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ON GRANULAR DERIVATIVES AND THE SOLUTION OF A GRANULAR INITIAL VALUE PROBLEM

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Perceptions about function changes are represented by rules like "*If* X *is SMALL then* Y *is QUICKLY INCREASING*." The consequent part of a rule describes a granule of directions of the function change when X is increasing on the fuzzy interval given in the antecedent part of the rule. Each rule defines a granular differential and a rule base defines a granular derivative. A reconstruction of a fuzzy function given by the granular derivative and the initial value given by the rule is similar to Euler's piecewise linear solution of an initial value problem. The solution method is based on a granulation of the directions of the function change, on an extension of the initial value in directions and on a propagation of fuzzy constraints given in antecedent parts of rules on possible function values. The proposed method is illustrated with an example.

Keywords: fuzzy differential, fuzzy granule, initial value problem, cylindrical extension

1. Introduction

In spite of the great success of crisp mathematics in the description and modeling of quantitative processes, the natural language has often been used for these purposes till now. The uncertainty in the understanding of described processes, the complexity of processes and the absence of resources for a detailed description are only some of the reasons behind using such rough and qualitative tool as the language, instead of exact and fine mathematical methods developed during the last centuries. Besides, in many real tasks it is sufficient to have a qualitative description of a system and a qualitative solution instead of some crisp mathematical result. In such situations the methodology of computing with words tolerant for imprecision to achieve tractability, robustness, a low solution cost and better rapport with reality may be considered as an alternative or additional tool with respect to traditional mathematical methods of modeling (Zadeh, 1997; 1999).

Computing with words is based on a translation of propositions expressed in a natural language into propositions expressed as a generalized constraint, and a fuzzy graph constraint is often used for these purposes (Zadeh, 1999). The most important step in the explicitation of generalized constraints is a fuzzy information granulation which involves a decomposition of the whole into parts such that the resulting granules are clumps of physical or mental objects drawn together by indistinguishability, similarity, proximity or functionality (Zadeh, 1997). This approach was used in (Batyrshin and Panova, 2001), where the new type of rules describing the shapes of dependencies between variables was introduced and the methods of representation of such rules by granular directions were discussed. The rules are often represented as follows:

$$\textit{If } X \textit{ is } A \textit{ then } Y \textit{ is } B, \tag{1}$$

where X and Y are variables and A, B are constraining fuzzy relations. The following are examples of rules discussed in (Batyrshin and Panova, 2001):

R_1 : *If TEMPERATURE is LOW then DENSITY is SLOWLY INCREASING*, (2)

R_2 : *If TEMPERATURE is HIGH then DENSITY is QUICKLY DECREASING*. (3)

The rules (2) and (3) are considered as linguistic expressions of dependencies between variables $Y = DENSITY$ and $X = TEMPERATURE$, such that Y is a *SLOWLY INCREASING* function of X on the fuzzy interval *LOW* and Y is a *QUICKLY DECREASING* function of X on the fuzzy interval *HIGH*. In this paper the set of rules such as (2) and (3) is translated into rule-based derivatives.

Differential equations play an important role in mathematical modeling. But often the values of the variables used in the problem considered are uncertain. Moreover, the functional dependencies between variables may be unknown. In the first case the model of the process may be based on fuzzy differential equations, i.e. on differential equations with fuzzy parameters (Ma *et al.*, 1999; Nieto, 1999; Park and Han, 2000; Song and Wu, 2000; Vorobiev and Seikkala, 2002). In the second case the model of the process may be based on a qualitative description which uses the signs of derivatives instead of the derivatives or, equivalently, the labels "increasing", "steady" and "decreasing" (De Kleer and Brawn 1984, Forbus 1984, Kuipers 1984). If the first approach requires a crisp description of quantitative dependencies between variables, the second approach uses very poor information about the dependencies. Fuzzy differentiation based on the extension principle was considered by Dubois and Prade (1982).

The rule-based approach to representation of derivatives considered here occupies an intermediate position between the two approaches considered above. The problem of the reconstruction of a function based on the set of rules considered and on the initial value given by a rule such as "If X is *APPROXIMATELY* 5 then Y is *APPROXIMATELY* 10" is considered here as a granular initial-value problem. The method of solving the problem discussed in this paper may be considered as a granular generalization of Euler's method of solving an initial value problem for an ordinary differential equation.

