Int. J. Appl. Math. Comput. Sci., 2003, Vol. 13, No. 1, 29–44

STABILITY ANALYSIS OF SOLUTIONS TO AN OPTIMAL CONTROL PROBLEM ASSOCIATED WITH A GOURSAT-DARBOUX PROBLEM[†]

DARIUSZ IDCZAK*, MAREK MAJEWSKI*
STANISŁAW WALCZAK*

* University of Łódź, Faculty of Mathematics
ul. S. Banacha 22, 90–238 Łódź, Poland
e-mail: `{idczak, marmaj, stawal}@imul.math.uni.lodz.pl`

In the present paper, some results concerning the continuous dependence of optimal solutions and optimal values on data for an optimal control problem associated with a Goursat-Darboux problem and an integral cost functional are derived.

Keywords: Goursat-Darboux problem, optimal control, continuous dependence

1. Introduction

The question of the continuous dependence of solutions to an optimal control problem on data (the so-called stability analysis of solutions or well-posedness) is very important from the point of view of practical applications of theory. Indeed, if the problem is related to a physical phenomenon, its data can be considered only an arbitrarily close approximation to exact values. Consequently, if the solution does not depend continuously on the data, it is not actually determined.

Example 1. Let us consider the one-dimensional system with a scalar parameter $\omega \in [1/2, 1]$

$$\begin{cases} \dot{x}^1(t) = x^2(t), \\ \dot{x}^2(t) = u(t) - \omega^2 x^1(t), \\ u(t) \in [0,1] \end{cases}$$

with boundary conditions

$$x^1(0) = 0, \qquad x^1(\pi) = 0,$$

$$x^2(0) \text{ and } x^2(\pi) \text{ are free,}$$

and the cost functional

$$J_\omega(x,u) = \int_0^\pi x^1(t)\big(x^1(t) - 10^3\sqrt{2\pi}\big)\,\mathrm{d}t \to \inf.$$

In a way analogous to (Idczak, 1998) one can show that for any parameter $\omega \in [1/2, 1]$ the above optimal control problem possesses an optimal solution (x^*_ω, u^*_ω).

One can show that for any $\omega \in [1/2, 1)$

$$J_\omega(x^*_\omega, u^*_\omega) \geq -4 \times 10^3 \pi\sqrt{2\pi}.$$

If $\omega = 1$, then the control system has a solution x_ω (not unique) only for $u \equiv 0$. Moreover,

$$J_\omega(x^*_\omega, u^*_\omega) = -4 \times 10^6.$$

So, we see that as $\omega \to 1$, the optimal value has a jump, i.e. it is not continuous with respect to ω at the point $\omega = 1$. In this case, we say that the optimal control problem is ill posed. ♦

The stability analysis of solutions to finite-dimensional mathematical programming problems was investigated, e.g. in (Bank *et al.*, 1983; Fiacco, 1981; Levitin, 1975; Robinson, 1974). A survey of the results for the case of abstract Banach spaces is given in (Malanowski, 1993).

We study the continuous dependence of solutions and optimal values on data for the optimal control problem associated with the Goursat-Darboux problem

$$\begin{cases} \dfrac{\partial^2 w}{\partial x \partial y} = g\Big(x, y, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, u\Big), \\ \qquad (x,y) \in P = [0,1] \times [0,1] \text{ a.e.,} \\ w(x,0) = \varphi(x),\ w(0,y) = \psi(y),\ x, y \in [0,1], \\ I(w,u) = \displaystyle\int_0^1\!\!\int_0^1 G(x,y,w,u)\,\mathrm{d}x\,\mathrm{d}y \to \min, \\ u \in \mathcal{U}_M = \big\{u \in L^2(P);\ u(x,y) \in M, \\ \qquad (x,y) \in P \text{ a.e.}\big\} \end{cases} \tag{NH}$$

(in linear and nonlinear cases).

[†] Supported by grant 7 T11A 004 21 of the State Committee for Scientific Research, Poland.

Systems of this type (in control theory they are called two-dimensional (2D) continuous Fornasini-Marchesini systems) have been investigated by many authors (see, e.g. (Bergmann *et al.*, 1989; Pulvirenti and Santagati, 1975; Surryanarayama, 1973)). They can be applied to describe the absorption gas phenomenon (Idczak and Walczak, 1994; Tikhonov and Samarski, 1958).

Our aim is to obtain stability results analogous to those obtained in (Walczak, 2001) for an ordinary problem. The main tools used in the study of the stability question in the case of abstract Banach spaces are: the implicit function theorem for generalized equations (Robinson, 1980), the open mapping theorem for set-valued maps (Robinson, 1976), and composite optimization (Ioffe, 1994). Our proofs make no appeal to the above approaches.

First, we consider Problem (NH) in the case when the system is linear, autonomous, and the set M does not vary. The approach used here is based on the Cauchy formula for a solution to a linear autonomous system. Since, in the nonlinear case, we have no formula for solving system (NH), in this case we use a different method, based on the Gronwall lemma for functions of two variables and the continuity of the mapping $M \longmapsto \mathcal{U}_M$ (in the Hausdorff sense). Let us point out the fact that this method cannot be applied in the linear case.

2. Space of Solutions to the Goursat–Darboux Problem

By a function of an interval (Łojasiewicz, 1988) we mean a mapping F defined on the set of all closed intervals $[x_1, x_2] \times [y_1, y_2]$ contained in P, with values in $\mathbb{R}$. We say that F is additive if

$$F(Q \cup R) = F(Q) + F(R)$$

for any closed intervals $Q, R \subset P$ such that $Q \cup R$ is an interval contained in P and Int$Q \cap$ Int$R = \emptyset$.

Let a function $z : P \to \mathbb{R}$ of two variables be given. The function F_z of an interval given by

$$F_z\big([x_1, x_2] \times [y_1, y_2]\big) = z(x_2, y_2) - z(x_1, y_2) - z(x_2, y_1) + z(x_1, y_1)$$

for $[x_1, x_2] \times [y_1, y_2] \subset P$ is called the function of an interval associated with z.

We say that a function $z : P \to \mathbb{R}$ of two variables is absolutely continuous (Walczak, 1987) if $z(0, \cdot), z(\cdot, 0)$ are absolutely continuous functions on $[0, 1]$ and F_z is an absolutely continuous function of an interval.

It can be shown (Walczak, 1987) that $z : P \to \mathbb{R}$ is absolutely continuous if and only if there exist functions $l \in L^1(P, \mathbb{R}), l^1, l^2 \in L^1([0, 1], \mathbb{R})$ and a constant $c \in \mathbb{R}$ such that

$$z(x, y) = \int_0^x \int_0^y l(s, t)\, \mathrm{d}s\, \mathrm{d}t + \int_0^x l^1(s)\, \mathrm{d}s + \int_0^y l^2(t)\, \mathrm{d}t + c \tag{1}$$

for $(x, y) \in P$.

The above integral formula implies that the partial derivatives

$$\frac{\partial z}{\partial x}(x, y), \quad \frac{\partial z}{\partial y}(x, y), \quad \frac{\partial^2 z}{\partial x \partial y}(x, y)$$

exist a.e. on P and

$$\frac{\partial z}{\partial x}(x, y) = \int_0^y l(x, t)\, \mathrm{d}t + l^1(x),$$

$$\frac{\partial z}{\partial y}(x, y) = \int_0^x l(s, y)\, \mathrm{d}s + l^2(y),$$

$$\frac{\partial^2 z}{\partial x \partial y}(x, y) = l(x, y)$$

for $(x, y) \in P$ a.e. (Idczak, 1990).

Moreover, it is easy to see that a function $z : P \to \mathbb{R}$ is absolutely continuous and satisfies the conditions

$$z(0, y) = 0 \quad \text{for} \quad y \in [0, 1],$$
$$z(x, 0) = 0 \quad \text{for} \quad x \in [0, 1],$$

if and only if there exists a function $l \in L^1(P, \mathbb{R})$ such that

$$z(x, y) = \int_0^x \int_0^y l(s, t)\, \mathrm{d}s\, \mathrm{d}t \tag{2}$$

for $(x, y) \in P$.

By $AC^2(P, \mathbb{R})$ we denote the set of all functions $z : P \to \mathbb{R}$, each of them having the integral representation (1) with functions $l \in L^2(P, \mathbb{R}), l^1, l^2 \in L^2([0, 1], \mathbb{R})$.

By $AC_0^2(P, \mathbb{R})$ we denote the set of all functions $z : P \to \mathbb{R}$ each of them having the integral representation (2) with function $l \in L^2(P, \mathbb{R})$.

By $AC^2(P, \mathbb{R}^n)$ $(AC_0^2(P, \mathbb{R}^n))$ we denote the set of all vector-valued functions $z = (z^1, z^2, \ldots, z^n) : P \to \mathbb{R}^n$ such that each coordinate function $z^i : P \to \mathbb{R}$ belongs to $AC^2(P, \mathbb{R})$ $(AC_0^2(P, \mathbb{R}))$.

It is easy to see that $AC_0^2(P,\mathbb{R}^n)$ with the scalar product

$$\langle z,w\rangle_{AC_0^2(P,\mathbb{R}^n)} = \int_P \left\langle l\left(s,t\right), k\left(s,t\right)\right\rangle_{\mathbb{R}^n} \mathrm{d}s\,\mathrm{d}t,$$

where

$$z\left(x,y\right) = \int_0^x\int_0^y l\left(s,t\right)\mathrm{d}s\,\mathrm{d}t,$$
$$w\left(x,y\right) = \int_0^x\int_0^y k\left(s,t\right)\mathrm{d}s\,\mathrm{d}t$$

and $AC^2(P,\mathbb{R}^n)$ with the scalar product

$$\begin{aligned}\langle z,w\rangle_{AC^2(P,\mathbb{R}^n)} &= \int_P \left\langle l\left(s,t\right), k\left(s,t\right)\right\rangle_{\mathbb{R}^n} \mathrm{d}s\,\mathrm{d}t\\ &+ \int_0^1 \left\langle l^1\left(s\right), k^1\left(s\right)\right\rangle_{\mathbb{R}^n}\mathrm{d}s\\ &+ \int_0^1 \left\langle l^2\left(t\right), k^2\left(t\right)\right\rangle_{\mathbb{R}^n}\,\mathrm{d}t + \langle c,d\rangle_{\mathbb{R}^n},\end{aligned}$$

where

$$\begin{aligned}z(x,y) &= \int_0^x\int_0^y l(s,t)\,\mathrm{d}s\,\mathrm{d}t + \int_0^x l^1(s)\,\mathrm{d}s\\ &+ \int_0^y l^2(t)\,\mathrm{d}t + c,\end{aligned}$$

$$\begin{aligned}w(x,y) &= \int_0^x\int_0^y k(s,t)\,\mathrm{d}s\,\mathrm{d}t + \int_0^x k^1(s)\,\mathrm{d}s\\ &+ \int_0^y k^2(t)\,\mathrm{d}t + d\end{aligned}$$

are Hilbert spaces. In much the same way as in (Idczak, 1996) it can be proved that if $z_n \rightharpoonup z_0$ weakly in $AC_0^2\left(P,\mathbb{R}^n\right)$, then $z_n \rightrightarrows z_0$ uniformly on P.

