerical integration of the rigidity matrices needs six Gauss points.

The meshes for the domain Ω are generated by using the Modulef code. The domain is covered with right-angled triangles such that the sides opposite the hypotenuse of each triangle are parallel to the y^1 and y^2 coordinates. This allows us to perform uniformly the mesh refinement by respecting the asymptotic curves.

The surface is defined by the mapping (8) with

$$\vec{r}\left(y^1, y^2\right) = \left(y^1, y^2, y^1 y^2\right),$$

so that the surface is a hyperbolic paraboloid satisfying all the required hypotheses.

The material is isotropic and homogeneous, with Young's modulus 28500 Nm^{-2} and Poisson's ratio 0.4. The thickness is equal to 10^{-4}.

In both cases, the numerical experiment involves 14400 triangles, 7381 nodes and 66429 degrees of freedom.

4.1. First Example of Loading

In the case of Section 3.1, we take $L = 4$, $a^1 = c^2 = 1$ and $b^2 = 2$ (Fig. 1) and $F(y^1) = 1$. Below we give and explain the behavior of u_3^ε in different sections.

Figure 3 shows u_3^ε in the section $y^1 = 0.5$, i.e. in the region $(0 < y^1 < a^1 = 1)$ on the left of the loading. We observe that this function is nearly vanishing except

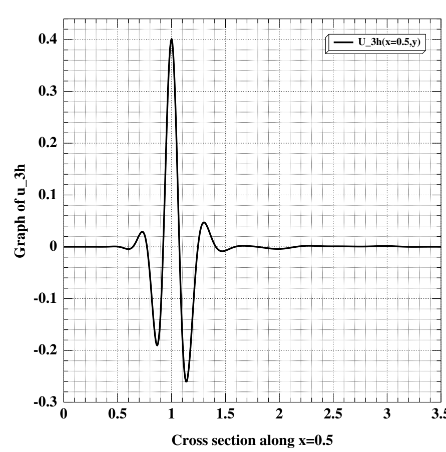

Fig. 3. The first example of loading, Sec. 3.1. The graph of u_3 for $y^1 = 1.5$ manifesting a propagated δ''-like singularity at $y^2 = 1$.

in the neighbourhood of $y^2 = c^2 = 1$, where it manifests a behaviour analogous to (53) with $p = 2$. Indeed, the moments m^0 and m^1 with respect to $x = y^2 - 1$ are clearly small and $m_2 \neq 0$. This perfectly agrees with the structure of the singularity in δ'' of u_3 in (35). Of course, as the section is on the left of the loading, the singularity is propagated in the sense of Remark 4.

Figure 4 shows u_3^ε in the section $y^1 = 1.5$, which cuts the support of the loading. The behaviour is exactly the same as in Fig. 3 but quantitatively larger.

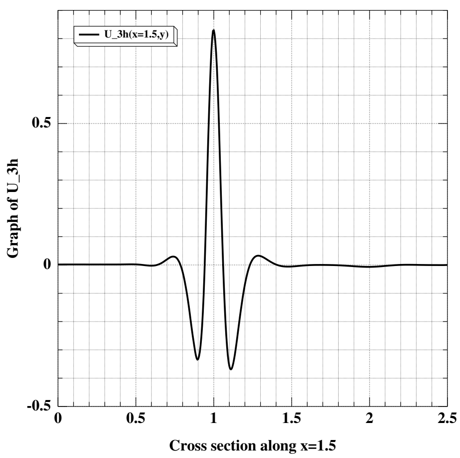

Fig. 4. The first example of loading, Sec. 3.1. The graph of u_3 for $y^1 = 1.5$ manifesting a non-propagated singularity at $y^2 = 1$.

Figure 5 shows u_3^ε in the section $y^2 = 0.5$, which cuts the characteristics $y^1 = a^1 = 1$ and $y^1 = b^1 = 2$ bearing the propagated singularities in δ', cf. (45). We observe that the function manifests in the neighbourhoods of $y^1 = 1$ and $y^1 = 2$ a behaviour analogous to (53) with $p = 1$. Indeed, the moment m^0 is clearly small and $m^1 \neq 0$.

Remark 6. Figure 5 also shows a δ' singularity in the vicinity of $y^1 = 3$. According to Fig. 1, with $c^2 = 1$, this corresponds to the section of the characteristic $y^1 = 3$, which bears the "pseudo-reflected" singularity of that along $y^2 = 1$ (Karamian, 1998b). Indeed, the δ''-singularity along $y^2 = 1$ intersects the non-characteristic boundary AB at the point $(3, 1)$ so that a singularity of the order lower by one, i.e. in δ', appears along $y^1 = 3$.

