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LINEAR REPETITIVE PROCESS CONTROL THEORY APPLIED TO A PHYSICAL EXAMPLE

KRZYSZTOF GAŁKOWSKI*, ERIC ROGERS**
WOJCIECH PASZKE*, DAVID H. OWENS***

* Institute of Control and Computation Engineering
University of Zielona Góra, 65–246 Zielona Góra, ul. Podgórna 50, Poland
e-mail: {K.Galkowski, W.Paszke}@issi.uz.zgora.pl

** Department of Electronics and Computer Science
University of Southampton, Southampton SO17 1BJ, U.K.
e-mail: etar@ecs.soton.ac.uk

*** Department of Automatic Control and Systems Engineering
University of Sheffield, Sheffield S1 3JD, U.K.
e-mail: d.h.owens@sheffield.ac.uk

In the case of linear dynamics, repetitive processes are a distinct class of 2D linear systems with uses in areas ranging from long-wall coal cutting and metal rolling operations to iterative learning control schemes. The main feature which makes them distinct from other classes of 2D linear systems is that information propagation in one of the two independent directions only occurs over a finite duration. This, in turn, means that a distinct systems theory must be developed for them for onward translation into efficient routinely applicable controller design algorithms for applications domains. In this paper, we introduce the dynamics of these processes by outlining the development of models for various metal rolling operations. These models are then used to illustrate some recent results on the development of a comprehensive control theory for these processes.

Keywords: repetitive dynamics, metal rolling, LMIs, delay differential system, stability

1. Introduction

The essential unique characteristic of a repetitive, or multipass, process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(t),\ 0 \le t \le \alpha$, generated on the pass k, acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(t),\ 0 \le t \le \alpha,\ k \ge 0$.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (Edwards, 1974; Smyth, 1992). Also in recent years there have occurred applications where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications of repetitive process theory include classes of iterative learning control, denoted by ILC in this paper, schemes (Amann *et al.*, 1998) and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle (Roberts, 2000).

Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in several very restrictive special cases) precisely because such an approach ignores the inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass and along a given pass, and the pass initial conditions are reset before the start of each new pass. In seeking a rigorous foundation for a control theory for these processes it is natural to attempt to exploit structural links which exist between, in particular, the class of the so-called discrete linear repetitive processes and 2D linear systems described by extensively studied Roesser (Roesser, 1975) or Fornasini Marchesini (Fornasini and Marchesini, 1978) state space models. Discrete linear repetitive processes are distinct from such 2D linear systems in the sense that information propagation in one of the two independent directions (along the pass) only occurs over a finite duration. This is also the distinction between the so-called differential linear repetitive processes and the 2D continuous-discrete linear systems studied by Kaczorek (1995) and others.

In this paper, we first introduce the essential unique features of repetitive processes by modelling metal rolling operations. This will introduce both differential and discrete linear repetitive processes and also highlight both the unique control problem for these processes and the 'rich' variety of dynamics which they can generate. Following this, two models arising from different aspects of metal rolling operations will be used to highlight today's progress in the development of a mature control theory for differential and discrete linear repetitive processes. In particular, recent work on the use of LMI (Linear Matrix Inequality) based methods in the design of control schemes for discrete linear repetitive processes will be highlighted by the application of the resulting theory to linear metal rolling dynamics modelled in the discrete domain. For differential processes, a version of metal rolling dynamics will be used to highlight links with classes of delay differential systems. Finally, some areas for further research will be briefly discussed.

2. Metal Rolling as a Repetitive Process

Metal rolling is an extremely common industrial process where, in essence, the deformation of the workpiece takes place between two rolls with parallel axes revolving in opposite directions. Figure 1 is a schematic diagram of the process where one approach is to pass the stock (i.e. the metal to be rolled to a pre-specified thickness (also termed the gauge or shape)) through a series of rolls for successive reductions, which can be costly in terms of equipment. A more economic route is to use a single two high stand, where this process is often termed 'clogging' (see also below). In practice, a number of models of this process can be developed depending on the assumptions made about the underlying dynamics and the particular mode of operation under consideration. Here we begin by developing a linearized model of the dynamics of the case shown schematically in Fig. 2. The particular task is to develop a simplified (but practically feasible) model relating the gauge on the current and previous passes through the rolls. These are denoted here by $y_k(t)$ and $y_{k-1}(t)$, respectively, and the other process variables and physical constants are defined as follows:

- $F_M(t)$ is the force developed by the motor;
- $F_s(t)$ is the force developed by the spring;

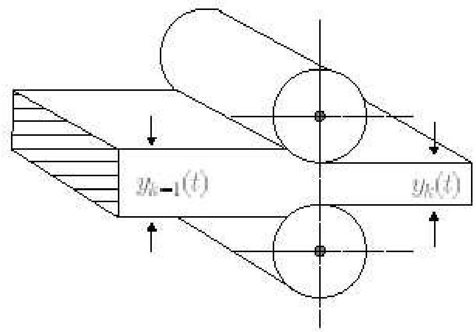

Fig. 1. Metal rolling process.

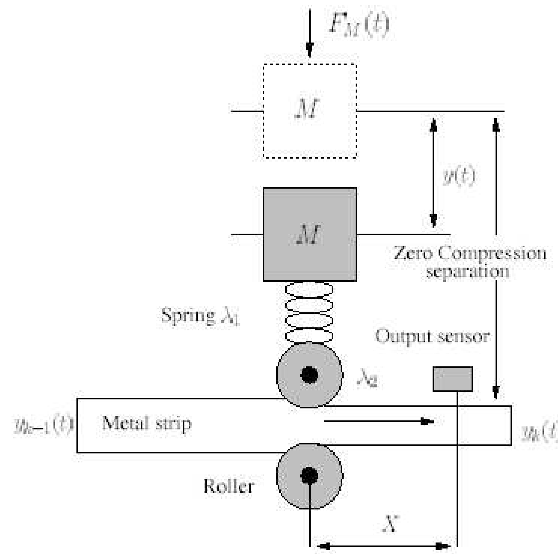

Fig. 2. Metal rolling process.

- M is the lumped mass of the roll-gap adjusting mechanism;
- λ_1 is the stiffness of the adjustment mechanism spring;
- λ_2 is the hardness of the metal strip;
- $\lambda = \lambda_1\lambda_2/(\lambda_1+\lambda_2)$ is the composite stiffness of the metal strip and the roll mechanism.