In the sequel, we denote by $AC^2([0,1],\mathbb{R}^n)$ the standard space of all absolutely continuous functions of one variable $\varphi : [0,1] \to \mathbb{R}^n$ such that $\dot{\varphi} \in L^2([0,1],\mathbb{R}^n)$. The scalar product in $AC^2([0,1],\mathbb{R}^n)$ is given by

$$\langle\varphi,\psi\rangle_{AC^2([0,1],\mathbb{R}^n)} = \int_0^1 \left\langle l\left(s\right), k\left(s\right)\right\rangle_{\mathbb{R}^n}\mathrm{d}s + \langle c,d\rangle_{\mathbb{R}^n},$$

where

$$\varphi\left(x\right) = \int_0^x l\left(s\right)\mathrm{d}s + c,$$
$$\psi\left(x\right) = \int_0^x k\left(s\right)\mathrm{d}s + d.$$

3. Linear System

3.1. Problem Formulation and Basic Assumptions

Consider the family of optimal control problems

$$\begin{cases}\dfrac{\partial^2 v}{\partial x\partial y}\left(x,y\right) + A_1^k\dfrac{\partial v}{\partial x}\left(x,y\right) + A_2^k\dfrac{\partial v}{\partial y}\left(x,y\right)\\ \qquad + A^k v\left(x,y\right) = B^k u\left(x,y\right), \quad \left(x,y\right)\in P \text{ a.e.},\\ v\left(x,0\right) = \varphi^k\left(x\right),\ v\left(0,y\right) = \psi^k\left(y\right)\\ \qquad \text{for all} \quad x,y\in[0,1],\\ J^k\left(v,u\right) = \displaystyle\int_P G^k\left(x,y,v\left(x,y\right),u\left(x,y\right)\right)\mathrm{d}x\,\mathrm{d}y,\end{cases} \tag{L^k}$$

where $A_1^k, A_2^k, A^k \in \mathbb{R}^{n\times n}, B^k \in \mathbb{R}^{n\times m}, \varphi^k, \psi^k \in AC^2([0,1],\mathbb{R}^n)$ and $\varphi^k(0) = \psi^k(0) = c^k$ for $k = 0,1,2,\ldots$. Problem (L^k) is considered in the space $AC^2(P,\mathbb{R}^n)$ of solutions v and in the set $\mathcal{U}_M = \{u \in L^2(P,\mathbb{R}^m) : u(x,y) \in M \text{ a.e.}\}$ of controls u. The set M is a convex and compact subset of $\mathbb{R}^m$.

It is easy to see that, using the substitution

$$z\left(x,y\right) = v\left(x,y\right) - \varphi^k\left(x\right) - \psi^k\left(y\right) + c^k,$$

we get the equivalent problem

$$\begin{cases}\dfrac{\partial^2 z}{\partial x\partial y}\left(x,y\right) + A_1^k\dfrac{\partial z}{\partial x}\left(x,y\right) + A_2^k\dfrac{\partial z}{\partial y}\left(x,y\right)\\ \qquad + A^k z\left(x,y\right) = B^k u\left(x,y\right) + b^k\left(x,y\right),\\ z\left(x,0\right) = z\left(0,y\right) = 0 \quad \text{for all} \quad \left(x,y\right)\in P,\\ J^k\left(z,u\right) = \displaystyle\int_P F^k\left(x,y,z\left(x,y\right),u\left(x,y\right)\right)\mathrm{d}x\,\mathrm{d}y,\end{cases} \tag{LH^k}$$

where

$$\begin{aligned}b^k\left(x,y\right) &= -A_1^k\frac{\mathrm{d}}{\mathrm{d}x}\varphi^k\left(x\right) - A_2^k\frac{\mathrm{d}}{\mathrm{d}x}\psi^k\left(y\right)\\ &\quad - A^k\varphi^k\left(x\right) - A^k\psi^k\left(x\right) + A^k c^k,\\ F^k\left(x,y,z,u\right) &= G^k\left(x,y,z+\varphi^k\left(x\right)+\psi^k\left(y\right)-c^k,u\right).\end{aligned}$$

Problem (LH^k) will be considered in the space $AC_0^2(P,\mathbb{R}^n)$ of solutions z and in the set $\mathcal{U}_M$ of controls u.

For simplicity, we prove our results for Problem (LH^k). However, Problems (LH^k) and (L^k) are equivalent, and therefore all results which are to be proved can be used for (L^k).

For the system

$$\begin{cases} \dfrac{\partial^2 z}{\partial x \partial y}(x,y) + A_1^k \dfrac{\partial z}{\partial x}(x,y) + A_2^k \dfrac{\partial z}{\partial y}(x,y) \\ \qquad + A^k z(x,y) = B^k u(x,y) + b^k(x,y), \\ \qquad (x,y) \in P \text{ a.e.}, \\ z(x,0) = z(0,y) = 0 \quad \text{for all} \quad x,y \in [0,1], \end{cases} \tag{3}$$

the following theorem holds (Bergmann *et al.*, 1989):

Theorem 1. *For any $u \in L^2(P,\mathbb{R}^m)$ the system (3) possesses a unique solution $z^k \in AC_0^2(P,\mathbb{R}^n)$ given by the formula*

$$z^k(x,y) = \int_0^x \int_0^y R^k(s,t,x,y)\left(B^k u(s,t) + b^k(s,t)\right) \mathrm{d}s\,\mathrm{d}t, \tag{4}$$

where the function $R^k : P \times P \to \mathbb{R}^{n\times n}$ (called the Riemann function) has the form

$$R^k(s,t,x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(s-x)^i}{i!}\frac{(t-y)^j}{j!} T_{i,j}^k,$$

and the sequence $T_{i,j}^k$ is defined by the recurrence formulae

$$\begin{aligned} T_{i,j}^k &= T_{i,j-1}^k A_1^k + T_{i-1,j}^k A_2^k - T_{i-1,j-1}^k A^k, \\ T_{0,0}^k &= I,\ T_{i,j}^k = 0 \text{ for } i=-1 \text{ or } j=-1, \end{aligned} \tag{5}$$

for $k = 0,1,2,\dots$.

We shall make the following assumptions:

(L0) $b^k \to b^0$ in $L^2(P,\mathbb{R}^n)$,

(L1) the function $P \ni (x,y) \mapsto F^k(x,y,z,u) \in \mathbb{R}$ is measurable for $(z,u) \in \mathbb{R}^n \times \mathbb{R}^m$, and $k = 0,1,2,\dots$,

(L2) the function $\mathbb{R}^n \times \mathbb{R}^m \ni (z,u) \mapsto F^k(x,y,z,u) \in \mathbb{R}$ is continuous for $(x,y) \in P$ a.e., $k = 0,1,2,\dots$,

(L3) the function $\mathbb{R}^m \ni u \mapsto F^k(x,y,z,u) \in \mathbb{R}$ is convex for $(x,y) \in P$ a.e., $z \in \mathbb{R}^n$ and $k = 0,1,2,\dots$,

(L4) for any bounded set $B \subset \mathbb{R}^n$, there exists a function $\gamma_B \in L^1(P,\mathbb{R}^+)$ such that

$$\left|F^k(x,y,z,u)\right| \le \gamma_B(x,y) + |u|$$

for $(x,y) \in P$ a.e., $z \in B, u \in \mathbb{R}^m$ and $k = 0,1,2,\dots$,

(L5) the sequences of matrices $(A^k)_{k\in\mathbb{N}}$, $(A_1^k)_{k\in\mathbb{N}}$, $(A_2^k)_{k\in\mathbb{N}}$ tend to matrices A^0, A_1^0, A_2^0, respectively, in the norm of $\mathbb{R}^{n\times n}$, and $(B^k)_{k\in\mathbb{N}}$ tends to B^0 in the norm of $\mathbb{R}^{n\times m}$ (by the norm of a matrix A we mean the value $\|A\| = (\sum_{i,j} a_{i,j}^2)^{\frac{1}{2}}$).

For $k = 0,1,2,\dots$, set

$$Z^k = \left\{ z^k \in AC_0^2(P,\mathbb{R}^n) : \quad \text{there exists } u \in \mathcal{U}_M \text{ such that } z^k \text{ is the solution of (3) corresp. to } u \right\}$$

and

$$m^k = \inf J^k(z,u)$$

with respect to (z,u) such that z is the solution of (3) corresponding to $u \in \mathcal{U}_M$.

In a standard way one can prove the following result:

Theorem 2. *Assume that (L1)–(L4) hold. Then, for any $k = 0,1,2,\dots$, there exists an optimal solution of Problem (LH^k), i.e. for any $k = 0,1,2,\dots$ there exist control $u_*^k \in \mathcal{U}_M$ and the trajectory $z_*^k \in Z^k$ corresponding to u_*^k, such that*

$$J^k(z_*^k, u_*^k) = m^k.$$

Write

$$\mathcal{A}^k = \left\{ (z_*^k, u_*^k) \in AC_0^2(P,\mathbb{R}^n) \times \mathcal{U}_M : z_*^k \text{ satisfies (3) with } u_*^k \text{ and } J^k(z_*^k,u_*^k) = m^k \right\}. \tag{6}$$

This set will be referred to as the set of optimal solutions, or the optimal set.

Lemma 1. *Let*

$$c = \sup_{k\in\{0,1,\dots\}} \max\{|A^k|,\ |A_1^k|,\ |A_2^k|, |B^k|, 1\}.$$

Then

1. *for any $k = 0,1,2,\dots$,*

$$\left|R^k(x,y,s,t)\right| \le e^{3c}$$

for $(x,y,s,t) \in P \times P$,

2. *if $\varepsilon > 0$ and K are such that, for any $k > K$,*

$$\left|A^k - A^0\right| \leq \frac{\varepsilon}{3}, \ \left|A_1^k - A_1^0\right| \leq \frac{\varepsilon}{3}, \ \left|A_2^k - A_2^0\right| \leq \frac{\varepsilon}{3},$$

then, for $k > K$, we have

$$\left|T_{i,j}^k - T_{i,j}^0\right| \leq \varepsilon 3^{i+j+1} c^{i+j} \quad \text{for} \quad i,j \in \{0,1,2,\dots\}$$

and, consequently,

$$\left|R^k(x,y,s,t) - R^0(x,y,s,t)\right| \leq 3\varepsilon e^{3c}$$

for any $(x,y,s,t) \in P \times P$ (i.e. $R^k \to R^0$ uniformly on $P \times P$ as $k \to \infty$).

The proof of this lemma (using the induction argument) has only a technical character and is very arduous.

Let us recall that the weak upper limit of a sequence of the sets $V^k \subset X$ (X is a Banach space) is defined as the set of all cluster points (with respect to the weak topology) of sequences $\left(v^k\right)$ where $v^k \in V^k$ for $k = 1,2,3,\dots$. We denote this set as $\operatorname{wLimsup} V^k$ (Aubin and Frankowska, 1990).

Theorem 3. *If*

1. *Problems (LH^k) satisfy the conditions (L0)–(L5),*

2. *the sequence of cost functionals $J^k(x,u)$ tends to $J^0(x,u)$ uniformly on $\widetilde{B} \times \mathcal{U}_M$ for any bounded set $\widetilde{B} \subset AC_0^2(P,\mathbb{R}^n)$,*

then

(a) *there exists a ball $\widetilde{B}(0,\rho) \subset AC_0^2(P,\mathbb{R}^n)$ such that $Z^k \subset \widetilde{B}(0,\rho)$ for $k = 0,1,2,\dots$, i.e. there exists $\rho > 0$ such that, for any $z^k \in Z^k$, we have $\|z^k\|_{AC_0^2(P,\mathbb{R}^n)} \leq \rho$,*

(b) *the sequence of optimal values m^k tends to an optimal value m^0,*

(c) *the weak upper limit of the optimal sets $\mathcal{A}^k \subset AC_0^2(P,\mathbb{R}^n) \times L^2(P,\mathbb{R}^m)$ is a non-empty set, and $\operatorname{wLimsup} \mathcal{A}^k \subset \mathcal{A}^0$.*

If the set $\mathcal{A}^k$ is a singleton, i.e. $\mathcal{A}^k = \{(z_*^k, u_*^k)\}$ for $k = 0,1,2,\dots$, then z_*^k tends to z_*^0 weakly in $AC_0^2(P,\mathbb{R}^n)$ and u_*^k tends to u_*^0 weakly in $L^2(P,\mathbb{R}^m)$.