In Section 2, we translate the consequent parts of rules (2) and (3) into linguistic values of derivatives. These values are also considered as evaluations of slopes of the tangent line to the curve of the function. The methods of the fuzzy granulation of such slopes are discussed and granular differentials defined by these slopes are considered. The solution of the initial-value problem based on the examined type of rules is discussed in Section 3. This procedure is based on the reconstruction of a function from rule to rule starting from an initial value similar to Euler's method. The procedures considered are illustrated with an example. In conclusions, we discuss possible applications and extensions of the proposed approach to the modeling of complex processes.

2. Granular Differentials

The linguistic label *SLOWLY INCREASING* in the consequent part of rule (2) may be interpreted as a linguistic evaluation of the speed of the change of the variable $Y = DENSITY$ when the variable $X = TEMPERATURE$ is increasing within the fuzzy interval *LOW*. Since the speed of the function change is related to the derivative of the function, the consequent part of this rule may be also considered as a linguistic evaluation of the derivative $\mathrm{d}Y/\mathrm{d}X$ on this interval. In terms of derivatives the rules (2) and (3) may be translated in the following form:

$$R_1 : \textit{If } X \textit{ is LOW then } \mathrm{d}Y/\mathrm{d}X \textit{ is POSITIVE SMALL}, \tag{4}$$

$$R_2 : \textit{If } X \textit{ is HIGH then } \mathrm{d}Y/\mathrm{d}X \textit{ is NEGATIVE LARGE}. \tag{5}$$

Since the value of the derivative is equal to the slope of the tangent line to the curve of a function, the linguistic labels in the consequent parts of rules may be considered also as linguistic evaluations of this slope or parameter p in the equation of the tangent line $y = px + q$. A granular direction of the function change defined by the tangent will be represented by a fuzzy clump of directions. From another point of view, the granule of directions defines fuzzy sets of differential values $\mathrm{d}Y$ corresponding to given crisp values of increment Δx as $\mathrm{d}Y = P\Delta x$, where P is a granular slope value defined by a rule. We will suppose that the range of crisp values of increment Δx (or differential $\mathrm{d}x$) is defined by the antecedent part of the corresponding rule. As a result, the granular differential $\mathrm{d}Y$ may be considered as a fuzzy function of the crisp argument Δx. For example, the rule (5) will define a fuzzy differential as a fuzzy function $\mathrm{d}Y = P\Delta x$, where P is a fuzzy set corresponding to the linguistic term *NEGATIVE LARGE* and Δx takes values in the fuzzy interval defined by the term *HIGH*.

For explicitation of rules it is necessary to define linguistic scales for linguistic variables used in the rules, to define a granulation of possible slope values and to establish a correspondence between the grades of scales and slope values.

The explicitation of consequent parts of rules can be based on perceptions about the graphical representation of dependencies between linguistic variables (Batyrshin and Panova, 2001). Such perceptions may arise as a result of a visual analysis of graphics representing the dependencies between the variables, and may denote the directions of the change of the variable Y with the change of the variable X. In this case, instead of the granulation of slope values, granulation of angles of the directions of function changes or granulation of arctangent of slopes may be used.

Suppose that the domain of slope values is equal to the interval $[-10, 10]$, and seven granules of slopes are defined by fuzzy sets with central modal values $p_i,\ i = 1, \ldots, 7$. The possible linguistic scales and centers of membership functions corresponding to linguistic grades of the scales are shown in Table 1. Each grade of the scale represents some fuzzy granule of directions that is a fuzzy clump of similar directions.

We consider two methods of construction of granular directions. The first method is called the *proportional*

Table 1. Linguistic scales of slope values.

l_i	Linguistic description of the speed of the function change	Linguistic value of the derivative (slope)	p_i
7	*QUICKLY INCREASING*	*POSITIVE LARGE*	9
6	*INCREASING*	*POSITIVE MIDDLE*	6
5	*SLOWLY INCREASING*	*POSITIVE SMALL*	3
4	*CONSTANT*	*ZERO*	0
3	*SLOWLY DECREASING*	*NEGATIVE SMALL*	-3
2	*DECREASING*	*NEGATIVE MIDDLE*	-6
1	*QUICKLY DECREASING*	*NEGATIVE LARGE*	-9

extension in direction. Suppose that P_i is a fuzzy slope value, e.g. a fuzzy set defined on the domain of slope values p. For each value $\Delta x > 0$ from the domain of increments $\mathrm{Dom}(\Delta x)$, the corresponding fuzzy set $\mathrm{d}Y_i$ of differential values $\mathrm{d}y$ associated with the direction l_i is defined by the extension principle of fuzzy logic from the equation $\mathrm{d}Y = P_i\Delta x$ as follows:

$$\mu_{\mathrm{d}Y_i}^{\mathrm{prop}}(\mathrm{d}y) = \mu_{P_i}(p)\,, \tag{6}$$

where $p = \mathrm{d}y/\Delta x$. The corresponding fuzzy relation is defined as follows:

$$\mu_{D_i}^{\mathrm{prop}}(\Delta x,\,\mathrm{d}y) = \mu_{P_i}\left(\frac{\mathrm{d}y}{\Delta x}\right). \tag{7}$$