To model the basic process dynamics, refer to Fig. 2 where the force developed by the motor is

$$F_M(t) = F_s(t) + M\ddot{y}(t), \tag{1}$$

(where $y(t)$ is defined in Fig. 2), and the force developed by the spring is given by

$$F_s(t) = \lambda_1\big[y(t) + y_k(t)\big]. \tag{2}$$

This last force is also applied to the metal strip by the rolls and hence

$$F_s(t) = \lambda_2\big[y_{k-1}(t) - y_k(t)\big]. \tag{3}$$

Thus the following linear differential equation models the relationship between $y_k(t)$ and $y_{k-1}(t)$ on the above assumptions:

$$\begin{aligned}\ddot{y}_k(t) + \frac{\lambda}{M}y_k(t) = {} & \frac{\lambda}{\lambda_1}\ddot{y}_{k-1}(t) + \frac{\lambda}{M}y_{k-1}(t)\\ & - \frac{\lambda}{M\lambda_2}F_M(t).\end{aligned} \tag{4}$$

Suppose now that differentiation in (4) is approximated by backward difference discretization with sampling period T. (See, e.g. (Gałkowski *et al.*, 2001) for a treatment of the numerics and related matters associated with the construction of discrete approximations to the dynamics of differential linear repetitive processes.) Then the resulting difference equation is

$$\begin{aligned}y_k(t) = {} & a_1y_k(t-T) + a_2y_k(t-2T) + a_3y_{k-1}(t)\\ & + a_4y_{k-1}(t-T) + a_5y_{k-1}(t-2T)\\ & + bF_M(t),\end{aligned} \tag{5}$$

with the coefficients

$$a_1 = \frac{2M}{\lambda T^2 + M}, \qquad a_2 = \frac{-M}{\lambda^2 T + M},$$

$$a_3 = \frac{\lambda}{\lambda T^2 + M}\left(T^2 + \frac{M}{\lambda_1}\right), \qquad a_4 = \frac{-2\lambda M}{\lambda_1(\lambda T^2 + M)},$$

$$a_5 = \frac{\lambda M}{\lambda_1(\lambda T^2 + M)}, \qquad b = \frac{-\lambda T^2}{\lambda_2(\lambda T^2 + M)}.$$

Now set $t = pT$ and $y_k(p) = y_k(pT)$. Then (5) can be written for $k \geq 1$ as

$$\begin{aligned}x_k(p+1) &= Ax_k(p) + Bu_k(p) + B_0y_{k-1}(p),\\ y_k(p) &= Cx_k(p) + Du_k(p) + D_0y_{k-1}(p),\end{aligned} \tag{6}$$

where $u_k(p) = F_M(p)$ and

$$x_k(p) = \begin{bmatrix} y_k(p-1) & y_k(p-2) & y_{k-1}(p-1) & y_{k-1}(p-2)\end{bmatrix}^T,$$

$$A = \begin{bmatrix} a_1 & a_2 & a_4 & a_5\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}, \quad B = \begin{bmatrix} b\\ 0\\ 0\\ 0\end{bmatrix},$$

$$B_0 = \begin{bmatrix} a_3\\ 0\\ 1\\ 0\end{bmatrix}, \quad C = \begin{bmatrix} a_1 & a_2 & a_4 & a_5\end{bmatrix},$$

$$D = b, \qquad D_0 = a_3.$$

The model of (6) is a particular example of that for discrete linear repetitive processes where, in the general case on the pass k, $x_k(p)$ is the $n \times 1$ state vector, $y_k(p)$ is the $m \times 1$ pass profile vector and $u_k(p)$ is the $l \times 1$ control input vector. To complete the process description, it is necessary to specify the pass length and the initial, or boundary, conditions, i.e. the pass state initial vector sequence and the initial pass profile. Here the boundary conditions will be specified in the following section where controller design is the subject. In these design studies, the data used (in a compatible set of units) are $\lambda_1 = 600$, $\lambda_2 = 2000$, $M = 100$ and $T = 0.1$. This yields $\lambda = 461.54$ and the following matrices in (6):

$$A = \begin{bmatrix} 1.9118 & -0.0047 & -1.4706 & 0.7353\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix},$$

$$B = \begin{bmatrix} -2.2059 \times 10^{-5}\\ 0\\ 0\\ 0\end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.7794\\ 0\\ 1\\ 0\end{bmatrix},$$

$$C = \begin{bmatrix} 1.9118 & -0.0047 & -1.4706 & 0.7353\end{bmatrix},$$

$$D = \; 2.2059 \times 10^{-5} \;, \qquad D_0 = 0.7794.$$

The use of differential linear repetitive processes in this application area can be introduced by first assuming that the local feedback of $y(t)$ (see Fig. 2) with proportional plus derivative (PD) action is used to control the gap-setting motor, i.e.

$$F_M(t) = k_a\big[r(t) - y(t)\big] - k_b\dot{y}(t), \tag{7}$$

where k_a and k_b are the proportional and derivative gains of the local loop PD controller and $r(t)$ denotes the desired value of the motor deflection from the unstressed position. Combining this last equation with the uncontrolled dynamics modelled above now yields

$$\begin{aligned}&\ddot{y}_k(t) + 2\zeta\omega_n\dot{y}_k(t) + \omega_n^2y_k(t)\\ &\quad = \frac{\lambda}{\lambda_1}\left[\ddot{y}_{k-1}(t) + 2\zeta\omega_n\dot{y}_{k-1}(t) + \omega_n^2y_{k-1}(t)\right]\\ &\qquad + \frac{\lambda^2}{\lambda_2 M}y_{k-1}(t) - \frac{k_a\lambda}{\lambda_2 M}r(t),\end{aligned} \tag{8}$$

where $\omega_n = \sqrt{(k_a+\lambda)/M}$ and $\zeta = k_b/2\omega_n M$ are the (angular) natural frequency and damping ratio of the local servomechanism loop.

This linear differential equation can be easily written as a special case of the following one, which is the state space model for differential linear repetitive processes. Such processes are the natural continuous domain (in the along-the-pass direction) counterparts of the discrete linear repetitive processes of (6) (where in the general case on the pass $k \geq 1$ the dimensions of the state, pass profile and control input vectors are $n \times 1$, $m \times 1$, and $l \times 1$, respectively):

$$\begin{aligned} \dot{x}_k(t) &= Ax_k(t) + Bu_k(t) + B_0 y_{k-1}(t) \\ y_k(t) &= Cx_k(t) + Du_k(t) + D_0 y_{k-1}(t). \end{aligned} \tag{9}$$

A feature of the metal rolling process is that it can also be used to highlight yet more distinct repetitive process dynamics and, in particular, exhibit connections between these processes and classes of delay differential systems. For example, in operation the work strip must be passed back and forth to allow the successive passes of the rolling process to take place. One way of doing this is to use a reversing stand. This can, however, be costly in terms of the required power, and a more economical approach is to assume that the strip is passed repeatedly through a non-reversing single stand where the roll-gap is reduced for each pass—the so-called ‘clogging’ operation (Edwards, 1974). Note, however, that this process is slow (in relative terms) and has a variable pass delay since the stock is usually passed over the top of the rolls.

The thickness of the incoming strip in this case can be related to the actual roll-gap thickness by the so-called interpass interaction equation:

$$y_{k-1}(t) = y_k(t-h_1), \tag{10}$$

where h_1 denotes the pass delay and can be related to the length of the metal strip, denoted by L, which varies from pass-to-pass.

The gauge thickness is normally controlled through the proportional feedback control action of the form

$$r(t) = -k_c\big(r_d(t) - y_k(t-h_2)\big), \tag{11}$$

where k_c is the loop gain and $r_d(t)$ is the adjustable reference setting for the desired strip thickness. The delay h_2 is the output sensor measurement delay given by $h_2 = X/v(t)$, where (see Fig. 2) X denotes the distance between the roll-gap and the output sensor, and $v(t)$ is the velocity of the metal strip, which may also vary from pass-to-pass.