Proof. (a) Let $z^k \in AC_0^2(P,\mathbb{R}^n)$ be the solution of (3) corresponding to u^k. From (4) we have

$$\begin{aligned}
\frac{\partial^2 z^k}{\partial x \partial y}(x,y) &= B^k u^k(x,y) + b^k(x,y) \\
&+ \int_0^x \frac{\partial}{\partial x} R^k(s,y,x,y)\left(B^k u^k(s,y) + b^k(s,y)\right) \mathrm{d}s \\
&+ \int_0^y \frac{\partial}{\partial y} R^k(x,t,x,y)\left(B^k u^k(x,t) + b^k(x,t)\right) \mathrm{d}t \\
&+ \int_0^x\int_0^y \frac{\partial^2}{\partial x \partial y} R^k(s,t,x,y) \\
&\times \left(B^k u^k(s,t) + b^k(s,t)\right) \mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

for $(x,y) \in P$.

Since $u^k(x,y) \in M$, which is bounded, $B^k \to B^0$ and R^k are analytic, from (L0) and Lemma 1 we get that there exists a constant $\rho > 0$ such that

$$\|z^k\|_{AC_0^2(P,\mathbb{R}^n)} = \Big(\int_P \left|\frac{\partial^2 z}{\partial x \partial y}(x,y)\right|^2 \mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}} \leq \rho$$

for $k = 0,1,2,\dots$.

(b) Let $\left(z_*^k, u_*^k\right) \in \mathcal{A}^k$ for $k = 1,2,3,\dots$. Then we have

$$m^0 \leq J^0\left(\tilde{z}_k^0, u_*^k\right) \quad \text{for} \quad k = 1,2,3,\dots,$$

where $\tilde{z}_k^0$ is the trajectory of (3) with $k = 0$, corresponding to u_*^k for $k = 1,2,\dots$. Let $\varepsilon > 0$. By (L0) and (L5), there exists a K such that, for any $k > K$,

$$\begin{aligned}
&\left|A^k - A^0\right| \leq \frac{\varepsilon}{3}, \quad \left|A_1^k - A_1^0\right| \leq \frac{\varepsilon}{3}, \\
&\left|A_2^k - A_2^0\right| \leq \frac{\varepsilon}{3}, \quad \left|B^k - B^0\right| < \varepsilon, \\
&\int_P \left|b^k(x,y) - b^0(x,y)\right| \mathrm{d}x\,\mathrm{d}y < \varepsilon^2.
\end{aligned} \tag{7}$$

By direct calculations, from Lemma 1 and (7) we obtain, for $k > K$ and $(x,y) \in P$,

$$\begin{aligned}
&\left|z_*^k(x,y) - \tilde{z}_k^0(x,y)\right| \\
&\leq \int_0^x\int_0^y \left|R^k(s,t,x,y) - R^0(s,t,x,y)\right| \\
&\qquad \left|B^k u_*^k(s,t) + b^k(s,t)\right| \mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

$$+\int_0^x\int_0^y \left|R^0(s,t,x,y)\right|\left|B^k-B^0\right|\left|u_*^k(s,t)\right|\mathrm{d}s\,\mathrm{d}t$$

$$+\int_0^x\int_0^y \left|R^0(s,t,x,y)\right|\left|b^k(s,t)-b^0(s,t)\right|\mathrm{d}s\,\mathrm{d}t\le\varepsilon\tilde{c},$$

where $\tilde{c}>0$. This means that $\sup_{(x,y)\in P}\left|z_*^k(x,y)-\tilde{z}_k^0(x,y)\right|$ is arbitrarily small (for a sufficiently large k). From the Scorza-Dragoni Theorem (Ekeland and Temam, 1976) applied to the function $F^0|_{P\times B\times M}$ (where B is the ball with radius ρ described in (a)) we have that for any $\eta>0$ there exists a compact set $P_\eta\subset P$ such that $\mu(P\backslash P_\eta)\le\eta$ and $F^0|_{P_\eta\times B\times M}$ is uniformly continuous. Thus for a sufficiently large k we have

$$\begin{aligned}
&\left|J^0\left(\tilde{z}_k^0,u_*^k\right)-J^0\left(z_*^k,u_*^k\right)\right|\\
&\le\int_{P_\eta}\left|F^0\left(x,y,\tilde{z}_k^0(x,y),u_*^k(x,y)\right)\right.\\
&\qquad\left.-F^0\left(x,y,z_*^k(x,y),u_*^k(x,y)\right)\right|\\
&\qquad+\int_{P\backslash P_h}\left|F^0\left(x,y,\tilde{z}_k^0(x,y),u_*^k(x,y)\right)\right.\\
&\qquad\left.-F^0\left(x,y,z_*^k(x,y),u_*^k(x,y)\right)\right|\\
&\le\mu(P_\eta)\,\eta+\eta\hat{c}=\bar{\varepsilon}
\end{aligned}$$

where $\hat{c}>0$ and $\bar{\varepsilon}$ is arbitrarily small.

Thus, for any $\bar{\varepsilon}>0$,

$$m^0\le J^0\left(\tilde{z}_k^0,u_*^k\right)\le J^0\left(z_*^k,u_*^k\right)+\bar{\varepsilon}\tag{8}$$

for a sufficiently large k.

From Assumption 2 and (a) we have

$$\left|J^k\left(z_*^k,u_*^k\right)-J^0\left(z_*^k,u_*^k\right)\right|<\bar{\varepsilon}\tag{9}$$

for a sufficiently large k. Consequently, by (8) and (9),

$$m^0\le J^k\left(z_*^k,u_*^k\right)+2\varepsilon=m^k+2\bar{\varepsilon}$$

for a sufficiently large k.

Similarly, we can prove that, for sufficiently large k,

$$m^k\le m^0+2\bar{\varepsilon}.$$

We have thus proved that $m^k\to m^0$ as $k\to\infty$.

(c) Let (z_*^k,u_*^k) be an optimal process for (LH^k), i.e. $(z_*^k,u_*^k)\in\mathcal{A}^k$ for $k=0,1,2,\ldots$. Since $u_*^k(x,y)\in M$ and M is compact, $(u_*^k)_{k\in\mathbb{N}}$ is bounded. Since $L^2(P,\mathbb{R}^m)$ is reflexive, the sequence $(u_*^k)_{k\in\mathbb{N}}$ is compact in the weak topology of the space $L^2(P,\mathbb{R}^m)$. Without loss of generality, we can assume that $u_*^k\rightharpoonup\bar{u}_*^0\in\mathcal{U}_M$ in the weak topology. From the formula of solution (4) we have that, for any $(x,y)\in P$,

$$\begin{aligned}
&z_*^k(x,y)\\
&=\int_0^x\int_0^y R^k(s,t,x,y)\left(B^k u_*^k(s,t)+b^k(s,t)\right)\mathrm{d}s\,\mathrm{d}t\\
&=\int_0^x\int_0^y\left(R^k(s,t,x,y)-R^0(s,t,x,y)\right)\\
&\qquad\times\left(B^k u_*^k(s,t)+b^k(s,t)\right)\mathrm{d}s\,\mathrm{d}t\\
&\quad+\int_0^x\int_0^y R^0(s,t,x,y)\Big(\left(B^k-B^0\right)u_*^k(s,t)\\
&\qquad+b^k(s,t)-b^0(s,t)\Big)\mathrm{d}s\,\mathrm{d}t\\
&\quad+\int_0^x\int_0^y R^0(s,t,x,y)\\
&\qquad\times\left(B^0u_*^k(s,t)+b^0(s,t)\right)\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

By virtue of Lemma 1 we have $R^k\rightrightarrows R^0$. By (L0), (L5) and from the boundedness of M the first and the second integral tend to zero. By the weak convergence of u_*^k, the last integral tends to

$$\begin{aligned}
&\int_0^x\int_0^y R^0(s,t,x,y)\\
&\qquad\times\left(B^0(s,t)\,u_*^0(s,t)+b^0(s,t)\right)\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

In this way we have proved that z_*^k tends pointwisely to some $\tilde{z}^0\in AC_0^2(P,\mathbb{R}^n)$ which is the solution to (LH^k) with $k=0$, corresponding to $\tilde{u}^0$. Further, we prove that $(\tilde{z}^0,\tilde{u}^0)$ is an optimal process for (LH^k) with $k=0$. Suppose that it is not true. Let (z_*^0,u_*^0) be an optimal process for (LH^k) with $k=0$. Let

$$J^0\left(\tilde{z}^0,\tilde{u}^0\right)-J^0\left(z_*^0,u_*^0\right)=\alpha>0.\tag{10}$$

Then we have

$$\begin{aligned}
m^k-m^0&=J^k\left(z_*^k,u_*^k\right)-J^0\left(z_*^0,u_*^0\right)\\
&=\left[J^k\left(z_*^k,u_*^k\right)-J^0\left(z_*^k,u_*^k\right)\right]\\
&\quad+\left[J^0\left(z_*^k,u_*^k\right)-J^0\left(\tilde{z}^0,\tilde{u}^0\right)\right]+\alpha.
\end{aligned}$$

By (b), $m^k\to m^0$. From Assumption (2) and (a) we get that the first component tends to zero as $k\to\infty$. Moreover, $\lim_{k\to\infty}J^0(z_*^k,u_*^k)\ge J^0(\tilde{z}^0,\tilde{u}^0)$ by (L2)–(L4). In this way we have a contradiction with (10) and the proof is complete. ■

3.2. Main Results for a Linear System

Based on Theorem 3, we obtain the following sufficient conditions for the stability of a two-dimensional optimal control system:

Corollary 1. *Suppose that, for any $k = 0, 1, \ldots$, Problem (LH^k) satisfies Assumptions (L0)–(L5) and, for any bounded set $B \subset \mathbb{R}^n$, there exists a sequence of functions $\gamma_B^k \in L^1(P, \mathbb{R}^+)$ such that*

$$\left|F^k(x, y, z, u) - F^0(x, y, z, u)\right| \le \gamma_B^k(x, y)$$

for $(x, y) \in P$ a.e., $(z, u) \in B \times M$ and $k = 0, 1, 2, \ldots$. Moreover, we assume that $\gamma_B^k \to 0$ in $L^1(P, \mathbb{R}^+)$. Then

(a) the sequence of optimal values m^k tends to an optimal value m^0 as $k \to \infty$,

(b) $\operatorname{wLimsup} \mathcal{A}^k \subset \mathcal{A}^0$ and $\operatorname{wLimsup} \mathcal{A}^k \neq \emptyset$.