If fuzzy sets are defined by generalized bell membership functions (GBMF) (Jang *et al*., 1997), then from (6) we obtain the following definition of the granular differential:

$$\mu_{\mathrm{d}Y_i}^{\mathrm{prop}}(\mathrm{d}y) = \frac{1}{1+\left|\frac{p-p_i}{a_i}\right|^{2b_i}}, \tag{8}$$

where a_i is the width of the fuzzy set on the level 0.5 and b_i is the steepness of the membership function. Examples of fuzzy clumps of proportional extensions of directions based on GBMF and trapezoidal membership functions are shown in Figs. 1(a), (b). The parameters (a, b, c, d) of the trapezoidal membership function (Jang *et al.*, 1997) are defined by means of the central slope values p_i as follows: $a = p_i - w_1,\ \ b = p_i - w_2,\ \ c = p_i + w_2,\ \ d = p_i + w_1$, where $w_1 > w_2 > 0$.

The corresponding fuzzy relations are considered as granular differentials which define for a given value of increment Δx a fuzzy set of differential values $\mathrm{d}Y$. Such a fuzzy relation may be considered as an extending fuzzy linear function representing granular differential values. A granular differential obtained by (6) will also be called a *proportional differential*. Since $\Delta x > 0$, the fuzzy set of differentials for increment $\Delta x = 0$ is not defined. Nevertheless, we can define fuzzy sets D_{i0} at the point $\Delta x = 0$ as singletons, such that $D_{i0}(\mathrm{d}y) = 1$ for $\mathrm{d}y = 0$ and $D_{i0}(\mathrm{d}y) = 0$ for all other values of $\mathrm{d}y$. These fuzzy sets D_{i0} defined for $\Delta x = 0$ will be called starting sets for proportional extensions of the direction l_i.

The "width" of proportional differentials $\mathrm{d}Y$ is an extending value with the increasing of the increment value Δx. If the extending "width" of the fuzzy differential $\mathrm{d}Y$ is not desirable, then we can use a *cylindrical extension in direction* (Zadeh, 1966; 1997) and, correspondingly, the *cylindrical differential*:

$$\mu_{D_i}^{\mathrm{cyl}}(\Delta x,\,\mathrm{d}y) = \mu_{\mathrm{d}Y_i}(\mathrm{d}y)\,, \tag{9}$$

where $\mathrm{d}y = p\Delta x$ (for all $\Delta x > 0$) and $\mathrm{d}Y_i$ is a given fuzzy set of differential values in the direction l_i. For example, the cylindrical extension of generalized bell membership functions for each value $\Delta x > 0$ will be defined as

$$\mu_{\mathrm{d}Y_i}^{\mathrm{cyl}}(\mathrm{d}y) = \frac{1}{1+\left|\frac{\mathrm{d}y-\mathrm{d}y_i}{a_i}\right|^{2b_i}}, \tag{10}$$

where $\mathrm{d}y_i = p_i\Delta x$. The fuzzy value of the cylindrical differential will have a constant cross-section. Examples of cylindrical differentials constructed by means of GBMF and trapezoidal membership functions are shown in Figs. 1(c), (d).

For $\Delta x = 0$ we define D_{i0}^{cyl} by (9) with $\mathrm{d}y_i = 0$, which will be called a starting set for the cylindrical extension of the direction l_i.

3. Solution of the Granular Initial-Value Problem

The total set of rules with granular derivatives in the consequent parts of rules may be considered as a granular description of the derivative $\mathrm{d}Y/\mathrm{d}X = F(X)$ of a function Y piecewise defined on the domain of the variable X. Each rule defines some piece of the derivative on the fuzzy interval corresponding to the value of X in the antecedent part of a rule. The use of linguistic values of X in the antecedent parts of rules implies that the set of terms of the linguistic variable X is defined (Zadeh, 1975). This set of terms can include the labels *VERY SMALL, SMALL, MIDDLE, LARGE, VERY LARGE, APPROXIMATELY* N*, BETWEEN* N *AND* M, *GREATER THAN* N, etc., where N and M are some real values or fuzzy numbers. The meaning of these terms may be explicitated by the definition of the corresponding fuzzy sets defined on X.