Appropriate substitutions and routine algebraic manipulations now show that the controlled (or closed-loop) system is this case is modelled by the following forced delay differential equation:

$$\ddot{y}_k(t) + f(\cdot) = \frac{c_1 k_a k_c}{M} r_d(t), \tag{12}$$

where

$$\begin{aligned} f(\cdot) = {} & 2\zeta\omega_n \dot{y}_k(t) + \omega_n^2 y_k(t) - c_3 \ddot{y}_k(t-h_1) \\ & - 2\zeta\omega_n c_3 \dot{y}_k(t-h_1) \\ & - \Big(\omega_n^2 c_3 + \frac{c_2}{M}\Big) y_k(t-h_1) \\ & + \frac{c_1 k_a k_c}{M} y_k(t-h_2), \end{aligned} \tag{13}$$

and

$$c_1 = \frac{\lambda}{\lambda_2}, \quad c_2 = \lambda c_1, \quad c_3 = \frac{\lambda}{\lambda_1}. \tag{14}$$

The structural links between this model and that for differential linear repetitive processes will be discussed in detail in Section 4.2.

In the next section, we introduce abstract model based stability theory for linear repetitive processes and then combine this theory with LMI based design tools to produce algorithms for the design of stabilizing control schemes for discrete linear repetitive processes. This is a key aspect in terms of applications for which few substantial results are yet available, and here we also demonstrate the relative ease to which they can be applied using the model (6) of the metal rolling process with the data given earlier in this section. Their extension to include stability margins is also discussed and illustrated.

3. Analysis and Control of Discrete Linear Repetitive Processes

Stability theory (Rogers and Owens, 1992; Rogers *et al.*, 2003) for linear constant pass length repetitive processes is based on the following abstract model of the underlying process dynamics where E_α is a suitably chosen Banach space with norm $\|\cdot\|$ and W_α is a linear subspace of E_α:

$$y_{k+1} = L_\alpha y_k + b_{k+1}, \quad k \geq 0. \tag{15}$$

Here $y_k \in E_\alpha$ is the pass profile on the pass k, $b_{k+1} \in W_\alpha$, and L_α is a bounded linear operator mapping E_α into itself. The term $L_\alpha y_k$ represents the contribution from the pass profile on the pass k to that on the pass $k+1$ and b_{k+1} represents known initial conditions, disturbances and control input effects which enter on pass $k+1$. We denote this model by S.

The linear repetitive process S is said to be asymptotically stable provided that there exists a real scalar

$\delta > 0$ such that, given any initial profile y_0 and any disturbance sequence $\{b_k\}_{k\geq 1} \subset W_\alpha$ bounded in norm (i.e. $\|b_k\| \leq c_1$ for some constant $c_1 \geq 0$ and $\forall\, k \geq 1$), the output sequence generated by the perturbed process

$$y_{k+1} = (L_\alpha + \gamma)y_k + b_{k+1}, \quad k \geq 0, \tag{16}$$

is bounded in norm whenever $\|\gamma\| \leq \delta$. This definition is easily shown to be equivalent to the requirement that there exist finite real numbers $M_\alpha > 0$ and $\lambda_\alpha \in (0,1)$ such that

$$\|L_\alpha^k\| \leq M_\alpha \lambda_\alpha^k, \quad k \geq 0, \tag{17}$$

where $\|\cdot\|$ is also used to denote the induced operator norm. A necessary and sufficient condition for (17) (for a proof of this and all other results relating to abstract model based stability theory, see (Rogers and Owens, 1992; Rogers *et al.*, 2003)) is that the spectral radius, $r(L_\alpha)$, of L_α satisfies

$$r(L_\alpha) < 1. \tag{18}$$

In order to apply this result to discrete linear repetitive processes of the form (6), it is necessary to specify the initial conditions, termed boundary conditions here, i.e. the pass state initial vector sequence and the initial pass profile. This is critical since it is known that the structure of these alone can cause instability.(That paper deals with differential linear repetitive processes but the results transfer in a natural manner to processes described by (6).) Here these are taken to be of the form

$$\begin{aligned} x_k(0) &= d_k, \quad k \geq 1, \\ y_0(p) &= y(p), \quad 0 \leq p \leq \alpha, \end{aligned} \tag{19}$$

where d_k is an $n \times 1$ vector with constant entries and $y(p)$ is an $m \times 1$ vector whose entries are known functions of p.

Given (19), a routine calculation now shows that asymptotic stability holds for processes described by (6) if, and only if, $r(D_0) < 1$. Also, if this property holds and the control input sequence $\{u_k\}_k$ applied converges strongly to u_∞ as $k \to \infty$, then the resulting output pass profile sequence $\{y_k\}_k$ converges strongly to y_∞—the so-called limit profile described, with $D = 0$ for ease of presentation, over $0 \leq p \leq \alpha$ by

$$\begin{aligned} x_\infty(p+1) &= \big[A + B_0(I_m - D_0)^{-1}C\big]x_\infty(p) \\ &\quad + Bu_\infty(p), \\ y_\infty(p) &= (I_m - D_0)^{-1}Cx_\infty(p), \\ x_\infty(0) &= d_\infty, \end{aligned} \tag{20}$$

where d_∞ is the strong limit of the sequence $\{d_k\}_k$.

In effect, this result states that if a process is asymptotically stable, then its repetitive dynamics can, after a 'sufficiently large' number of passes, be replaced by those of a 1D discrete linear system. Note, however, that this property does not guarantee that the limit profile is stable in the normal sense, i.e. $r(A + B_0(I_m - D_0)^{-1}C) < 1$— a point which is easily illustrated by examples. One such example is defined by $A = -0.5,\ B = 0,\ B_0 = 0.5 + \beta,\ C = 1, D = 0,\ D_0 = 0$, where β is a real scalar. Here the limit profile is given by

$$y_\infty(p+1) = \beta y_\infty(p) + u_\infty(p), \tag{21}$$

which is unstable if $|\beta| \geq 1$.

The reason why asymptotic stability does not guarantee a limit profile which is 'stable along the pass' is the finite pass length. In particular, asymptotic stability is easily shown to be a form of bounded-input bounded-output (BIBO) stability with respect to the finite and fixed pass length. Also in cases where the limit profile is unstable over the (finite and fixed) pass length, the stronger concept of stability along the pass must be used. In effect, for the abstract model (15), this requires that (17) holds uniformly with respect to the pass length α. One of several equivalent statements of this property is the requirement that there exist finite real scalars $M_\infty > 0$ and $\lambda_\infty \in (0,1)$ which are independent of α and satisfy

$$\|L_\alpha^k\| \leq M_\infty \lambda_\infty^k, \quad \forall\, \alpha > 0, \quad \forall\, k \geq 0. \tag{22}$$

In the case of processes described by (6) with boundary conditions (19), several equivalent sets of necessary and sufficient conditions for stability along the pass have been reported (Rogers and Owens, 1992; Rogers *et al.*, 2003) but here it is the following set that is required.

Theorem 1. *Discrete linear repetitive processes described by (6) and (19) are stable along the pass if, and only if, the 2D characteristic polynomial*

$$C(z_1, z_2) := \det \begin{bmatrix} I_n - z_1 A & -z_1 B_0 \\ -z_2 C & I_m - z_2 D_0, \end{bmatrix}, \tag{23}$$

satisfies

$$C(z_1, z_2) \neq 0, \quad \forall\ (z_1, z_2) \in \overline{U}^2, \tag{24}$$

where

$$\overline{U}^2 = \big\{\, (z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\big\}. \tag{25}$$

Note that (24) gives the necessary conditions that $r(D_0) < 1$ (asymptotic stability) and $r(A) < 1$, which should be verified before proceeding further with any stability analysis.