Proof. Let $\widetilde{B}$ be any bounded set in the space $AC_0^2(P, \mathbb{R}^n)$. It is easy to see that if $z \in \widetilde{B}$, then $z(x, y) \in B \subset \mathbb{R}^n$ for any $(x, y) \in P$, where B is bounded. By assumption, we have that, for any $(z, u) \in B \times \mathcal{U}_M$,

$$\begin{aligned}\left|J^k(z, u) - J^0(z, u)\right| &\le \int_P \big|F^k(x, y, z(x, y), u(x, y)) \\ &\qquad - F^0(x, y, z(x, y), u(x, y))\big| \\ &\le \int_P \gamma_B^k(x, y)\, \mathrm{d}x\, \mathrm{d}y.\end{aligned}$$

Since $\gamma_B^k \to 0$ in $L^1(P, \mathbb{R}^+)$, the sequence of cost functionals J^k converges uniformly to J^0 on $\widetilde{B} \times \mathcal{U}_M$ for any bounded set $\widetilde{B} \subset AC_0^2(P, \mathbb{R}^n)$. In this way, the assumptions of Theorem 3 are fulfilled and, by this theorem, (a) and (b) are true. ■

Corollary 2. *If*

1. *we have*

$$\begin{aligned}F^k(x, y, z, u) &= G_1\left(x, y, \omega^k(x, y), z\right) \\ &\quad + \left\langle G_2\left(x, y, \omega^k(x, y), z\right), u\right\rangle\end{aligned}$$

where $\omega^k(\cdot) \in L^p(P, \mathbb{R}^s)$, $p \ge 1$ and functions

$$P \times \mathbb{R}^s \times \mathbb{R}^n \ni (x, y, \omega, z) \mapsto G_1(x, y, \omega, z) \to \mathbb{R},$$

$$P \times \mathbb{R}^s \times \mathbb{R}^n \ni (x, y, \omega, z) \mapsto G_2(x, y, \omega, z) \to \mathbb{R}^m$$

are measurable with respect to (x, y) and continuous with respect to (ω, z),

2. *$\omega^k \to \omega^0$ in the norm topology of $L^p(P, \mathbb{R}^s)$ as $k \to \infty$,*

3. *for any bounded set $B \subset \mathbb{R}^n$, there exists $C > 0$ such that*

$$|G_i(x, y, \omega, z)| \le C\left(1 + |\omega|^p\right)$$

for $(x, y) \in P$ a.e., $\omega \in \mathbb{R}^s$, $z \in B$,

4. *Problems (LH^k) satisfy Assumptions (L0)–(L5),*

then the conditions (a) and (b) of Corollary 1 hold.

Proof. We shall prove that the sequence of cost functionals $J^k(z, u)$ tends to $J^0(z, u)$ uniformly on any set $\widetilde{B} \times \mathcal{U}_M$ where $\widetilde{B} \subset AC_0^2(P, \mathbb{R}^n)$ is bounded. Suppose that this is not true. Then there exist some bounded set $\widetilde{B} \subset AC_0^2(P, \mathbb{R}^n)$, $\varepsilon > 0$ and some sequence $(z^{k_i}, u^{k_i})_{i \in \mathbb{N}}$, such that $z^{k_i} \in \widetilde{B}$, $u^{k_i} \in \mathcal{U}_M$ and

$$\left|J^{k_i}\left(z^{k_i}, u^{k_i}\right) - J^0\left(z^{k_i}, u^{k_i}\right)\right| \ge \varepsilon \tag{11}$$

for $i = 1, 2, 3, \ldots$.

By the reflexivity of $AC_0^2(P, \mathbb{R}^n)$, we may assume (extracting, if necessary, a subsequence) that z^{k_i} tends to some z^0 uniformly on P as $i \to \infty$. We have

$$\begin{aligned}&\left|J^{k_i}\left(z^{k_i}, u^{k_i}\right) - J^0\left(z^{k_i}, u^{k_i}\right)\right| \\ &\le \int_P \Big|G_1\left(x, y, \omega^{k_i}(x, y), z^{k_i}(x, y)\right) \\ &\qquad - G_1\left(x, y, \omega^0(x, y), z^0(x, y)\right)\Big|\, \mathrm{d}x\, \mathrm{d}y \\ &+ \int_P \Big|G_1\left(x, y, \omega^0(x, y), z^0(x, y)\right) \\ &\qquad - G_1\left(x, y, \omega^0(x, y), z^{k_i}(x, y)\right)\Big|\, \mathrm{d}x\, \mathrm{d}y \\ &+ \int_P \Big|G_2\left(x, y, \omega^{k_i}(x, y), z^{k_i}(x, y)\right) \\ &\qquad - G_2\left(x, y, \omega^0(x, y), z^0(x, y)\right)\Big| \left|u^{k_i}(x, y)\right| \mathrm{d}x\, \mathrm{d}y \\ &+ \int_P G_2\left(x, y, \omega^0(x, y), z^0(x, y)\right) \\ &\qquad - G_2\left(x, y, \omega^0(x, y), z^{k_i}(x, y)\right) \left|u^{k_i}(x, y)\right| \mathrm{d}x\, \mathrm{d}y.\end{aligned}$$

Since $u^{k_i} \in \mathcal{U}_M$, it is commonly bounded. By Krasnoselskii's theorem on the continuity of the Nemytskii operator, the right-hand side of the above inequality tends to zero as $i \to \infty$. This contradicts (11), and we have thus proved that the sequence of cost functionals $J^k(z, u)$ tends to $J^0(z, u)$ uniformly on any set $\widetilde{B} \times \mathcal{U}_M$, where $\widetilde{B} \subset AC_0^2(P, \mathbb{R}^n)$ is bounded. Applying Theorem (3), we complete the proof. ■

Next, let us consider a mixed case, i.e. when the cost functional is of the form

$$J^k(z,u) = \int_P \left\langle G_1(x,y,z(x,y)), \omega^k(x,y)\right\rangle + \left\langle G_2\left(x,y,v^k(x,y),\right.\right. \left.\left. z(x,y)\right), u(x,y)\right\rangle \mathrm{d}x\,\mathrm{d}y, \tag{12}$$

where $G_1 : P \times \mathbb{R}^n \to \mathbb{R}^s$, $G_2 : P \times \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^m$, $k = 0, 1, 2, \dots$.

Corollary 3. *If*

1. *the cost functional is of the form (12),*
2. *the function G_1 is measurable with respect to (x,y) and continuous with respect to z (analogously G_2),*
3. *for any bounded set $B \subset \mathbb{R}^n$, there exist $\alpha(\cdot) \in L^p(P, \mathbb{R}^+)$ and $C > 0$, such that*
$$|G_1(x,y,z)| \le \alpha(x,y),$$
$$|G_2(x,y,v,z)| \le (1 + |v|^p)$$
for $(x,y) \in P$ a.e., $z \in B$ and $v \in \mathbb{R}^r$,
4. *ω^k tends to ω^0 in the weak topology of $L^q(P, \mathbb{R}^s)$ when $q \in [1,\infty)$, or weakly-$*$ when $q = \infty$; v^k tends to v^0 in the norm topology of $L^p(P, \mathbb{R}^s)$,*
5. *the optimal control problems (LH^k) satisfy Assumptions (L0)–(L5),*

then the optimal values and the sets of optimal processes satisfy the conditions (a) and (b) of Corollary 1.

Remark 1. The obtained results remain true for a family of problems (L^k), whereas the following additional assumption concerning the functions φ^k, ψ^k is satisfied:

$$\varphi^k \to \varphi^0,\ \psi^k \to \psi^0 \text{ in } AC^2([0,1], \mathbb{R}^n).$$

Example 2. Consider a two-dimensional continuous optimal control system with variable parameters

$$\begin{cases} z_{xy}(x,y) + A_1^k z_x(x,y) + \left(1 + A_2^k\right) z_y(x,y) \\ \qquad\qquad = \left(1 + B^k\right) u(x,y), \\ z(x,0) = \varphi^k(x), \quad z(0,y) = \psi^k(y), \\ u(x,y) \in [0,1], \end{cases} \tag{13}$$

$$J^k(z,u) = \int_{P^2} \Big[(x-2) z + \omega_1^k(x,y)\, \phi_1(x,y,z(x,y)) + \omega_2^k(x,y)\, \phi_2(x,y,u(x,y)) + \frac{1}{4}(1-x)\, u(x,y) + 4xy \Big] \mathrm{d}x\,\mathrm{d}y \to \min,$$

where $\varphi^k, \psi^k \in AC^2([0,1], \mathbb{R})$, ϕ_1 is continuous and ϕ_2 is continuous and convex with respect to $u \in [0,1]$, $u(\cdot) \in L^2(P, [0,1])$, $z \in AC^2(P, \mathbb{R})$, $\omega_1^k(\cdot), \omega_2^k(\cdot) \in L^1(P, [-1,1])$. By Theorem 2, the problem (13), (14) possesses at least one optimal solution but, in general, it is not easy to find an optimal process for this system. Suppose that $A_1^k, A_2^k, B^k \to 0$ in $\mathbb{R}$; $\varphi^k, \psi^k \to 0$ in $AC^2([0,1], \mathbb{R})$, $\omega_1^k, \omega_2^k \to 0$ in $L^1(P, \mathbb{R})$ as $k \to \infty$.

In the limit case, we obtain the problem

$$\begin{aligned} &z_{xy}(x,y) + z_y(x,y) = u(x,y), \\ &z(0,y) = z(0,y) = 0, \end{aligned} \tag{14}$$

$$J^k(z,u) = \int_P \Big[(x-2)\, z(x,y) + \frac{1}{4}(1-x)\, u(x,y) + 4xy \Big] \mathrm{d}x\,\mathrm{d}y. \tag{15}$$

By Theorem 2, the above problem possesses an optimal process and, applying the extremum principle, we are able to find effectively an optimal solution (z_*^0, u_*^0) and an optimal value m^0. In fact, the Lagrange function for the system (14), (15) is of the form

$$\begin{aligned} &\mathcal{L}(z,u) \\ &= \int_P \Big[(x-2)\, z(x,y) + \frac{1}{4}(1-x)\, u(x,y) + 4xy \\ &\quad + v(x,y)\, z_{xy}(x,y) + z_y(x,y) - u(x,y) \Big] \mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{16}$$

where $v \in L^2(P, \mathbb{R})$.

The extremum principle implies that $\mathcal{L}_z\left(z_*^0, u_*^0\right) h = 0$ for any $h \in AC_0^2(P, \mathbb{R}^n)$, and

$$\mathcal{L}\left(z_*^0, u_*^0\right) \le \mathcal{L}(z_*, u) \tag{17}$$

for any admissible control u.

Taking account of (16) and integrating by parts, we get

$$\begin{aligned} \mathcal{L}_z(z_*, u_*)\, h &= \int_{P^2} (x-2)\, h(x,y) \\ &\quad + v(x,y) \left(h_{xy}(x,y) + h_y(x,y)\right) \mathrm{d}x\,\mathrm{d}y \\ &= \int_P \Big[\int_x^1 \int_y^1 (x-2)\, \mathrm{d}x\,\mathrm{d}y + v(x,y) \\ &\quad + \int_x^1 v(x,y)\, \mathrm{d}x\, h_{xy}(x,y) \Big] \mathrm{d}x\,\mathrm{d}y = 0 \end{aligned}$$

for any $h \in AC_0^2(P, \mathbb{R}^n)$. Thus

$$v(x,y) + \int_x^1 v(x,y)\,\mathrm{d}x + (1-y)\left(-\frac{3}{2} + 2x - \frac{1}{2}x^2\right) = 0.$$

The above equation is of the Volterra type, and therefore there exists a unique solution v_* of this equation. By direct calculation, it is easy to check that $v_*(x,y) = (1-x)(1-y)$.