Generally, for the same rule base there may exist several different explicitations of linguistic values of X dependent on some parameter or context. The role of such a parameter or context may be played by another variable. The explicitation of granular slopes may also depend on the value of this parameter. In this case the rule base describes the parametric family of granular derivatives with

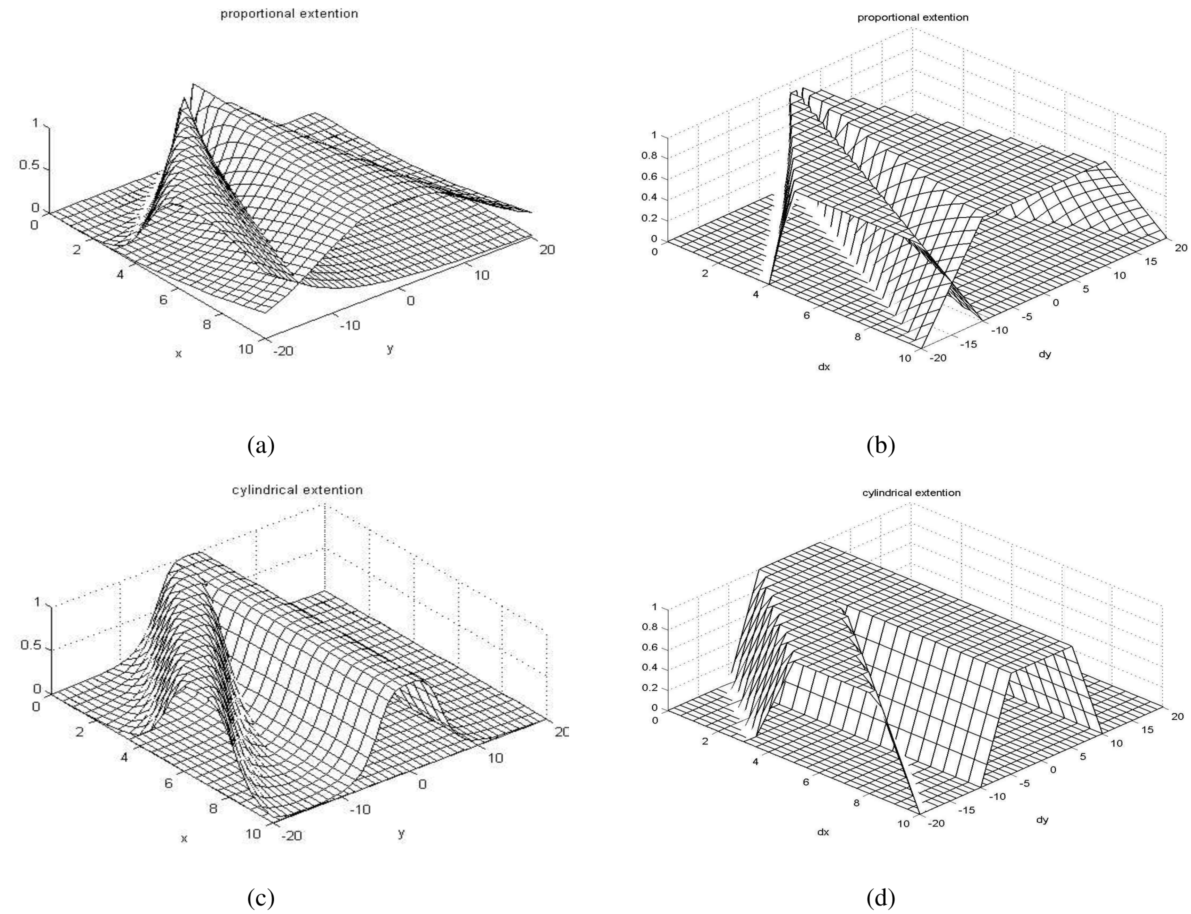

Fig. 1. Proportional ((a) and (b)) and cylindrical ((c) and (d)) differentials in directions 3 (“*SLOWLY DECREASING*”) and 4 (“*CONSTANT*”) based on generalized bell membership functions ((a) and (c)) and trapezoidal membership functions ((b) and (d)).

explicitation dependent on the value of this parameter. For example, the rules (4) and (5) may describe the derivative $\mathrm{d}\mathit{DENSITY}/\mathrm{d}\mathit{TEMPERATURE}$ for different values of the third parameter Z = *PRESSURE*, but the explicitation of this derivative may be different and will depend on the explicitation of linguistic values of X and the explicitation of slopes defined by the value of parameter Z.