It is easily shown that the particular numerical example considered here is asymptotically stable but unstable along the pass. Hence it is necessary to design a control scheme. To do this, we use an LMI based approach to stability analysis for discrete linear repetitive processes (see also (Gałkowski *et al.*, 2002b)). Despite the fact that this LMI approach uses sufficient, but not necessary, conditions for stability, it will be demonstrated that, unlike alternative stability tests, it provides a natural basis for controller design.

The following well-known lemma is central to the application of LMIs to the problems considered in the remainder of this section, cf. (Boyd *et al.*, 1994).

Lemma 1. *Given constant matrices* W, L *and* V *of appropriate dimensions, where* $W = W^T$ *and* $V = V^T > 0$, *we have*

$$W + L^T V L < 0 \tag{26}$$

if, and only if,

$$\begin{bmatrix} W & L^T \\ L & -V^{-1} \end{bmatrix} < 0, \tag{27}$$

or, equivalently,

$$\begin{bmatrix} -V^{-1} & L \\ L^T & W \end{bmatrix} < 0. \tag{28}$$

The matrix $W + L^T V L$ is known as the Schur complement. In this paper > 0 denotes a symmetric positive matrix and ≤ 0 a symmetric negative matrix.

Now, define the following matrices from the state space model (6):

$$\widehat{A}_1 = \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{A}_2 = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix}. \tag{29}$$

Then we have the following sufficient condition for stability along the pass of processes described by (6) and (19):

Theorem 2. *Discrete linear repetitive processes described by (6) and (19) are stable along the pass if there exist matrices* $P > 0$ *and* $Q > 0$ *satisfying the following LMI:*

$$\begin{bmatrix} \widehat{A}_1^T P \widehat{A}_1 + Q - P & \widehat{A}_1^T P \widehat{A}_2 \\ \widehat{A}_2^T P \widehat{A}_1 & \widehat{A}_2^T P \widehat{A}_2 - Q \end{bmatrix} < 0. \tag{30}$$

Proof. This result was established elsewhere (see, e.g. (Gałkowski *et al.*, 2002b)) but here we give an alternative shorter proof. In particular, using '$*$' to denote the complex conjugate transpose operation, pre-multiply (30) by $[z_1^* I_{n+m},\ z_2^* I_{n+m}]$ and post-multiply it by $[z_1^* I_{n+m},\ z_2^* I_{n+m}]^T$ to yield

$$|z_1|^2(Q-P) - |z_2|^2 Q + (z_1\widehat{A}_1 + z_2\widehat{A}_2)^* P (z_1\widehat{A}_1 + z_2\widehat{A}_2) < 0. \tag{31}$$

Hence, since (30) clearly requires that $P - Q > 0$ and $Q > 0$ by definition

$$-P < -|z_1|^2(P-Q) - |z_2|^2 Q \quad \forall\, (z_1, z_2) \in \overline{U}^2. \tag{32}$$

Using these last two facts we now have

$$(z_1\widehat{A}_1 + z_2\widehat{A}_2)^* P (z_1\widehat{A}_1 + \widehat{A}_2) - P < 0, \tag{33}$$

and therefore

$$r(z_1\widehat{A}_1 + z_2\widehat{A}_2) < 1, \quad \forall\, (z_1, z_2) \in \overline{U}^2 \tag{34}$$

implies

$$\det(I_{n+m} - z_1\widehat{A}_1 - z_2\widehat{A}_2) \neq 0, \quad \forall\, (z_1, z_2) \in \overline{U}^2, \tag{35}$$

i.e. (24) holds and the proof is complete. ■

In terms of the design of control schemes for discrete linear repetitive processes, most work has been done in the ILC area (see, e.g. (Amann *et al.*, 1998)). Here it has become clear that a very powerful control action comes from using a (state) feedback action on the current pass augmented by a feedforward action from the previous pass. Here we consider a control law of the following form over $0 \leq p \leq \alpha,\ k \geq 0$:

$$u_{k+1}(p) = K_1 x_{k+1}(p) + K_2 y_k(p) := K \begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}, \tag{36}$$

where K_1 and K_2 are appropriately dimensioned matrices to be designed. Note here that, in implementation terms, the above control law does assume that all elements in the current pass state vector can actually be measured. In practice, some of these will have to be estimated with a suitably structured observer but, given the relatively low volume of work currently available on the structure and design of control laws for repetitive processes, this is not a severe restriction.

The following is the necessary and sufficient condition for closed loop stability along the pass (simply interpret (24) for the resulting closed loop system):

$$C_c(z_1, z_2) \neq 0, \quad \forall\, (z_1, z_2) \in \overline{U}^2, \tag{37}$$

where

$$C_c(z_1, z_2) := \det \begin{bmatrix} I_n - z_1(A + BK_1) & -z_1(B_0 + BK_2) \\ -z_2(C + DK_1) & I_m - z_2(D_0 + DK_2) \end{bmatrix}. \tag{38}$$

Now introduce the matrices

$$\widehat{B}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \widehat{B}_2 = \begin{bmatrix} 0 \\ D \end{bmatrix}. \tag{39}$$

Then we have the following result which follows immediately from interpreting Theorem 2 in terms of the closed loop system:

Theorem 3. *Suppose that a discrete linear repetitive process of the form described by (6) and (19) is subjected to a control law of the form (36). Then the resulting closed loop process is stable along the pass if there are matrices* $P > 0$ *and* $Q > 0$ *such that*

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} < 0, \tag{40}$$

where

$$X_{11} = (\widehat{A}_1^T + K^T \widehat{B}_1^T) P (\widehat{A}_1 + \widehat{B}_1 K) + Q - P,$$

$$X_{12} = (\widehat{A}_1^T + K^T \widehat{B}_1^T) P (\widehat{A}_2 + \widehat{B}_2 K),$$

$$X_{21} = (\widehat{A}_2^T + K^T \widehat{B}_2^T) P (\widehat{A}_1 + \widehat{B}_1 K),$$

$$X_{22} = (\widehat{A}_2^T + K^T \widehat{B}_2^T) P (\widehat{A}_2 + \widehat{B}_2 K) - Q.$$

The difficulty with the condition of Theorem 3 is that it is nonlinear in its parameters. It can, however, be converted into the following result, where the inequality is a strict LMI with a linear constraint which also gives a formula for computing K in (36).

Theorem 4. *The condition of Theorem 3 is equivalent to the requirement that there exist matrices* $Y > 0$, $Z > 0$, *and a matrix* N *such that the following LMI holds:*

$$\begin{bmatrix} Z - Y & 0 & Y \widehat{A}_1^T + N^T \widehat{B}_1^T \\ 0 & -Z & Y \widehat{A}_2^T + N^T \widehat{B}_2^T \\ \widehat{A}_1 Y + \widehat{B}_1 N & \widehat{A}_2 Y + \widehat{B}_2 N & -Y \end{bmatrix} < 0. \tag{41}$$

Also, if this condition holds, then a stabilizing K *for the control law (36) is given by*

$$K = N Y^{-1}. \tag{42}$$

Proof. Apply the Schur complement formula (Lemma 1) to (40), followed by congruence transformation defined by $\mathrm{diag}(\; P^{-1}, \; P^{-1}, \; I \;)$. Then introduce the substitutions $Z = P^{-1} Q P^{-1} > 0$, $Y = P^{-1} > 0$ to obtain

$$\begin{bmatrix} Z - Y & 0 & Y(\widehat{A}_1^T + K^T \widehat{B}_1^T) \\ 0 & -Z & Y(\widehat{A}_2^T + K^T \widehat{B}_2^T) \\ (\widehat{A}_1 + \widehat{B}_1 K) Y & (\widehat{A}_2 + \widehat{B}_2 K) Y & -Y \end{bmatrix} < 0.$$

The use of (42) now completes the proof. ■

Note also that a family of solutions is available via this LMI setting and further work is obviously required on how to select the best one in a given situation.