The minimum condition (17) takes the form

$$\int_P (1-x)\left(y - \frac{3}{4}\right) u_*^0(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq \int_P (1-x)\left(y - \frac{3}{4}\right) u(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for all $u(x,y) \in [0,1]$. This implies

$$u_*^0(x,y) = \begin{cases} 1 & \text{for } x \in [0,1], \quad y \in \left[0, \frac{3}{4}\right] \\ 0 & \text{for } x \in [0,1], \quad y \in \left[\frac{3}{4}, 1\right] \end{cases} \tag{18}$$

and that, by (14) and (15),

$$z_*^0(x,y) = \begin{cases} (1-e^{-x})\,y & \text{for } x \in [0,1], \quad y \in \left[0, \frac{3}{4}\right] \\ (1-e^{-x})\,\frac{3}{4} & \text{for } x \in [0,1], \quad y \in \left[\frac{3}{4}, 1\right] \end{cases} \tag{19}$$

and

$$m^0 = J^0\left(z_*^0, u_*^0\right) = \frac{55}{64}.$$

Applying Theorem 2 to our example, we see that, for any $k = 1, 2, 3, \ldots$, there exists at least one optimal process (z_*^0, u_*^0) for the system (13), (14), and the sequence $(u_*^k)_{k\in\mathbb{N}}$ tends to u_*^0 weakly in L^2, $(z_*^k)_{k\in\mathbb{N}}$ tends to z_*^0 weakly in $AC_0^2(P, \mathbb{R}^n)$, where u_*^0 and z_*^0 are defined by (18) and (19), respectively. Moreover, the sequence (m^k) of the optimal values for the systems (13), (14) tends to $m^0 = 55/64$ and the sequence $\left(z_*^k\right)_{k\in\mathbb{N}}$ of the optimal trajectories tends to z_*^0 uniformly on P.

In this way, we deduce that, in general, it is difficult to find an optimal solution for (13), (14), but the process (z_*^0, u_*^0) given by (18), (19) and the optimal value $m^0 = 55/65$ are a good approximation for (z_*^k, z_*^k) and m^k with a sufficiently large k. ♦

4. Nonlinear System

4.1. Preliminaries

In this part we give a definition of a Hausdorff metric and prove some generalization of the Gronwall lemma regarding the case of functions of two variables.

Let (X, ρ) be a metric space. We define a distance from a point $x_0 \in X$ to a bounded set $A \subset X$ as

$$\operatorname{dist}(x_0, A) = \inf\{\rho(x_0, a);\ a \in A\}.$$

The Hausdorff distance $\rho_H(A,B)$ between the bounded sets $A, B \subset X$ is defined as

$$\rho_H(A,B) = \inf\{\varepsilon > 0;\ A \subset N(B,\varepsilon),\ B \subset N(A,\varepsilon)\}$$

where, for a given $\varepsilon > 0$ and a bounded set $C \subset X$,

$$N(C,\varepsilon) = \{x \in X;\ \operatorname{dist}(x,C) \leq \varepsilon\}.$$

The function ρ_H restricted to the set $Z\times Z$, where Z is the family of all closed bounded subsets of X, is a metric in Z (Kisielewicz, 1991). It is called the Hausdorff metric.

Lemma 2. *If $k, c > 0$, $\upsilon : P \to \mathbb{R}$ is continuous and*

$$0 \leq \upsilon(x,y) \leq \int_0^x\int_0^y k\upsilon(s,t)\,\mathrm{d}s\,\mathrm{d}t + c, \quad (x,y) \in P,$$

then

$$\upsilon(x,y) \leq e^{kxy}c, \quad (x,y) \in P.$$

Proof. Write

$$u(x,y) = \int_0^x\int_0^y k\upsilon(s,t)\,\mathrm{d}s\,\mathrm{d}t + c, \quad (x,y) \in P,$$

$$w(x,y) = e^{-kxy}u(x,y), \quad (x,y) \in P.$$

From the continuity of υ it follows that u possesses the partial derivatives $\partial u(x,y)/\partial x$ and $\partial u(x,y)/\partial y$ everywhere on P. Moreover,

$$\begin{aligned}\frac{\partial u}{\partial x}(x,y) &= \int_0^y k\upsilon(x,t)\,\mathrm{d}t \\ &\leq \int_0^y k\Big(\int_0^x\int_0^t k\upsilon(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau + c\Big)\,\mathrm{d}t \\ &= k^2\int_0^x\int_0^y (y-\tau)\upsilon(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau + kcy\end{aligned}$$

for $(x,y)\in P$. Consequently,

$$\begin{aligned}\frac{\partial w}{\partial x}(x,y) &= -kye^{-kxy}u(x,y)+e^{-kxy}\frac{\partial u}{\partial x}(x,y)\\ &\le -kye^{-kxy}u(x,y)\\ &\quad + e^{-kxy}k^2y\int_0^x\int_0^y \upsilon(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau\\ &= -e^{-kxy}k^2\int_0^x\int_0^y \tau\upsilon(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau+e^{-kxy}kcy\\ &= -kye^{-kxy}u(x,y)+kye^{-kxy}u(x,y)\\ &\quad -k^2e^{-kxy}\int_0^x\int_0^y \tau\upsilon(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau\le 0\end{aligned}$$

for $(x,y)\in P$ and, analogously,

$$\frac{\partial w}{\partial y}(x,y)\le 0,\quad (x,y)\in P.$$

Consequently,

$$w(x,y)\le w(0,0)=c$$

for $(x,y)\in P$. This implies

$$\upsilon(x,y)\le u(x,y)=e^{kxy}w(x,y)\le e^{kxy}c$$

for $(x,y)\in P$. ■

4.2. Main Result

In this part we consider the family of homogeneous problems

$$\begin{cases}\dfrac{\partial^2 z}{\partial x\partial y}=f^k\Big(x,y,z,\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y},u\Big),\ (x,y)\in P \text{ a.e.},\\ z(x,0)=0,\quad z(0,y)=0,\quad x,y\in[0,1],\\ J^k(z,u)=\displaystyle\int_0^1\int_0^1 F^k(x,y,z,u)\,\mathrm{d}x\,\mathrm{d}y\to\min,\\ u\in\mathcal{U}^k=\{u\in L^2(P,\mathbb{R}^m);\ u(x,y)\in M^k,\\ \qquad (x,y)\in P \text{ a.e.}\},\end{cases}\tag{NH^k}$$

$k=0,1,\dots$. Problem (NH^0) will be referred to as the ‘limit problem’. In the sequel, we shall assume that the functions

$$f^k: P\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times M\to\mathbb{R}^n,$$

$$F^k: P\times\mathbb{R}^n\times M\to\mathbb{R},$$

where $M=\bigcup_{k=0}^{\infty}M^k$, and the sets $M^k\subset\mathbb{R}^m$, $k=0,1,\dots$, satisfy the following conditions:

(N1) the sets M^k are compact and $M^k\xrightarrow[k\to\infty]{}M^0$ in $\mathbb{R}^m$ with respect to the Hausdorff metric;

(N2) the functions f^k are measurable in $(x,y)\in P$ a.e., continuous in $u\in M$ and there exists a constant $L>0$ such that

$$\begin{aligned}&\left|f^k(x,y,z,z_x,z_y,u)-f^k(x,y,w,w_x,w_y,u)\right|\\ &\qquad\le L\left(|z-w|+|z_x-w_x|+|z_y-w_y|\right)\end{aligned}$$

for $(x,y)\in P$ a.e., $z,z_x,z_y,w,w_x,w_y\in\mathbb{R}^n, u\in M$, $k=0,1,\dots$;

(N3) there exist constants $a,b>0$ such that

$$\left|f^k(x,y,z,z_x,z_y,u)\right|\le a\,|z|+b$$

for $(x,y)\in P$ a.e., $z,z_x,z_y\in\mathbb{R}^n,u\in M$, $k=0,1,\dots$;

(N4) for any bounded set $A\subset\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$, there exists a sequence $(\varphi_A^k)_{k\in\mathbb{N}}\subset L^2(P,\mathbb{R}^+)$ such that $\varphi_A^k\xrightarrow[k\to\infty]{}0$ in $L^2(P,\mathbb{R}^+)$ and

$$\begin{aligned}&\left|f^k(x,y,z,z_x,z_y,u)-f^0(x,y,z,z_x,z_y,u)\right|\\ &\qquad\le\varphi_A^k(x,y)\end{aligned}$$

for $(x,y)\in P$ a.e., $k=1,2,\dots$;

(N5) the function f^0 is of the type

$$\begin{aligned}&f^0(x,y,z,z_x,z_y,u)\\ &\quad=\alpha^0(x,y,z,z_x,z_y)+\beta^0(x,y,z,z_x,z_y)u,\end{aligned}$$

where β^0 is measurable in $(x,y)\in P$, continuous in $(z,z_x,z_y)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ and, for any bounded set $A\subset\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$, there exists a function $\gamma_A\in L^2(P,\mathbb{R}^+)$ such that

$$\left|\beta^0(x,y,z,z_x,z_y)\right|\le\gamma_A(x,y)$$

for $(x,y)\in P$ a.e., $(z,z_x,z_y)\in A$;

(N6) the functions F^k are measurable in $(x,y)\in P$, continuous in $(z,u)\in\mathbb{R}^n\times\mathbb{R}^m$ and, for each bounded set $B\subset\mathbb{R}^n$, there exists a function $\nu_B\in L^1(P,\mathbb{R}^+)$ such that

$$\left|F^k(x,y,z,u)\right|\le\nu_B(x,y)$$

for $(x,y)\in P$ a.e., $z\in B$, $u\in M$, $k=0,1,\dots$;

(N7) for any bounded set $B \subset \mathbb{R}^n$, there exists a sequence $\left(\psi_B^k\right)_{k\in\mathbb{N}} \subset L^1(P, \mathbb{R}^+)$ such that $\psi_B^k \xrightarrow[k\to\infty]{} 0$ in $L^1(P, \mathbb{R}^+)$ and

$$\left|F^k(x,y,z,u) - F^0(x,y,z,u)\right| \le \psi_B^k(x,y)$$

for $(x,y) \in P$ a.e., $z \in B$, $u \in M$, $k = 1, 2, \dots$.

Remark 2. From Assumption (N1) it follows that the sets $M^k, k = 0, 1, \dots$, are commonly bounded in $\mathbb{R}^m$.

Remark 3. From Assumption (N2) it follows that the functions f^k are continuous in $(z, z_x, z_y, u) \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times M$.

In much the same way as in (Idczak *et al*, 1994), one can show that there exist $l \in \mathbb{N}$ and $\alpha \in (0,1)$ such that each operator

$$\mathcal{F}_u^k : L^2(P, \mathbb{R}^n) \ni g$$

$$\longmapsto f^k\left(x, y, \int_0^x\!\!\int_0^y g, \int_0^y g, \int_0^x g, u(x,y)\right) \in L^2(P, \mathbb{R}^n),$$

where $u \in \mathcal{U}^k$, $k = 0, 1, \dots$, is contracting with the constant $\alpha \in (0,1)$ with respect to the Bielecki norm in $L^2(P, \mathbb{R}^n)$ given by

$$\|g\|_l = \left(\int_0^1\!\!\int_0^1 e^{-2l(x+y)}\, |g(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}},$$

$$g \in L^2(P, \mathbb{R}^n).$$

Consequently, $\mathcal{F}_u^k$ possesses a unique fixed point $g_u^k \in L^2(P, \mathbb{R}^n)$. This means that the system

$$\frac{\partial^2 z}{\partial x \partial y} = f^k\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, u\right)$$

has a unique solution z_u^k in the space $AC_0^2(P, \mathbb{R}^n)$. This solution is given by

$$z_u^k(x,y) = \int_0^x\!\!\int_0^y g_u^k(s,t)\,\mathrm{d}s\,\mathrm{d}t, \quad (x,y) \in P,$$

and

$$\frac{\partial^2 z_u^k}{\partial x \partial y}(x,y) = g_u^k(x,y), \quad (x,y) \in P \text{ a.e.}$$

Let us recall that the weak convergence in $AC_0^2(P, \mathbb{R}^n)$ implies the uniform convergence. Using the standard arguments, one can also easily show that if $z_n \longrightarrow z_0$ in $AC_0^2(P, \mathbb{R}^n)$, then $\partial z_n/\partial x \longrightarrow \partial z_0/\partial x$ in $L^2(P, \mathbb{R}^n)$ as $n \to \infty$.