Let us consider the way of solving the granular ordinary differential equation

$$\mathrm{d}Y/\mathrm{d}X = F(X) \tag{11}$$

satisfying the initial condition

$$\text{“}\mathit{If}\ X\ \mathit{is}\ X_0\ \mathit{then}\ Y\ \mathit{is}\ Y_0\text{”}, \tag{12}$$

where X_0 and Y_0 are fuzzy sets defined on X and Y, respectively, and (11) is given by the rule base

$$R_i : \mathit{If}\ X\ \mathit{is}\ A_i\ \mathit{then}\ \mathrm{d}Y/\mathrm{d}X\ \mathit{is}\ P_i,\quad i = 1, \dots, m, \tag{13}$$

with piecewise description of the derivative of Y. Suppose that all A_i’s in (13) are normal and convex fuzzy sets defined on the domain $\mathrm{Dom}(X)$ of X, and the set of intervals A_i, $i = 1, \dots, m$ defines some fuzzy partition of $\mathrm{Dom}(X)$, i.e., the following conditions are fulfilled: $\sup_x (A_j \cap A_k)(x) = s_1$, $\inf_{x\in X}((\cup_{i=1}^m A_i)(x)) = s_2$, where s_1 and s_2 belong to $[0, 1]$ such that $s_1 < 1$ and $s_2 > 0$. Since the cores of the fuzzy intervals in a fuzzy partition do not intersect, these fuzzy intervals may be linearly ordered in such a way that $A_j < A_k$ iff $x_j < x_k$ for some points x_j and x_k from the cores of A_j and A_k, respectively. We will suppose that this ordering coincides with the numbering of rules such that $A_i < A_{i+1}$ for all $i = 1, \dots, m$.

The problem of solving the granular differential equation (11) with initial condition (12) will be called a granular initial-value problem.

With no loss of generality we will suppose that the intersection of the initial value X_0 with the fuzzy interval A_1 from the first rule is a normal fuzzy set. The solution of the granular initial-value problem based on cylindrical extension in directions defined by the slope values will include the following steps:

1. Find the core $[x_{11}, x_{12}]$ of the fuzzy set $X_0 \cap A_1$, $k = 1$.
2. Select a starting point x_0 in $[x_{11}, x_{12}]$, e.g., as follows: $x_0 = (x_{11} + x_{12})/2$.
3. Construct a fuzzy set Y_0 in x_0.
4. Choose a fuzzy set Y_0 as a starting fuzzy set D_{i0} for the direction l_i determined by the slope P_1.
5. Construct a granular extension D_1 in the direction l_i based on the initial fuzzy set D_{i0}. Set $k = 2$.
6. Select a starting point x_{k-1} in the interval $[x_{k1}, x_{k2}]$ maximizing the intersection of fuzzy sets A_{k-1} and A_k.
7. Cut the granular extension in the direction D_{k-1} at the point x_{k-1}. The result will give a fuzzy set $D_{Yk-1}(y) = D_{k-1}(y, x_{k-1})$.
8. Construct a granular extension D_k based on a fuzzy set D_{Yk-1} and on a slope value defined by P_k. Set $k = k + 1$.
9. Repeat Steps 6–8 while $k \leq m$.
10. Construct cylindrical extensions of constraints A_k, $k = 1, \ldots, m$ along the Y axis, i.e. $C_Y(A_k)(x, y) = A_k(x)$.
11. Propagate the cylindrical extensions of constraints A_k, $k = 1, \ldots, m$ on the corresponding granular directions D_k.
12. Aggregate in overall fuzzy graph the constrained directions obtained in Step 11.

As a result of the above procedure, a fuzzy relation R on $X \times Y$ which will give a solution $Y_R(X)$ to the granular initial-value problem will be constructed. The calculation of the function value for a given fuzzy value X^* of the input variable X represented by a fuzzy set A^* can be performed as a result of the following steps:

13. Construct a cylindrical extension $C_Y(A^*)$ of A^* along the Y axis.
14. Calculate a granular solution $Y_R(X^*) = C_Y(A^*) \cap R$.
15. Find a projection $B^* = P_Y(Y_R(X*))$ on the Y axis.
16. Find a linguistic retranslation of the fuzzy set $Y(X^*) = B^*$.
17. Find a numerical solution $y^* = \text{Defuz}(Y(X^*))$ as a result of the defuzzification procedure.