In the particular numerical example considered here, the underlying LMI test is feasible and the resulting Z, Y and N matrices are

$$Z = \begin{bmatrix} 12.6481 & 0.0178 & 0.0876 & -4.0263 & -0.0607 \\ 0.0178 & 41.1928 & 0.0528 & -0.3824 & 0.0282 \\ 0.0876 & 0.0528 & 11.9133 & 0.8867 & 1.8210 \\ -4.0263 & -0.3824 & 0.8867 & 38.7197 & -0.5789 \\ -0.0607 & 0.0282 & 1.8210 & -0.5789 & 7.5567 \end{bmatrix},$$

$$Y = \begin{bmatrix} 38.5634 & 0.1567 & -2.0385 & -16.4307 & 0.4870 \\ 0.1567 & 81.7970 & -0.1663 & -1.1286 & 0.1043 \\ -2.0385 & -0.1663 & 45.8157 & 11.0620 & 3.6070 \\ -16.4307 & -1.1286 & 11.0620 & 79.7523 & -3.0545 \\ 0.4870 & 0.1043 & 3.6070 & -3.0545 & 24.0251 \end{bmatrix},$$

$$N = 1 \times 10^6 \begin{bmatrix} 2.98951 & 0 & -2.5303 & 0 & 0.5673 \end{bmatrix}.$$

Hence the following K gives stability along the pass closed loop:

$$K = 1 \times 10^4 \begin{bmatrix} 8.5536 & -0.0046 & -6.0744 & 2.7369 & 3.4478 \end{bmatrix}.$$

The resulting closed loop system is again of the form (6) where B and D are as before, but now

$$A = \begin{bmatrix} 0.0249 & -0.0057 & -0.1307 & 0.1316 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.0189 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0249 & -0.0057 & -0.1307 & 0.1316 \end{bmatrix},$$

$$D_0 = 0.0189.$$

In the design of control laws for discrete linear repetitive processes, stability along the pass will often only be the minimal requirement. In particular, a key task will be to ensure that the example under consideration retains this stability property in the presence of process parameter variations. The analysis which follows in the remainder of this section addresses this problem area from the stability margin standpoint and again uses the metal rolling model as an illustrative example.

As for 2D discrete linear systems described by the Roesser and Fornasini Marchesini state space models (see, e.g. (Agathoklis *et al.*, 1982)), the stability margin for discrete linear repetitive processes was defined (Rogers and Owens, 1992; Rogers *et al.*, 2003) as a measure of the degree to the shortest distance between the singularity of the process and the stability along the pass limit which is the boundary of the unit bidisc. Hence, the stability margin is a measure of the degree to which the process will remain stable along the pass under variations in the process state space model matrices which define this property.

The so-called generalized stability margin for discrete linear repetitive processes of the form described by (6) and (19) is defined as follows.

Definition 1. The *generalized stability margin*, denoted by σ_β, for discrete linear repetitive processes of the form described by (6) and (19) is defined as the largest bidisc in which the 2D characteristic polynomial of (23) satisfies

$$C(z_2, z_2) \neq 0, \tag{43}$$

in

$$\overline{U}^2_{\sigma_\beta} = \big\{(z_1, z_2)\colon |z_1| \leq 1+(1-\beta)\sigma_\beta, |z_2| \leq 1+\beta\sigma_\beta\big\},$$

where $0 \leq \beta \leq 1$.

Note that when $\beta = 0, 1$ and 0.5, respectively, the set $\overline{U}^2_{\sigma_\beta}$ here reduces (with obvious changes in the notation) to

$$\overline{U}^2_{\sigma_1} = \{(z_1, z_2) : |z_1| \leq 1 + \sigma_1,\ |z_2| \leq 1\}, \tag{44}$$

$$\overline{U}^2_{\sigma_2} = \{(z_1, z_2) : |z_1| \leq 1,\ |z_2| \leq 1 + \sigma_2\}, \tag{45}$$

$$\overline{U}^2_{\sigma} = \{(z_1, z_2) : |z_1| \leq 1 + \sigma,\ |z_2| \leq 1 + \sigma\}, \tag{46}$$

introduced and studied, e.g. in (Agathoklis *et al.*, 1982) for 2D discrete linear systems described by the Roesser state space model. (Note that (46) follows from setting $\sigma_{0.5} = 2\sigma$ in (43).) In particular, $(1-\beta)\sigma_\beta$ and $\beta\sigma_\beta$ give the stability margins corresponding to z_1 and z_2, respectively, i.e. along the pass and pass-to-pass, respectively. In what follows we will also need the following easily proven result:

Lemma 2. *Given $q_i \in \mathbb{R}$, $q_i > 0$, $i = 1, 2$ suppose that*

$$\widehat{C}(z_1, z_2) := \det\begin{bmatrix} I_n - z_1(1+q_1)A & -z_1(1+q_1)B_0 \\ -z_2(1+q_2)C & I_m - z_2(1+q_2)D_0 \end{bmatrix} \neq 0, \tag{47}$$

in $\overline{U}^2$. Then

$$C(z_1', z_2') = \det\begin{bmatrix} I_n - z_1'A & -z_1'B_0 \\ -z_2'C & I_m - z_2'D_0 \end{bmatrix} \neq 0, \tag{48}$$

in $\overline{U}^2_q$, where

$$\overline{U}^2_q = \{(z_1', z_2') : |z_1'| \leq 1 + q_1, \quad |z_2'| \leq 1 + q_2\}. \tag{49}$$

Now we can give the first main result on lower bounds for the stability margins defined above. A proof of this result can again be found in (Rogers *et al.*, 2003).

Theorem 5. *For a given β such that $0 \leq \beta \leq 1$, a lower bound for the generalized stability margin σ_β is given by the solution of the following quasi-convex optimization problem: Maximize σ_β subject to $P > 0$, $Q > 0$, $\sigma_\beta > 0$, and the LMI*

$$\begin{bmatrix} Q - P & 0 & (1+(1-\beta)\sigma_\beta)\,\widehat{A}_1^T P \\ 0 & -Q & (1+\beta\sigma_\beta)\,\widehat{A}_2^T P \\ X_{31} & (1+\beta\sigma_\beta)\,P\widehat{A}_2 & -P \end{bmatrix} < 0, \tag{50}$$

where $X_{31} = (1+(1-\beta)\sigma_\beta)\,P\widehat{A}_1$.

For a detailed explanation of the term 'quasi-convex optimization problem' used above, see (Boyd *et al.*, 1994).

In the numerical example used here, we have

$$\sigma_1 = 1.4033 \quad (\beta = 0),$$

$$\sigma_2 = 2.6666 \quad (\beta = 1),$$

$$\sigma = 0.7017 \quad (\beta = 0.5).$$

Also, in the closed loop case one possible additional objective is to achieve stability along the pass with a prescribed lower bound on the stability margins σ_1 and σ_2. Here we denote such bounds by σ_1^* and σ_2^*, respectively, and we have the following result (again proved in (Rogers *et al.*, 2003)).