In the proof of the main theorem, we shall use the following three lemmas:

Lemma 3. *There exists a constant r such that*

$$\left|\frac{\partial^2 z_u^k}{\partial x \partial y}(x,y)\right|, \left|\frac{\partial z_u^k}{\partial x}(x,y)\right|, \left|\frac{\partial z_u^k}{\partial y}(x,y)\right| \le r$$

for $(x,y) \in P$ a.e. and $u \in \mathcal{U}^k$, $k = 0, 1, \dots$.

Proof. From Assumption (N3) we have

$$\left|\frac{\partial^2 z_u^k}{\partial x \partial y}(x,y)\right|$$

$$= \left|f^k\left(x, y, z_u^k(x,y), \frac{\partial z_u^k}{\partial x}(x,y), \frac{\partial z_u^k}{\partial y}(x,y), u(x,y)\right)\right|$$

$$\le a\left|z_u^k(x,y)\right| + b, \quad (x,y) \in P \text{ a.e.}$$

for $u \in \mathcal{U}^k$, $k = 0, 1, \dots$. Hence

$$\left|z_u^k(x,y)\right| = \left|\int_0^x\!\!\int_0^y \frac{\partial^2 z_u^k}{\partial x \partial y}(s,t)\,\mathrm{d}s\,\mathrm{d}t\right|$$

$$\le a \int_0^x\!\!\int_0^y \left|z_u^k(s,t)\right| \mathrm{d}s\,\mathrm{d}t + b, \quad (x,y) \in P$$

for $u \in \mathcal{U}^k$, $k = 0, 1, \dots$. Applying the previous lemma with $v(x,y) = \left|z_u^k(x,y)\right|$, $k = a$, $c = b$, we obtain

$$\left|z_u^k(x,y)\right| \le e^a b, \quad (x,y) \in P,$$

for $u \in \mathcal{U}^k$, $k = 0, 1, \dots$. Thus

$$\left|\frac{\partial^2 z_u^k}{\partial x \partial y}(x,y)\right| \le a e^a b + b, \quad (x,y) \in P \text{ a.e.},$$

for $u \in \mathcal{U}^k$, $k = 0, 1, \dots$. The remaining part of the assertion follows from the fact that

$$\frac{\partial z_u^k}{\partial x}(x,y) = \int_0^y \frac{\partial^2 z_u^k}{\partial x \partial y}(x,t)\,\mathrm{d}t, \quad (x,y) \in P \text{ a.e.},$$

$$\frac{\partial z_u^k}{\partial y}(x,y) = \int_0^x \frac{\partial^2 z_u^k}{\partial x \partial y}(s,y)\,\mathrm{d}s, \quad (x,y) \in P \text{ a.e.},$$

for $u \in \mathcal{U}^k$, $k = 0, 1, \dots$. ■

As an immediate consequence of Lemma 3, we obtain the following result:

Corollary 4. *The set $\{z_u^k \in AC_0^2(P, \mathbb{R}^n)(P) : u \in \mathcal{U}^k,\ k = 0, 1, \dots\}$ is bounded in $AC_0^2(P, \mathbb{R}^n)$.*

Lemma 4. $\mathcal{U}^k \xrightarrow[k\to\infty]{} \mathcal{U}^0$ *in* $L^\infty(P,\mathbb{R}^m)$ *with respect to the Hausdorff metric.*

Proof. Fix $\varepsilon > 0$. Let $k_0 \in \mathbb{N}$ be such that

$$M^k \subset N(M^0,\varepsilon) \text{ and } M^0 \subset N(M^k,\varepsilon)$$

for $k \geq k_0$. This implies that, for any fixed function $u^k \in \mathcal{U}^k \ \ (k \geq k_0)$,

$$\operatorname{dist}(u^k(x,y), M^0) \leq \varepsilon, \quad (x,y) \in P \text{ a.e.}$$

Using the theorem on the measurable selection (Kisielewicz, 1991, Thm. 3.13), we conclude that there exists a function $u^0 \in \mathcal{U}^0$ such that

$$\left|u^k(x,y) - u^0(x,y)\right| \leq \varepsilon, \quad (x,y) \in P \text{ a.e.},$$

that is,

$$\left\|u^k - u^0\right\|_{L^\infty(P)} \leq \varepsilon.$$

Therefore, $\mathcal{U}^k \subset N(\mathcal{U}^0,\varepsilon)$ for $k \geq k_0$.

In the same way, we check that $\mathcal{U}^0 \subset N(\mathcal{U}^k,\varepsilon)$ for $k \geq k_0$. ∎

Lemma 5. *For each* $\varepsilon > 0$, *there exist* $\overline{k} \in \mathbb{N}$ *and* $\delta \in (0,\varepsilon)$ *such that*

$$\mathcal{U}^k \subset N(\mathcal{U}^0,\delta) \quad \textit{and} \quad \mathcal{U}^0 \subset N(\mathcal{U}^k,\delta)$$

for $k \geq \overline{k}$, *and if* $u^k \in \mathcal{U}^k \ (k \geq \overline{k}), u^0 \in \mathcal{U}^0$ *are such that* $\|u^k - u^0\|_{L^\infty(P)} < \delta$, *then*

$$\|z^k_{u^k} - z^0_{u^o}\|_{AC^2_0(P,\mathbb{R}^n)} < \varepsilon.$$

Proof. Let us observe that if $u^k \in \mathcal{U}^k, u^0 \in \mathcal{U}^0$ $z^k_{u^k} \in AC^2_0(P,\mathbb{R}^n)$ is a solution of the system

$$\frac{\partial^2 z}{\partial x \partial y} = f^k\Big(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},u^k\Big),$$

satisfying the zero-boundary conditions, and $z^0_{u^0} \in AC^2_0(P,\mathbb{R}^n)$ is a solution of the system

$$\frac{\partial^2 z}{\partial x \partial y} = f^0\Big(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},u^0\Big),$$

which also satisfies the zero-boundary conditions, then $g^k_{u^k} = \partial^2 z^k_{u^k}/\partial x\partial y$ is a fixed point of the operator $\mathcal{F}^k_{u^k}$ and $g^0_{u^0} = \partial^2 z^0_{u^0}/\partial x\partial y$ is a fixed point of the operator $\mathcal{F}^0_{u^0}$. Moreover,

$$\begin{aligned}
\left\|g^k_{u^k} - g^0_{u^0}\right\|_l &= \left\|\mathcal{F}^k_{u^k}(g^k_{u^k}) - \mathcal{F}^0_{u^0}(g^0_{u^0})\right\|_l \\
&\leq \left\|\mathcal{F}^k_{u^k}(g^k_{u^k}) - \mathcal{F}^k_{u^k}(g^0_{u^0})\right\|_l \\
&\quad + \left\|\mathcal{F}^k_{u^k}(g^0_{u^0}) - \mathcal{F}^0_{u^0}(g^0_{u^0})\right\|_l \\
&\leq \alpha \left\|g^k_{u^k} - g^0_{u^0}\right\|_l \\
&\quad + \left\|\mathcal{F}^k_{u^k}(g^0_{u^0}) - \mathcal{F}^0_{u^0}(g^0_{u^0})\right\|_l .
\end{aligned}$$

Hence

$$\left\|g^k_{u^k} - g^0_{u^0}\right\|_l \leq \frac{1}{1-\alpha}\left\|\mathcal{F}^k_{u^k}(g^0_{u^0}) - \mathcal{F}^0_{u^0}(g^0_{u^0})\right\|_l .$$

Since

$$\left\|g^k_{u^k} - g^0_{u^0}\right\| \leq e^{2l}\left\|g^k_{u^k} - g^0_{u^0}\right\|_l ,$$

$$\left\|\mathcal{F}^k_{u^k}(g^0_{u^0}) - \mathcal{F}^0_{u^0}(g^0_{u^0})\right\|_l \leq \left\|\mathcal{F}^k_{u^k}(g^0_{u^0}) - \mathcal{F}^0_{u^0}(g^0_{u^0})\right\| ,$$

we obtain

$$\begin{aligned}
&\left\|z^k_{u^k} - z^0_{u^0}\right\|_{AC^2_0(P)} = \left\|g^k_{u^k} - g^0_{u^0}\right\| \\
&\leq \frac{e^{2l}}{1-\alpha}\left\|\mathcal{F}^k_{u^k}(g^0_{u^0}) - \mathcal{F}^0_{u^0}(g^0_{u^0})\right\| \\
&\leq \frac{e^{2l}}{1-\alpha}\left\|\mathcal{F}^k_{u^k}(g^0_{u^0}) - \mathcal{F}^0_{u^k}(g^0_{u^0})\right\| \\
&\quad + \frac{e^{2l}}{1-\alpha}\left\|\mathcal{F}^0_{u^k}(g^0_{u^0}) - \mathcal{F}^0_{u^0}(g^0_{u^0})\right\| \\
&\leq \frac{e^{2l}}{1-\alpha}\Bigg(\int_0^1\!\!\int_0^1 \Big|f^k\Big(x,y,\int_0^x\!\!\int_0^y g^0_{u^0},\int_0^y g^0_{u^0},\int_0^x g^0_{u^0},u^k(x,y)\Big) \\
&\quad - f^0\Big(x,y,\int_0^x\!\!\int_0^y g^0_{u^0},\int_0^y g^0_{u^0},\int_0^x g^0_{u^0},u^k(x,y)\Big)\Big|^2 \,\mathrm{d}x\,\mathrm{d}y\Bigg)^{\frac{1}{2}} \\
&\quad + \frac{e^{2l}}{1-\alpha}\Bigg(\int_0^1\!\!\int_0^1 \Big|\beta^0\Big(x,y,\int_0^x\!\!\int_0^y g^0_{u^0},\int_0^y g^0_{u^0},\int_0^x g^0_{u^0}\Big)\Big|^2 \\
&\quad \times \left|u^k(x,y) - u^0(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

Now, let us fix $\varepsilon > 0$ and choose $\delta \in (0,\varepsilon)$ to be such that

$$\Big(\frac{e^{2l}}{1-\alpha} + \frac{e^{2l}}{1-\alpha}\|\gamma_C\|_{L^2(P,\mathbb{R}^+)}\Big)\delta < \varepsilon$$

(C is the ball in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ centred at 0 with radius $3r$, where r is described in Lemma 3). Let $k_1 \in \mathbb{N}$ be such that (cf. Lemma 4)

$$\mathcal{U}^k \subset N(\mathcal{U}^0,\delta) \text{ and } \mathcal{U}^0 \subset N(\mathcal{U}^k,\delta)$$

for $k \geq k_1$, and let k_2 satisfy (cf. Assumption (N4))

$$\left\|\varphi^k_C\right\|^2_{L^1(P,\mathbb{R}^+)} < \delta$$

for $k \geq k_2$.