The linguistic value of the function Y obtained as a result of the retranslation of the fuzzy set $Y(X^*) = B^*$ may be considered as a reply to the query "*What is the value of Y if X is A^*?*".

Let us discuss some steps of the procedure considered. If we use fuzzy intervals A_k with strict monotonic membership functions from both the sides of the cores, then in Step 6 each interval $[x_{k1}, x_{k2}]$ will contain only one point.

Steps 11 and 12 can be realized by several methods. A max-min aggregation of rules is based on the intersection of each granular direction with the corresponding cylindrical extension of the fuzzy constraint A_k in Step 11 and the aggregation of results obtained for each rule with the union operation in Step 12 as follows:

$$R = \bigcup_{k=1}^{m} \big(D_k \cap C_Y(A_k)\big).$$

For this method Steps 11 and 12 are reduced to

$$R(x, y) = \max_{k=1,\ldots,m} \big(\min\big(Dk(x, y), Ak(x)\big)\big) .$$

Another method is based on the weighting of granular directions by the corresponding membership values of the cylindrical extensions of fuzzy constraints A_k in Step 11 and on the averaging of results in Step 12:

$$R(x, y) = \frac{\sum_{k=1}^{m} (D_k(x, y) A_k(x))}{\sum_{k=1}^{m} A_k(x)}.$$

This method will be called a weighted-average aggregation. It gives a smoother overall graph than the first method and is illustrated in Fig. 2 with the example considered below. As it follows from the last formulas, for both the methods it is not necessary to calculate cylindrical extensions of constraints A_k.

The procedure of linguistic retranslation in Step 16 can be based on linguistic approximation procedures (Zadeh, 1975) and will not be discussed here.

If it is necessary, a defuzzification procedure in Step 17 can be applied. Different types of such procedures are described in (Jang *et al.*, 1997).

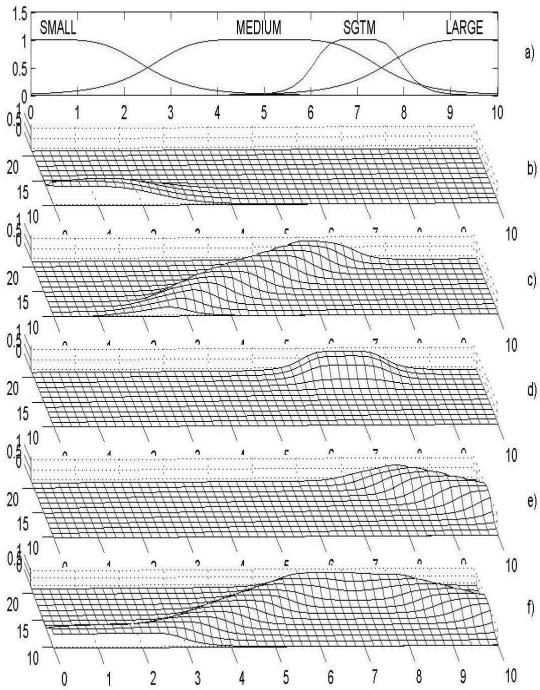

Fig. 2. Construction of the overall graph defined by the fuzzy rule set from example: (a) The fuzzy constraints on the variable X; (b)–(e) the propagation of constraints from X on the corresponding granular directions defined by fuzzy rules: (b) R_1; (c) R_2; (d) R_3; (e) R_4; (f) the overall fuzzy graph is obtained as weighted-average aggregation of the constrained directions.

4. Example

Consider the following rule base describing the dependency between the variables Y and X:

R_1 : *If* X *is SMALL then* Y *is SLOWLY INCREASING*,

R_2 : *If* X *is MIDDLE then* Y *is QUICKLY INCREASING*,

R_3 : *If* X *is SLIGHTLY GREATER THAN MIDDLE then* Y *is CONSTANT*,

R_4 : *If* X *is LARGE then* Y *is QUICKLY DECREASING*.