Theorem 6. *Discrete linear repetitive processes of the form described by (6) and (19) are stable along the pass under control laws of the form (36) with K defined by (42) and with prescribed lower bounds on the stability margins σ_1^*, σ_2^*, corresponding to z_1 and z_2, respectively, if there exist symmetric matrices $Y > 0$, $Z > 0$, and a matrix N such that*

$$\begin{bmatrix} Z - Y & 0 & \hat{Y}_{13} \\ 0 & -Z & \hat{Y}_{23} \\ \hat{Y}_{13}^T & \hat{Y}_{23}^T & -Y \end{bmatrix} < 0, \tag{51}$$

where

$$\hat{Y}_{13} = (1+\sigma_1^*)\left(Y\widehat{A}_1^T + N^T\widehat{B}_1^T\right),$$

$$\hat{Y}_{23} = (1+\sigma_2^*)\left(Y\widehat{A}_2^T + N^T\widehat{B}_2^T\right).$$

4. Control of Differential Linear Repetitive Processes

In this section we address the control of differential linear repetitive processes again using the metal rolling as a physical example. First, we consider the direct application of delay differential systems theory and then follow this by linking it to stability theory for differential linear repetitive processes of the form considered here which again follows as a special case of that based on the abstract model (15). This leads to the so-called 2D Lyapunov equation based sufficient condition for stability along the pass which, in turn, leads to a new result which opens up the possibility of developing a very powerful LMI based approach to control system specification and design for these processes—a subject which has so far received relatively little attention in the literature.

4.1. Direct Application of Delay Differential Theory

In this part, we directly apply the existing delay differential systems theory to the model considered and then in the next subsection we establish links for differential linear repetitive processes. The proofs of the results used in this section can be found in standard references in the delay differential systems area, such as (Hale, 1977).

First note that the repetitive process model of (12)–(14) can also be treated as a delay differential system of the neutral type with two noncommensurate delays h_1 and h_2. (Delays $h_1, \dots, h_q$ are termed noncommensurate if there exist no integers $l_1, \dots, l_q$ (not all of them zero) such that $\sum_{i=1}^{q} l_i h_i = 0$. The underlying delay differential system is termed commensurate if $q = 1$.) Also introduce $z_i := e^{-h_i s}$, $i = 1, 2$, i.e. z_i is a left shift operator of duration h_i, and s denotes the Laplace transform variable. Then the characteristic polynomial associated with this model is a two-variable polynomial of the form

$$\rho(s, z_1, z_2) = s^2 + \sum_{i=0}^{2} \sum_{j_1=0}^{1} \sum_{j_2=0}^{1} c_{ij_1j_2} s^i z_1^{j_1} z_2^{j_2}. \quad (52)$$

It is also possible to treat (12)–(14) as a special case of the generalized linear system

$$G(z_1, z_2)\dot{x}(t) = H_1(z_1, z_2)x(t) + H_2(z_1, z_2)u(t), \quad (53)$$

where $G, H_1 \in \mathbb{R}^{p\times p}[z_1, z_2]$ and $H_2 \in \mathbb{R}^{p\times b}[z_1, z_2]$, z_i is (in this representation) a delay operator of duration h_i, $i = 1, 2$, and $\mathbb{R}[z_1, z_2]$ denotes the ring of polynomials in (z_1, z_2) with coefficients in $\mathbb{R}$. Also $\mathbb{R}[z_1, z_2]$ is, in general, a commutative ring and in the commensurate case (see the next subsection) it is also a principal ideal domain.

To detail this representation for the particular case considered here, take $z_{1,2}$ to be the shift operators defined by $z_i y_k(t) := y_k(t - h_i)$, $i = 1, 2$, and also introduce $x_1(t) := y_k(t)$, $x_2(t) := \dot{y}_k(t)$. Then, with $x(t) := [x_1(t), x_2(t)]^T$, (12)–(14) can be modelled by

$$G(z_1, z_2)\dot{x}(t) = H_1(z_1, z_2)x(t) + H_2(z_1, z_2)r_d(t), \quad (54)$$

where G, H_1 and H_2 are over the ring $\mathbb{R}[z_1, z_2]$ and, in detail,

$$G(z_1, z_2) = \begin{bmatrix} 1 & 0 \\ 0 & (1 - c_3 z_1) \end{bmatrix},$$

$$H_1(z_1, z_2) = \begin{bmatrix} 0 & 1 \\ g_1 & -2\zeta\omega_n(1 - c_3 z_1) \end{bmatrix},$$

$$H_2(z_1, z_2) = \begin{bmatrix} 0 \\ \dfrac{c_1 k_a k_c}{M} \end{bmatrix}, \quad (55)$$

$$g_1 = \omega_n^2 - \Big(\frac{c_2}{M} + \omega_n^2 c_3\Big) z_1 + \frac{c_1 k_a k_c}{M} z_2.$$

Now we require the following definition and result (for the general case).

Definition 2. A matrix $G(z_1, z_2) \in \mathbb{R}^{p\times p}[z_1, z_2]$ is said to be *atomic* at zero if $(G(0))$ is nonsingular over the field of real numbers.

Here we will always assume that this property holds. Also let $\overline{U}_\delta^2$ denote the closed bidisc $\overline{U}_\delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_i| \le 1 + \delta,\ 1 \le i \le 2\}$. Then the formal stability of $G(z_1, z_2)$ is defined as follows.

Definition 3. (Byrnes *et al.*, 1984) $G(z_1, z_2)$ is said to be *formally stable* if $\det(G(z_1, z_2)) \ne 0$, $\forall\ (z_1, z_2) \in \overline{U}_\delta^2$ for some $\delta > 0$.

Clearly, the particular case of $G(z_1, z_2)$ defined by (55) is atomic at zero and is formally stable since $c_3 = \lambda_2/(\lambda_1 + \lambda_2) < 1$. Hence we can invert $G(z_1, z_2)$ over the closed unit bidisc and (54) can be written as

$$\dot{x}(t) = H_3(z_1, z_2)x(t) + H_4(z_1, z_2)r_d(t), \quad (56)$$

where

$$H_3(z_1, z_2) = \begin{bmatrix} 0 & 1 \\ \hat{g}_1 & \hat{g}_2 \end{bmatrix},$$

$$H_4(z_1, z_2) = \begin{bmatrix} 0 \\ \hat{g}_3 \end{bmatrix}, \quad (57)$$

with

$$\hat{g}_1 = \frac{\left[\omega_n^2 - \left(\frac{c_2}{M} + \omega_n^2 c_3\right) z_1 + \frac{c_1 k_a k_c}{M} z_2\right]}{(1 - c_3 z_1)},$$

$$\hat{g}_2 = -2\zeta\omega_n, \tag{58}$$

$$\hat{g}_3 = \frac{c_1 k_a k_c}{M(1 - c_3 z_1)}.$$

Now set $h = \max h_i,\ i = 1, 2$, and (for the remainder of this subsection) let B denote the Banach space of continuous functions $[-h, 0] \mapsto \mathbb{R}$ with the norm defined by $\|f\| = \sup_{\sigma\in[-h,0]} |f(\sigma)|$, for any $f \in B$. Suppose also that $y(t)$ denotes the output of an autonomous delay differential system of the form considered here. Also let $y_t \in B$ denote the function segment defined by $y_t(\sigma) = y(t + \sigma),\ \sigma \in [-h, 0]$, and take the initial condition as $f(t),\ t \in [-h, 0],\ f \in B$. Then the asymptotic stability of this delay differential system is defined and characterized as follows.