Set $\overline{k} = \max\{k_1,k_2\}$ and fix $k \geq \overline{k}$. If $u^k \in \mathcal{U}^k, u^0 \in \mathcal{U}^0$ are such that $\|u^k - u^0\|_{L^\infty(P,\mathbb{R}^m)} < \delta$, then we obtain

$$\begin{aligned}&\left\|z^k_{u^k} - z^0_{u^0}\right\|_{AC^2_0(P,\mathbb{R}^n)} \\ &\qquad \le \frac{e^{2l}}{1-\alpha}\delta + \frac{e^{2l}}{1-\alpha}\left\|\gamma_C\right\|_{L^2(P,\mathbb{R}^+)}\delta < \varepsilon\end{aligned}$$

and the proof is complete. ■

In the sequel, we shall assume that each Problem (NH^k), $k = 0, 1, \ldots$ has a solution. As in the previous part, the set of all solutions to Problem (NH^k) will be denoted by $\mathcal{A}^k$. The minimal value of Problem (NH^k) will be denoted by m^k.

By an upper limit of sets $\mathcal{A}^k \subset AC^2_0(P,\mathbb{R}^n) \times L^2(P,\mathbb{R}^m)$ we mean the set $\limsup_{k\to\infty}\mathcal{A}^k$ of all cluster points (in the space $AC^2_0(P,\mathbb{R}^n) \times L^2(P,\mathbb{R}^m)$) of sequences $((z^k_{u^k_*}, u^k_*))_{k\in\mathbb{N}}$, where $(z^k_{u^k_*}, u^k_*) \in \mathcal{A}^k$.

Now, we shall prove the main result of this part:

Theorem 4. *If Assumptions (N1)–(N7) are satisfied and the sets* $\mathcal{A}^k$, $k = 0, 1, \ldots$, *are non-empty, then*

(a) $m^k \xrightarrow[k\to\infty]{} m^0$,

(b) $\limsup\limits_{k\to\infty}\mathcal{A}^k \subset \mathcal{A}^0$ *(in the space* $AC^2_0(P,\mathbb{R}^n) \times L^2(P,\mathbb{R}^m)$*).*

Proof. Fix $\eta > 0$. From the Scorza-Dragoni Theorem (Ekeland and Temam, 1976), applied to the function $F^0\mid_{P\times C\times M}$ (C is described in the proof of Lemma 5), it follows that there exists a compact set $P_\eta \subset P$ such that $\mu(P\backslash P_\eta) \le \eta$ (the Lebesgue measure) and $F^0\mid_{P_\eta\times C\times M}$ is uniformly continuous. In particular, this means that there exists $\sigma > 0$ such that

$$\left|F^0(x,y,z,u) - F^0(x,y,w,v)\right| < \eta,$$

provided that $|z-w|, |u-v| < \sigma, \quad (x,y) \in P_\eta$.

Now, for any positive integer k, we fix a pair $(z^k_{u^k_*}, u^k_*) \in \mathcal{A}^k$. Let $\overline{k}, \delta$ be the constants from Lemma 5 applied to $\varepsilon = \sigma$. Then, for each $k \ge \overline{k}$, there exists $u^0_{(k)} \in \mathcal{U}^0$ such that

$$\left|u^k_*(x,y) - u^0_{(k)}(x,y)\right| < \sigma, \quad (x,y) \in P \text{ a.e.},$$

and

$$\begin{aligned}\left|z^k_{u^k_*}(x,y) - z^0_{u^0_{(k)}}(x,y)\right| \le \left\|z^k_{u^k_*} - z^0_{u^0_{(k)}}\right\|_{AC^2_0(P)} < \sigma,\\ (x,y) \in P.\end{aligned}$$

Consequently,

$$\begin{aligned}&\Big|F^0\big(x,y,z^k_{u^k_*}(x,y),u^k_*(x,y)\big)\\ &\quad - F^0\big(x,y,z^0_{u^0_{(k)}}(x,y),u^0_{(k)}(x,y)\big)\Big| < \eta,\\ &\qquad\qquad (x,y) \in P_\eta \quad \text{a.e.}\end{aligned}$$

for $k \ge \overline{k}$.

Thus

$$\begin{aligned}&\left|J^0(z^k_{u^k_*}, u^k_*) - J^0(z^0_{u^0_{(k)}}, u^0_{(k)})\right|\\ &\le \int_{P_\eta}\Big|F^0\big(x,y,z^k_{u^k_*}(x,y),u^k_*(x,y)\big)\\ &\quad - F^0\big(x,y,z^0_{u^0_{(k)}}(x,y),u^0_{(k)}(x,y)\big)\Big|\,\mathrm{d}x\,\mathrm{d}y\\ &\quad + \int_{P\backslash P_\eta}\Big|F^0\big(x,y,z^k_{u^k_*}(x,y),u^k_*(x,y)\big)\\ &\quad - F^0\big(x,y,z^0_{u^0_{(k)}}(x,y),u^0_{(k)}(x,y)\big)\Big|\,\mathrm{d}x\,\mathrm{d}y\\ &\le \mu(P_\eta)\eta + \int_{P\backslash P_\eta} 2\nu_B(x,y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for $k \ge \overline{k}$ (B is the ball in $\mathbb{R}^n$ centred at 0 with radius r described in Lemma 3). Writing $\overline{\eta} = \eta + \int_{P\backslash P_\eta} 2\nu_B(x,y)\,\mathrm{d}x\,\mathrm{d}y$, we have

$$m^0 = J^0(z^0_{u^0_*}, u^0_*) \le J^0(z^0_{u^0_{(k)}}, u^0_{(k)}) \le J^0(z^k_{u^k_*}, u^k_*) + \overline{\eta}$$

for $k \ge \overline{k}$. From Assumption (N7) it follows that there exists $\widetilde{k}$ such that

$$\left|J^k(z^k_{u^k_*}, u^k_*) - J^0(z^k_{u^k_*}, u^k_*)\right| \le \overline{\eta}$$

for $k \ge \widetilde{k}$. So, for $k \ge \max\{\overline{k}, \widetilde{k}\}$, we have

$$m^0 \le J^k(z^k_{u^k_*}, u^k_*) + 2\overline{\eta} = m^k + 2\overline{\eta}.$$

In the same way, we show that

$$m^k \le m^0 + 2\overline{\eta}$$

for sufficiently large parameters k. Indeed, for each $k \ge \overline{k}$, there exists $u^k_{(0)} \in \mathcal{U}^k$ such that

$$\left|u^0_*(x,y) - u^k_{(0)}(x,y)\right| < \sigma, \quad (x,y) \in P \text{ a.e.},$$

and

$$\begin{aligned}\left|z^0_{u^0_*}(x,y) - z^k_{u^k_{(0)}}(x,y)\right| \le \left\|z^0_{u^0_*} - z^k_{u^k_{(0)}}\right\|_{AC^2_0(P)} < \sigma,\\ (x,y) \in P.\end{aligned}$$

Consequently,

$$\left|J^0(z^0_{u^0_*}, u^0_*) - J^0(z^k_{u^k_{(0)}}, u^k_{(0)})\right| \le \overline{\eta}$$

for $k \ge \overline{k}$ and

$$\begin{aligned}m^k = J^k(z^k_{u^k_*}, u^k_*) &\le J^k(z^k_{u^k_{(0)}}, u^k_{(0)})\\ &\le J^0(z^k_{u^k_{(0)}}, u^k_{(0)}) + \overline{\eta}\\ &\le J^0(z^0_{u^0_*}, u^0_*) + 2\overline{\eta}\end{aligned}$$

for $k \geq \max\{\overline{k}, \widetilde{k}\}$, which completes the proof of (a) (we have used here the fact that if $l \in L^1(P)$, then the set function $S \longmapsto \int_S l\,\mathrm{d}\mu$ is absolutely continuous, i.e. $\int_S l\,\mathrm{d}\mu \longrightarrow 0$ as $\mu(S) \to 0$).

Now, assume that $(\overline{z}, \overline{u})$ is a cluster point in $AC_0^2(P, \mathbb{R}^n) \times L^2(P, \mathbb{R}^m)$ of some sequence $((z^k_{u^k_*}, u^k_*))_{k\in\mathbb{N}}$ such that $(z^k_{u^k_*}, u^k_*) \in \mathcal{A}^k$. Without loss of generality, we may assume that $z^k_{u^k_*} \longrightarrow \overline{z}$ in $AC_0^2(P, \mathbb{R}^n)$ and $u^k_* \longrightarrow \overline{u}$ in $L^2(P, \mathbb{R}^m)$ as $k \to \infty$.

We shall show that $(\overline{z}, \overline{u}) \in \mathcal{A}^0$, i.e. $\overline{u} \in \mathcal{U}^0$, $\overline{z} = z^0_{\overline{u}}$ and $m^0 = J^0(\overline{z}, \overline{u})$. Fix $\varepsilon > 0$. Assumption (N1) implies that there exists $\overline{k}$ such that

$$u^k_*(x, y) \in N(M^0, \varepsilon), \quad (x, y) \in P \text{ a.e.}$$

for $k \geq \overline{k}$, i.e.

$$u^k_* \in \mathcal{U}_{N(M^0,\varepsilon)} = \big\{u \in L^2(P, \mathbb{R}^m);\ u(x, y) \in N(M^0, \varepsilon), \\ (x, y) \in P \text{ a.e.}\big\}.$$

Since M^0 is compact, so is $N(M^0, \varepsilon)$. Thus

$$\overline{u}(x, y) \in N(M^0, \varepsilon), \quad (x, y) \in P \text{ a.e.}$$

In particular,

$$\overline{u}(x, y) \in N(M^0, 1/n), \quad (x, y) \in P \text{ a.e.}$$

for $n = 1, 2, \dots$. Fix a point $(x, y) \in P$ which satisfies the above relation for $n = 1, 2, \dots$ (of course, the set of such points has a full measure). There exists a sequence $(v_n)_{n\in\mathbb{N}}$ of points belonging to M^0, such that

$$|\overline{u}(x, y) - v_n| \leq \frac{1}{n}$$

for $n = 1, 2, \dots$. In other words,

$$\overline{u}(x, y) = \lim_{n\to\infty} v_n,$$

i.e. $\overline{u}(x, y) \in M^0$ (by the closedness of M^0).

Now, we shall prove that $\overline{z} = z^0_{\overline{u}}$. Indeed,

$$\begin{aligned}
& z^k_{u^k_*}(x, y) \\
&= \int_0^x\!\!\int_0^y f^k\Big(s, t, z^k_{u^k_*}(s, t), \frac{\partial z^k_{u^k_*}}{\partial x}(s, t), \frac{\partial z^k_{u^k_*}}{\partial y}(s, t), u^k_*(s, t)\Big)\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^x\!\!\int_0^y \Big(f^k\Big(s, t, z^k_{u^k_*}(s, t), \frac{\partial z^k_{u^k_*}}{\partial x}(s, t), \frac{\partial z^k_{u^k_*}}{\partial y}(s, t), u^k_*(s, t)\Big) \\
&\quad - f^0\Big(s, t, z^k_{u^k_*}(s, t), \frac{\partial z^k_{u^k_*}}{\partial x}(s, t), \frac{\partial z^k_{u^k_*}}{\partial y}(s, t), u^k_*(s, t)\Big)\Big)\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^x\!\!\int_0^y f^0\Big(s, t, z^k_{u^k_*}(s, t), \frac{\partial z^k_{u^k_*}}{\partial x}(s, t), \frac{\partial z^k_{u^k_*}}{\partial y}(s, t), u^k_*(s, t)\Big)\mathrm{d}s\,\mathrm{d}t \\
&\xrightarrow[k\to\infty]{} 0 + \int_0^x\!\!\int_0^y f^0\Big(s, t, \overline{z}(s, t), \frac{\partial \overline{z}}{\partial x}(s, t), \frac{\partial \overline{z}}{\partial y}(s, t), \overline{u}(s, t)\Big)\mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

for $(x, y) \in P$. Consequently,

$$\begin{aligned}
\overline{z}(x, y) &= \lim_{k\to\infty} z^k_{u^k_*}(x, y) \\
&= \int_0^x\!\!\int_0^y f^0\Big(s, t, \overline{z}(s, t), \frac{\partial \overline{z}}{\partial x}(s, t), \frac{\partial \overline{z}}{\partial y}(s, t), \overline{u}(s, t)\Big)\mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

for $(x, y) \in P$. This means that $\overline{z} \in AC_0^2(P, \mathbb{R}^n)$ and

$$\begin{aligned}
&\frac{\partial^2 \overline{z}}{\partial x \partial y}(x, y) \\
&\quad = f^0\Big(x, y, \overline{z}(x, y), \frac{\partial \overline{z}}{\partial x}(x, y), \frac{\partial \overline{z}}{\partial y}(x, y), \overline{u}(x, y)\Big)
\end{aligned}$$

for $(x, y) \in P$ a.e.