4.2. Second Example of Loading

In the case of Section 3.2, we take $l = 4$, $a^1 = 1$ and $b^2 = 2$ (Fig. 2) and $F(y^1, y^2) = 1$.

Figure 6 shows u_3^ε in the section $y^1 = 0.5$, i.e. in the region $\big(0 < y^1 < a^1 = 1\big)$ on the left of the loading.

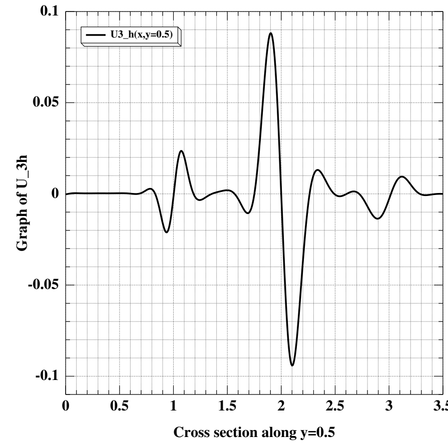

Fig. 5. The first example of loading, Sec. 3.1. The graph of u_3 for $y^2 = 0.5$ manifesting propagated δ'-like singularities at $y^1 = 1$ nad $y^1 = 2$.

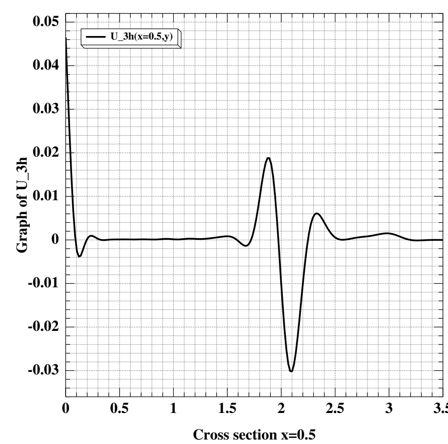

Fig. 6. The second example of loading, Sec. 3.2. The graph u_3 for $y^1 = 0.5$ manifesting a propagated δ'-like singularity at $y^2 = 2$ and a propagated boundary layer at $y^2 = 0$.

As has been explained in Section 3.2, the singularity along $y^2 = 2$ is in δ' for u_3. Its section by $y^1 = 0.5$ clearly appears in the figure, which also shows in the vicinity of $y^1 = 0$ the boundary layer mentioned in Remark 5. Both singularities are propagated from the support of $\vec{f}$.

Figure 7 shows the section $y^2 = 0.5$ and manifests δ' singularities at $y^1 = 1$ and $y^1 = 2$. This perfectly agrees with the description given in Section 3.2. The graph is analogous to that of Fig. 5 except for the pseudo-reflected singularities along $y^1 = 3$, which do not exist in the present case (cf. Remark 5). It should be noticed that the singularities in Fig. 5 are propagated, whereas those in Fig. 7 are not. Nevertheless, the shapes are closely similar. The fact that there is a loading between $y^1 = 1$ and $y^1 = 2$ in Fig. 7 is not relevant. Only the discontinuities at the extremities of its support yield significant singularities.

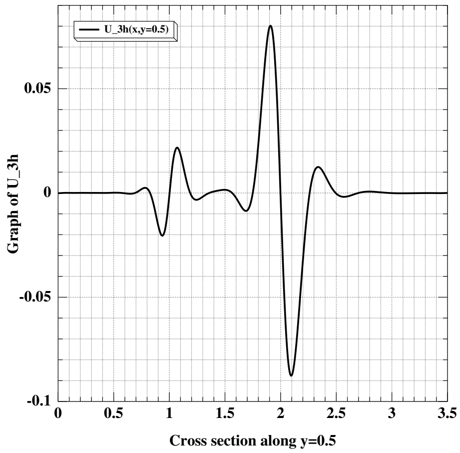

Fig. 7. The second example of loading, Sec. 3.2. The graph of u_3 for $y^2 = 0.5$ manifesting two propagated δ'-like singularities at $y^1 = 1$ and $y^1 = 2$.

References

Bernadou M. (1994): *Méthodes d'Éléments Finis pour les Problèmes de Coques Minces*. — Paris: Masson.