Definition 4. A delay differential system of the form considered here is said to be *asymptotically stable* if $\exists\, M, \gamma > 0$ such that for each $f \in B$ the solution $y(t)$ with initial condition f satisfies

$$\|y(t)\| \le M|f|e^{-\gamma t}, \quad \forall\, t \ge 0. \tag{59}$$

Also, if (59) holds $\forall\, h_i \ge 0,\ 1 \le i \le 2$, then this property is termed *stability independent of delay* (denoted by i.o.d.).

Theorem 7. *A delay differential system of the form considered here is asymptotically stable if, and only if, its characteristic polynomial satisfies*

$$\rho(s, e^{-h_1 s}, e^{-h_2 s}) \neq 0, \quad \mathrm{Re}\, s \ge 0 \tag{60}$$

and is asymptotically stable independent of delay (i.o.d.) if, and only if, (60) holds $\forall\, h_i \ge 0,\ \ i = 1, 2$.

Corollary 1. *A delay differential system of the form considered here is asymptotically stable if, and only if,*

$$\det\left(sI_2 - H_3(e^{-h_1 s}, e^{-h_2 s})\right) \neq 0, \quad \forall\, s \in \overline{D} \tag{61}$$

and asymptotically stable i.o.d. if, and only if, (61) holds $\forall\, h_i \ge 0,\ i = 1, 2$*, where* $\overline{D}$ *denotes the closed right-half of the* s *plane.*

The following result (Herz *et al.*, 1984) gives an analytic test for asymptotic stability i.o.d.

Theorem 8. *A delay differential system of the form considered here is asymptotically stable i.o.d. if, and only if,*
(a) $\rho(s, 1) \neq 0,\ s \in \overline{D}$,
(b) $\rho(s, -1) \neq 0, s \in R\ \ (s \neq 0)$; *and*
(c)

$$(1 + sT)^2 \rho\left(s, \frac{1 - sT}{1 + sT}\right) \neq 0, \quad s \in R, \ \ \forall\, T > 0, \tag{62}$$

where R *denotes the imaginary axis of the* s *plane.*

Pointwise asymptotic stability is a stronger concept than that above and is defined as follows.

Definition 5. A delay differential system of the form considered here with a characteristic polynomial $\rho(s, z_1, z_2)$ is said to be pointwise asymptotically stable if, and only if,

$$\rho(s, z_1, z_2) \neq 0, \quad \forall (s, z_1, z_2) \in \overline{D} \times \overline{U}^2. \tag{63}$$

Now it follows immediately that a delay differential system of the form considered here is pointwise asymptotically stable if, and only if,

$$\det\left(sI_2 - H_3(z_1, z_2)\right) \neq 0, \quad \forall\, (s, z_1, z_2) \in \overline{D} \times \overline{U}^2. \tag{64}$$

4.2. Links to Repetitive Process Stability Theory

The starting point in what follows is the calculation of bounds for the PD controller gains k_a, k_b and k_c to guarantee the stability of the controlled process described by the delay differential equation (12)–(14). Here h_1 and h_2 are, in general, not constant and hence asymptotic stability in delay intervals and asymptotic stability i.o.d. should be considered in this case. A major difficulty here is that, in general, it is very difficult to check any of the resulting stability conditions for the noncommensurate case. Consequently, we have to consider pointwise asymptotic stability using (64).

To proceed, note that the condition of (64) can be further simplified due to the fact that the regularity of $H_3(z_1, z_2)$ in (56) over the closed unit bidisc $\overline{U}^2$ implies that its eigenvalues, denoted by λ_i, are regular functions of (z_1, z_2). Hence if $\mathrm{Re}\, \lambda_i > 0$ for some $(z_1^o, z_2^o) \in \overline{U}^2$, then $\mathrm{Re}\ \lambda_i > 0,\ \forall\, (z_1, z_2)$ in an open neighborhood of (z_1^o, z_2^o). Thus instability in the pointwise asymptotic sense here can be detected by checking the eigenvalues of $H_3(z_1, z_2)$ on the distinguished boundary of the unit bidisc, i.e. $T^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_i| = 1,\ i = 1, 2\}$. Also, this can be implemented for a given example by a suitably fine partition of T^2, which is a finite computation. Alternatively, eigenvalue location tests can be implemented by applying known stability tests (see, e.g. (Barnett, 1983)) for polynomials with complex coefficients to its characteristic polynomial.

Suppose now that the pass delay is an integer multiple of the output sensor measurement delay or the sensor delay can be neglected. Then in these cases the delay differential models of the previous section will be commensurate and here we assume that the latter of these cases holds. Then, with an additional assumption of zero reference input, the delay differential equation (12)–(14) becomes

$$\begin{aligned}\ddot{y}_k(t) &+ 2\zeta\omega_n\dot{y}_k(t) + \Big(\omega_n^2 + \frac{c_1k_ak_c}{M}\Big)y_k(t)\\ &- c_3\ddot{y}_k(t-h_1) - 2\zeta\omega_n c_3\dot{y}_k(t-h_1)\\ &- \Big(\omega_n^2c_3 + \frac{c_2}{M}\Big)y_k(t-h_1) = 0 \qquad (65)\end{aligned}$$

with the corresponding characteristic polynomial

$$\begin{aligned}\rho(s,z_1) &= s^2 + 2\zeta\omega_n s + \Big(\omega_n^2 + \frac{c_1k_ak_c}{M}\Big)\\ &- c_3s^2z_1 - 2\zeta\omega_n c_3 s z_1\\ &- \Big(\omega_n^2c_3 + \frac{c_2}{M}\Big)z_1 = 0, \qquad (66)\end{aligned}$$

where we also assume that the gains k_a, k_b and k_c are positive.

The following result follows immediately from applying the analytic stability test of Theorem 8.

Theorem 9. *Consider the repetitive process (65), where the gains k_a, k_b and k_c are assumed to be positive. Then this commensurate metal rolling process is asymptotically stable i.o.d. if, and only if,*

$$k_a > \frac{c_2 - \lambda(1-c_3)}{(k_c\lambda/\lambda_2) + 1 - c_3} \qquad (67)$$

and

$$k_ak_c < \lambda_1 + \frac{\lambda_1(1-c_3^2)}{2c_2M}k_b^2. \qquad (68)$$

Proof. The first condition of Theorem 8 is easily seen to be equivalent to

$$\omega_n^2 + \frac{c_1k_ak_c - c_2}{(1-c_3)M} > 0, \qquad (69)$$

and (67) here follows immediately.