This rule base gives a context-insensitive initial data set (IDS). For the explicitation of this IDS we should define the corresponding fuzzy sets for the variable X and granular directions for the variable Y. Moreover, we should also define an initial fuzzy point (X_0, Y_0) that will define the starting point for the process of the reconstruction of the function Y. Suppose that the initial point is given by the rule

R_5 : *If* X *is APPROXIMATELY* 0 *then* Y *is APPROXIMATELY* 10,

with an appropriate definition of the fuzzy sets *APPROXIMATELY* 0 and *APPROXIMATELY* 10. Figure 2 illustrates the solution to the initial-value problem given by this example based on some explicitation of the membership functions used in the model. In all constructions generalized bell membership functions were used. In the rule R_5 the parameters c_i are equal to 0 and 10 for GBMF X_0 and Y_0, respectively. The point $(x_1, y_1) = (0, 10)$ is used as the starting point for construction of a granular direction in the rule R_1. The parameters of Y_0 are used as parameters of the cylindrical extension of granular directions. The weighted average method of aggregation was used. For each rule the corresponding granular directions weighted by fuzzy sets given in the antecedent parts of rules are shown in Figs. 2(b)–(e). The overall fuzzy graph is shown in Fig. 2(f).

The calculation of a reply to the query

Q : *What is the value of* Y *if* X *is VERY LARGE?*

is based on Steps 13–17 described above and illustrated in Fig. 3.

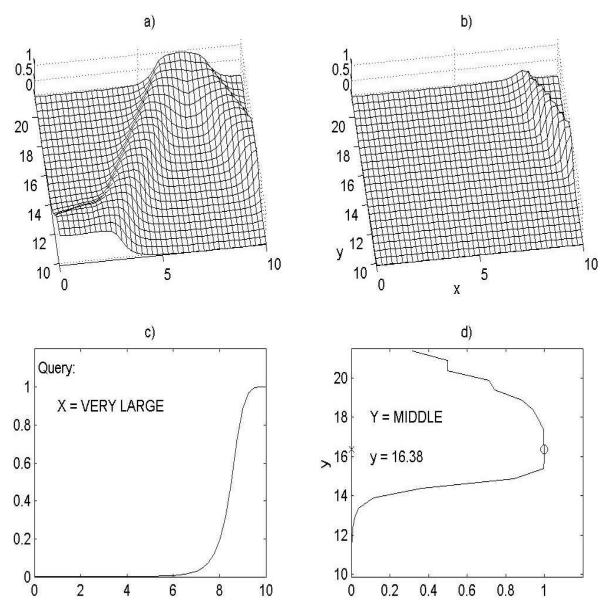

Fig. 3. Calculation of a reply to a query: (a) fuzzy graph defined by the fuzzy rule base; (c) constraint on X in a query: "X is *VERY LARGE*"; (b) result of intersection of a fuzzy graph with cylindrical extension of a constraint on X; (d) projection of the result onto the Y axis with possible retranslation "Y is *MIDDLE*" or with defuzzification result $y = 16.38$.

The retranslation of the resulting fuzzy set $Y(X^*)$ on Y in the linguistic form "Y *is MIDDLE*" was obtained as a result of linguistic approximation. The real value y from the fuzzy set $Y(X^*)$ was obtained as the mean of the modal values of $Y(X^*)$.

The answer to the query

$Q2$: *What is a maximal value of* Y*?*

is considered for a general case in (Zadeh, 1997). For our rule base, as the maximal value of Y the projection of the constrained granular direction corresponding to the rule R_3 (see Fig. 2(d)) on the Y axis can be used.

5. Conclusions

The methods of translating perceptions about function changes into rule based derivatives, and the methods of constructing granular derivatives and granular directions of function changes have been discussed. The problem of function reconstruction from a rule-based derivative and the initial value given by the rule is called the granular initial-value problem. The solution of this problem is based on a sequential rule by rule reconstruction of the function starting from a given initial value. The method can be considered as a generalization of Euler's method of piecewise linear solution to the crisp initial-value problem for ordinary differential equations. The method uses the basic procedures of computing with words introduced by Zadeh (1966; 1997; 1999), such as information granulation, extension principles, cylindrical extension along an axis and along a direction, projections etc. Additionally, new methods of aggregation of fuzzy rules were proposed.

When solving a granular initial-value problem, the initial data set (IDS) given by linguistic expressions about dependencies between variables is transformed into a terminal data set (TDS) which gives a linguistic reply to a query about the value of the function for a linguistically given value of the independent variable. This procedure of inference of the linguistic reply from propositions expressed in a natural language uses the steps of computing with words such as the explicitation of IDS into an initial constraint set (ICS) given by fuzzy relations, constraint propagation from ICS to a derived constraint set (DCS) which gives a fuzzy reply to the query, and a retranslation of DCS into TDS as a result of linguistic approximation (Zadeh, 1997).