Chapelle D. and Bathe K.J. (1998): *Fundamental considerations for the finite element analysis of shell structures*. — Comp. Struct., Vol. 66, No. 1, pp. 19–36.

Ciarlet P.G. (2000): *Mathematical Elasticity, Vol. III, Theory of Shells*. — Amsterdam: North Holland.

Egorov Yu.V. and Shubin M.A. (1992): *Linear partial differential equations. Foundations of the classical theory*, In: Encyclopaedia of Mathematical Sciences, Vol. 30 (Part. Diff. Eqs. I). – New York: Springer, pp. 345–375.

Gérard P. (1988): *Solutions conormales analytiques d'équations hyperboliques non linéaires*. — Comm. Part. Diff. Eqns., Vol. 13, No. 3, pp. 345–375.

Gérard P. and Sanchez Palencia É. (2000): *Sensitivity phenomena for certain thin elastic shells with edges*. — Math. Meth. Appl. Sci., Vol.23, No. 4, pp. 379–399.

Goldenveizer A. L. (1962): *Theory of Thin Elastic Shells*. — New York: Pergamon.

Karamian P., Sanchez-Hubert J. and Sanchez Palencia É. (2000): *A model problem for boundary layers of thin elastic shells*. — Math. Modell. Num. Anal., Vol. 34, No. 1, pp. 1–30.

Karamian P. (1998a): *Nouveaux résultats numériques concernant les coques minces hyperboliques inhibées: Cas du paraboloïde hyperbolique*. — Compt. Rend. Acad. Sci., Paris, Série IIb, Vol. 326, No. 11, pp. 755–760.

Karamian P. (1998b): *Réflexion des singularités dans les coques hyperboliques inhibées*. — Compt. Rend. Acad. Sci., Paris, Série IIb, Vol. 326, No. 1, pp. 609–614.

Karamian P. (1999) *Coques élastiques minces hyperboliques inhibées: calcul du problème limite par éléments finis et non reflexion des singularités*. — Ph. D. thesis, Université de Caen.

Karamian P. and Sanchez-Hubert J. (2002): *Boundary layers in thin elastic shells with developable middle surface*. — Euro. J. Mech. /A solids, Vol. 21, No. 1, pp. 13–47.

Leguillon D., Sanchez-Hubert J. and Sanchez Palencia É. (1999): *Model problem of singular perturbation without limit in the space of finite energy and its computation*. — C. R. Acad. Sci. Paris, Série IIb, Vol. 327, No. 5, pp. 485–492.

Lions J.L. (1973): *Perturbations Singulières dans les Problèmes aux Limites et en Contrôle Optimal*. — Berlin: Springer.

Pitkaranta J., Matache A.M. and Schwab C. (1998): *Fourier mode analysis of layers in shallow shell deformation*. — Res. rep., No. 98–18, seminar für Angewandte Mathematik, Technische Hochschule Zürich.

Sanchez-Hubert J. and Sanchez Palencia É. (1989), *Vibration and Coupling of Continuous Systems. Asymptotic Methods*. — Berlin: Springer.

Sanchez-Hubert J. and Sanchez Palencia É. (1998): *Pathological phenomena in computation of thin elastic shells*. — Trans. Can. Mech. Eng., Vol. 22, No. 4B, pp. 435–446.

Sanchez-Hubert J. and Sanchez Palencia É. (2001a): *Singular perturbations with non-smooth limit and finite element approximation of layers for model problems of shells*, In: Partial Differential Equations in Multistructures (F. Ali Mehmeti, J. von Below and S. Nicaise, Eds.). — New York: Dekker.

Sanchez-Hubert J. and Sanchez Palencia É. (2001b): *Anisotropic finite element estimates and local locking for shells: parabolic case*. — Compt. Rend. Acad. Sci., Paris, Série IIb, Vol. 329, No. 2, pp.153–159.

Sanchez-Hubert J. and Sanchez Palencia É (1997): *Coques Élastiques Minces. Propriétés Asymptotiques*. — Paris: Masson.

Sanchez Palencia É. (2000): *On a singular perturbation going out of the energy space*. — J. Math. Pures Appl., Vol. 79, No. 8, pp. 591–602.

Sanchez Palencia É. (2001) *New cases of propagation of singularities along characteristic boundaries for model problems of shell theory*. — Compt. Rend. Acad. Sci., Paris, Série IIb, Vol. 329, No. 5, pp. 315–321.