To prove (68), the second condition of Theorem 8 in this case requires that

$$\omega_n^2 + \frac{c_1k_ak_c + c_2}{(1+c_3)M} > 0, \qquad (70)$$

which holds for any positive k_a and k_c. Finally, the third condition of Theorem 8 requires (after some routine analysis) in this case that

$$s^3 + \hat{a}_1s^2 + \hat{a}_2s + \hat{a}_3 > 0, \quad \forall\, s \in \overline{D}, \qquad (71)$$

(i.e. the left-hand side be positive in the closed right-half of the s plane) where

$$\begin{aligned}\hat{a}_1 &= 2\zeta\omega_n + \frac{(1-c_3)}{(1+c_3)T},\\ \hat{a}_2 &= \omega_n^2 + 2\zeta\omega_n\frac{(1-c_3)}{(1+c_3)T} + \frac{c_1k_ak_c + c_2}{(1+c_3)M},\\ \hat{a}_3 &= \omega_n^2\frac{(1-c_3)}{(1+c_3)T} + \frac{c_1k_ak_c - c_2}{(1+c_3)MT}. \qquad (72)\end{aligned}$$

Some routine analysis now yields that (71) is equivalent to (68) here, and the proof is complete. ■

An autonomous delay differential system of the commensurate type can also be modelled by a 2D state space model of the form

$$\begin{bmatrix}\dot{x}_1(t)\\ x_2(t+\gamma)\end{bmatrix} = \begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}, \qquad (73)$$

where $x_1(t) \in \mathbb{R}^{n_1}$ denotes the differential state vector, $x_2(t) \in \mathbb{R}^{n_2}$ denotes the delay state vector, and for differential linear repetitive processes of the form considered here γ is equal to the pass length α. The characteristic polynomial for (73) is defined as

$$\rho_c(s,z) := \det\begin{bmatrix}sI_{n_1} - A_1 & -A_2\\ -zA_3 & I_{n_2} - zA_4\end{bmatrix}, \qquad (74)$$

which can be written in the form (52) in the special case of (12)–(14). In particular, the process under consideration here is a special case of (74) with

$$\begin{aligned}A_1 &= \begin{bmatrix}-\zeta\omega_n & (\zeta^2\omega_n^2 - \omega_n^2 - \frac{c_1k_ak_c}{M})\\ 1 & -\zeta\omega_n\end{bmatrix},\\ A_2 &= \begin{bmatrix}0\\ \hat{a}\end{bmatrix},\\ A_3 &= \begin{bmatrix}1 & 0\end{bmatrix},\\ A_4 &= c_3, \qquad (75)\\ \hat{a} &= c_3(c_2 - c_1k_ak_c)/(\omega_n^2M + c_1k_ak_c - \zeta^2\omega_n^2M).\end{aligned}$$

Suppose now that all eigenvalues of the matrix A_1 have strictly negative real parts and note that $A_4 < 1$. Then, when assuming zero initial conditions, (73) (with $\gamma = \alpha$) can in the special case of (12)–(14) be rewritten in the form

$$\begin{aligned}x_2(t+\alpha) &= \big(A_3(sI_2 - A_1)^{-1}A_2 + A_4\big)x_2(t)\\ &= G_1(s)x_1(t), \qquad (76)\end{aligned}$$

where $G_1(s)$ is the so-called interpass transfer function. We will also require the following definition.

Definition 6. A proper rational function $G(s)$ is termed *strictly continuous bounded real* (SCBR) if, and only if,
(a) $G(s)$ is analytic in $\operatorname{Re} s \geq 0$, and
(b) $1 - G^T(-\imath\omega)G(\imath\omega) > 0, \ \forall\, \omega \in \mathbb{R}$.

Stability along the pass for processes described by (76) can now be stated (Rogers and Owens, 1992; Rogers *et al.*, 2003) as follows.

Definition 7. The repetitive process (73) is *stable along the pass* if, and only if,

$$\|G_1^k(\imath\omega)\| \to 0 \quad \text{as } k \to \infty, \quad 0 \leq \omega < \infty, \tag{77}$$

where $\|G_1(\imath\omega)\| := \sup_{0\leq\omega<\infty} |G(\imath\omega)|$, i.e. each frequency component is attenuated from pass-to-pass.

It is easily shown (by inspecting the roots of the characteristic polynomial (i.e. $\det(sI_2 - A_1)$) that $G_1(s)$ is analytic in $\overline{D}$. Also, it can be shown (Foda and Agathoklis, 1989) that a necessary and sufficient condition for stability along the pass is that $G_1(s)$ is SCBR. Hence it is clear that stability along the pass is equivalent to $G_1(s)$ being SCBR. Since the characteristic polynomial $\rho(s, z)$ is first order in the delay operator, it follows that pointwise asymptotic stability is equivalent to $G_1(s)$ being SCBR. Hence stability along the pass for SISO differential linear repetitive processes is equivalent to pointwise asymptotic stability for delay differential systems—which is a strong concept of stability i.o.d.

The following result (first obtained in the delay differential systems literature (Agathoklis and Foda, 1989)) expresses stability along the pass in terms of a so-called 2D Lyapunov equation.

Theorem 10. *Differential linear repetitive processes which can be expressed in the form (73) are stable along the pass if there exist matrices $W = W_1 \oplus W_2 > 0$ and $Q > 0$ such that the following 2D Lyapunov equation holds:*

$$\tilde{A}^T W^{1,0} + W^{1,0}\tilde{A} + \tilde{A}^T W^{0,1}\tilde{A} - W^{0,1} = -Q, \tag{78}$$

where $\tilde{A}$ is the so-called augmented plant matrix given by

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \tag{79}$$

$W^{1,0} := W_1 \oplus 0,\ W^{0,1} := 0_{2\times 2} \oplus W_2$, *and $\oplus$ denotes the direct sum of two matrices, i.e. for matrices F_1 and F_2, $F_1 \oplus F_2 := \operatorname{diag}\{F_1, F_2\}$.*

Note that in the SISO case the result of Theorem 10 is both necessary and sufficient.

A major implication of this last result is that it can be used to provide a basis on which to begin the development of an LMI based approach to the analysis and design of control schemes for the differential linear repetitive processes considered here. To develop an LMI solution of the 2D Lyapunov equation of the previous result, first note that (78) can be rewritten in the form

$$\widetilde{A}_2^T \widetilde{W}_2 \widetilde{A}_2^T - W^{0,1} + \widetilde{A}_1^T W^{1,0} + W^{1,0}\widetilde{A}_1 < 0, \tag{80}$$

where $\widetilde{W}_2 = W_3 \oplus W_2,\ W_3$ is an arbitrary symmetric positive definite $n \times n$ matrix and

$$\widetilde{A}_1 = \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{A}_2 = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix}. \tag{81}$$

Now apply the Schur complement and then pre- and post-multiply (81) by the matrix $(I \oplus \widetilde{W}_2)$ to yield the equivalent condition

$$\begin{bmatrix} -W^{0,1} + \widetilde{A}_1^T W^{1,0} + W^{1,0}\widetilde{A}_1 & \widetilde{A}_2^T \widetilde{W}_2 \\ \widetilde{W}_2 \widetilde{A}_2 & -\widetilde{W}_2 \end{bmatrix} < 0, \tag{82}$$

which is clearly in the LMI form, and we have the following result.

Theorem 11. *The differential linear repetitive processes of the form considered here are stable along the pass if the LMI of (82) is feasible.*

Currently, the implications and full exploitation of this last result is under investigation and has already led to major progress—especially in the specification and design of control laws. Some early results in this area can be found in (Gałkowski *et al.*, 2002a).

5. Conclusions

This paper has considered the application of theory developed for the control of differential and discrete linear repetitive processes using models arising in metal rolling operations as examples. The new major feature is the emergence of LMI based analysis as a potentially very powerful analysis base for, in particular, the specification and design of control laws, which is an area of critical importance but which has so far seen relatively little progress.

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