To complete the proof, it suffices to show that $m^0 = J^0(\overline{z}, \overline{u})$. We have

$$\begin{aligned}
m^0 &= \lim_{k\to\infty} m^k = \lim_{k\to\infty} J^k(z^k_{u^k_*}, u^k_*) \\
&= \lim_{k\to\infty} \int_0^1\!\!\int_0^1 F^k\big(x, y, z^k_{u^k_*}(x, y), u^k_*(x, y)\big)\mathrm{d}x\,\mathrm{d}y \\
&= \lim_{k\to\infty} \int_0^1\!\!\int_0^1 \Big(F^k\big(x, y, z^k_{u^k_*}(x, y), u^k_*(x, y)\big) \\
&\quad - F^0\big(x, y, z^k_{u^k_*}(x, y), u^k_*(x, y)\big)\Big)\mathrm{d}x\,\mathrm{d}y \\
&\quad + \lim_{k\to\infty} \int_0^1\!\!\int_0^1 F^0\big(x, y, z^k_{u^k_*}(x, y), u^k_*(x, y)\big)\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

The first term equals zero in view of Assumption (N7). The other is equal to

$$\int_0^1\int_0^1 F^0(x,y,\overline{z}(x,y),\overline{u}(x,y))\,\mathrm{d}x\,\mathrm{d}y$$

on the basis of a generalization of the Krasnoselskii theorem (Idczak and Rogowski, 2003). Hence

$$m^0=\int_0^1\int_0^1 F^0(x,y,\overline{z}(x,y),\overline{u}(x,y))\mathrm{d}x\,\mathrm{d}y=J^0(\overline{z},\overline{u}),$$

which completes the proof. ■

4.3. Nonhomogeneous Problem

Now, consider the family of problems

$$\begin{cases}\dfrac{\partial^2 w}{\partial x\partial y}=g^k\Big(x,y,w,\dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y},u\Big),\ (x,y)\in P \text{ a.e.},\\ w(x,0)=\varphi^k(x),\quad w(0,y)=\psi^k(y),\\ x,y\in[0,1],\quad \varphi^k(0)=\psi^k(0)=c^k,\\ I^k(w,u)=\displaystyle\int_0^1\int_0^1 G^k(x,y,w,u)\,\mathrm{d}x\,\mathrm{d}y\to\min,\\ u\in\mathcal{U}^k=\big\{u\in L^2(P,\mathbb{R}^m);\ u(x,y)\in M^k,\\ \qquad (x,y)\in P \text{ a.e.}\big\},\end{cases}\tag{N^k}$$

$k=0,1,\dots$. By using the substitution

$$z(x,y)=w(x,y)-\varphi^k(x)-\psi^k(y)+c^k,\quad (x,y)\in P,$$

it is easy to see that Problems (N^k) and (NH^k) with

$$\begin{aligned}f^k&(x,y,z,z_x,z_y,u)\\&=g^k\big(x,y,z+\varphi^k(x)+\psi^k(y)+c^k,\\&\qquad z_x+\dot{\varphi}^k(x),z_y+\dot{\psi}^k(y),u\big),\end{aligned}$$

$$F^k(x,y,z,u)=G^k\big(x,y,z+\varphi^k(x)+\psi^k(y)-c^k,u\big)$$

are equivalent, i.e. $w\in AC^2(P,\mathbb{R}^n)$ is a solution to Problem (N^k) if and only if $z\in AC_0^2(P,\mathbb{R}^n)$, given by the above substitution, is a solution to Problem (NH^k), $k=0,1,\dots$. Moreover, the minimal values of functionals $I^k(w,u)$ and $J^k(z,u)$ are the same.

Next, by $C^1([0,1],\mathbb{R}^m)$ we mean the space of continuously differentiable functions $\omega:[0,1]\to\mathbb{R}^n$, with the norm given by

$$\begin{aligned}\|\omega\|_{C^1[0,1]}&=\max\big\{\,|\omega(t)|\,;\ t\in[0,1]\big\}\\&\quad+\max\big\{\big|\dot{\omega}(t)\big|\,,\ t\in[0,1]\big\}\,.\end{aligned}$$

Theorem 5. *If the functions g^k,G^k, $k=0,1,\dots$, and the sets M^k, $k=0,1,\dots$, satisfy Assumptions (N0)–(N7), the sets $\mathcal{B}^k$, $k=0,1,\dots$ of solutions to Problems (N^k) are non-empty and the functions φ^k,ψ^k tend to φ^0,ψ^0, respectively, in the space $C^1([0,1],\mathbb{R}^m)$, then*

(a) $m^k\xrightarrow[k\to\infty]{}m^0$,

(b) $\limsup_{k\to\infty}\mathcal{B}^k\subset\mathcal{B}^0$ *(in the space $AC^2(P,\mathbb{R}^n)\times L^2(P,\mathbb{R}^m)$).*

Proof. It is easy to show that the functions f^k,F^k given above satisfy Assumptions (N2)–(N7). Part (a) of the assertion is obvious. To prove Part (b), assume that $(w^k_{u^k_*},u^k_*)\in\mathcal{B}^k$, $k=1,2,\dots$ and

$$(w^k_{u^k_*},u^k_*)\xrightarrow[k\to\infty]{}(\overline{w},\overline{u})\text{ in }AC^2(P,\mathbb{R}^n)\times L^2(P,\mathbb{R}^m).$$

Then $\overline{u}\in\mathcal{U}^0$, as shown in the proof of Theorem 4. Moreover, $(z^k_{u^k_*},u^k_*)\in\mathcal{A}^k$, $k=1,2,\dots$, where

$$z^k_{u^k_*}(x,y)=w^k_{u^k_*}(x,y)-\varphi^k(x)-\psi^k(y)+c^k,\qquad (x,y)\in P,$$

and

$$(z^y_{u^k_*},u^k_*)\xrightarrow[k\to\infty]{}(\overline{z},\overline{u})\text{ in }AC^2(P,\mathbb{R}^n)\times L^2(P,\mathbb{R}^m),$$

where

$$\overline{z}(x,y)=\overline{w}(x,y)-\varphi^0(x)-\psi^0(y)+c^0,\quad (x,y)\in P.$$

Consequently, Theorem 4 implies $(\overline{z},\overline{u})\in\mathcal{A}^0$. This means that $(\overline{w},\overline{u})\in\mathcal{B}^0$. ■

References

Aubin J.P. and Frankowska H. (1990): *Set-Valued Analysis*. — Boston: Birkhäuser.

Bank B., Guddat J., Klatte D., Kummer B. and Tammer K. (1983): *Non-Linear Parametric Optimization*. — Basel: Birkhäuser.

Bergmann J., Burgemeier P. and Weiher P. (1989): *Explicit controllability criteria for Goursat problem*. — Optim., Vol. 20, No. 3, pp. 281–290.

Ekeland I. and Temam R. (1976): *Convex Analysis and Variational Problems*. — Amsterdam: North-Holland Publishing Company.

Fiacco A.V. (1981): *Sensitivity analysis for nonlinear programming using penalty methods*. — Math. Programm., Vol. 10, No. 3, pp. 287–311.

Idczak D. (1990): *Optimization of systems described by partial differential equations*. — Ph. D. Thesis, University of Łódź, Poland, (in Polish).

Idczak D. (1996): *The generalization of the Du Bois-Reymond lemma for functions of two variables to the case of partial derivatives of any order*. — Topology in Nonlinear Analysis, Banach Center Publications, Vol. 35, pp. 221–236.

Idczak D. (1998): *Optimal control of a coercive Dirichlet Problem*. — SIAM J. Contr. Optim., Vol. 36, No. 4, pp. 1250–1276.

Idczak D. and Rogowski A., (2003): *On a generalization of Krasnoselskii's theorem*. — J. Austral. Math. Soc., Ser. A, (to appear).

Idczak D. and Walczak S. (1994): *On Helly's theorem for functions of several variables and its applications to variational problems,* Optim., Vol. 30, No. 4, pp. 331–343.

Idczak D., Kibalczyc K. and Walczak S. (1994): *On an optimization problem with cost of rapid variation of control*. — J. Austral. Math. Soc., Ser. B, Vol. 86, No.1, pp. 117–131.

Ioffe A. (1994): *On sensitivity analysis of nonlinear programs in Banach spaces: the approach via composite unconstrained optimization*. — SIAM J. Optim., Vol. 4, No. 1, pp. 1–43.

Kisielewicz M., (1991): *Differential Inclusions and Optimal Control*. — Dordrecht-Warsaw: Kluwer Academic Publishers–Polish Scientific Publishers.

Levitin E.S. (1975): *On the local perturbation theory of a problem of mathematical programming in a Banach space*. — Soviet Math. Dokl., Vol. 16, No. 5, pp. 1354–1358.

Łojasiewicz S. (1988): *An Introduction to the Theory of Real Functions*. — Chichester: Wiley.

Malanowski K. (1993): *Stability and sensitivity analysis of solutions to infinite-dimensional optimization problems*. — Proc. 16-th IFIP Conf. *System Modelling and Optimization*, Compiegne, France, pp. 109–127.

Pulvirenti G. and Santagati G. (1975): *Sulla controllabilita completa in Luogo determinato ed in luogo variabile por un processo di controllo con parametri dostributi*. — Boll. UMI, Vol. 11, No. 3, pp. 316–328, (in Italian).

Robinson S.M. (1974): *Perturbed Kuhn-Tucker points and rates of convergence for a class of nonlinear systems*. — Math. Programm., 7, pp. 1–16.

Robinson S.M. (1976): *Stability theory for system of inequalities, Part II: Differentiable nonlinear systems*. — SIAM J. Numer. Anal. Vol. 13, No. 4, pp. 497–513.

Robinson S.M. (1980): *Strongly regular generalized equations*. — Math. Oper. Res., Vol. 5, No. 1, pp. 43–62.

Surryanarayama M.B. (1973): *Necessary conditions for optimization problems with hyperbolic partial differential equations*. — SIAM J. Contr., Vol. 11, No. 4, pp. 130–147.

Tikhonov A.N. and Samarski A.A. (1958): *Equations of Mathematical Physics*. — Moscow: GITTL, (in Russian).

Walczak S. (1987): *Absolutely continuous functions of several variables and their applications to differential equations*. — Bull. Pol. Acad. Sci., Ser. Math., Vol. 35, No. 11–12, pp. 733–744.

Walczak S. (2001): *Well-posed and ill-posed optimal control problems*. — JOTA, Vol. 109, No. 1, pp. 169–185.