This work may be considered as an initial step in the modeling and solution of granular differential equations based on the new types of rules. The approach considered can be extended to systems of differential equations, higher-order differential equations and partial differential equations. The proposed approach can be applied to the modeling and control of complex technological processes with several inputs and outputs when the description of dependencies between input and output parameters may be given in the form of rules considered in this paper. An example of fuzzy expert system modeling such a process with another type of rules describing similar dependencies between variables was considered in (Batyrshin *et al.*, 1994). It would be interesting to combine the proposed approach with the methods developed in qualitative reasoning about processes and systems based on the use of the signs of derivatives (De Kleer and Brawn, 1984; Forbus, 1984; Kuipers, 1984).

The rules containing some specific features of the shape of the output variable in consequent parts can be used for granular shape analysis. The granular models with the rules of type (2) and (3) can be also more suitable for representing the knowledge base invariant to the change of parameters of some problem area or insensitive to the context (Batyrshin and Fatkullina, 1995). For example, such models may describe the parametric family of the fuzzy graphs $Y(X)$ when the value of some variable Z is considered as a (perhaps hidden) parameter.

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References

Batyrshin I. and Fatkullina R. (1995): *Context-dependent fuzzy scales and context-free rules for dependent variables*. — Proc. 6-th Int. *Fuzzy Systems Association World Congress, IFSA'95*, Sao Paulo, Brasil, Vol. 1, pp. 89–91.

Batyrshin I. and Panova A. (2001): *On granular description of dependencies*. — Proc. 9-th Zittau Fuzzy Colloquium, Zittau, Germany, pp. 1–8.

Batyrshin I., Zakuanov R. and Bikushev G. (1994): *Expert system based on algebra of uncertainties with memory in process optimization*. — Proc. 1-st Int. Workshop *Fuzzy Logic and Intelligent Technologies in Nuclear Science*, Mol, Belgium, World Scientific, pp. 156–159.

De Kleer J. and Brawn J. (1984): *A qualitative physics based on confluences*. — Artif. Intell., Vol. 24, pp. 7–83.

Dubois D. and Prade H. (1982): *Towards fuzzy differential calculus: Part 3: Differentiation*. — Fuzzy Sets Syst., Vol. 8, pp. 225–233.

Forbus K.D. (1984): *Qualitative process theory*. — Artif. Intell., Vol. 24, pp. 85–168.

Jang J.S.R., Sun C.T. and Mizutani E. (1997): *Neuro-Fuzzy and Soft Computing. A Computational Approach to Learning and Machine Intelligence*. — New York: Prentice Hall.

Kuipers B. (1984): *Commonsense reasoning about causality: deriving behavior from structure*. — Artif. Intell., Vol. 24, pp. 169–203.

Ma M., Friedman M. and Kandel A. (1999): *Numerical solutions of fuzzy differential equations*. — Fuzzy Sets Syst., Vol. 105, pp. 133–138.

Nieto J.J. (1999): *The Cauchy problem for continuous fuzzy differential equations*. — Fuzzy Sets and Syst., Vol. 102, pp. 259–262.

Park J.K. and Han H.K. (2000): *Fuzzy differential equations*. — Fuzzy Sets Syst., Vol. 110, pp. 69–77.

Song S. and Wu C. (2000): *Existence and uniqueness of solutions to Cauchy problem of fuzzy differential equations*. — Fuzzy Sets Syst., Vol. 110, pp. 55–67.

Vorobiev D. and Seikkala S. (2002): *Towards the theory of fuzzy differential equations*. — Fuzzy Sets Syst., Vol. 125, pp. 231–237.

Zadeh L.A. (1966): *The shadows of fuzzy sets*. — Problems of Information Transmission, Vol. 2, pp. 37–44 (in Russian).

Zadeh L.A. (1975): *The concept of a linguistic variable and its application to approximate reasoning, Part I*. — Inf. Sci., Vol. 8, pp. 199–249; *Part II*. — Inf. Sci., Vol. 8, pp. 301–357; *Part III* — Inf. Sci., Vol. 9, pp. 43–80.

Zadeh L.A. (1997): *Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic*. — Fuzzy Sets Syst., Vol. 90, pp. 111–127.

Zadeh L.A. (1999): *From computing with numbers to computing with words – from manipulation of measurements to manipulation of perceptions*. — IEEE Trans. Circ. Syst. – 1: Fund. Th. Applic., Vol. 45, No. 1, pp. 105–119.