In memory of Prof. A. A. Pervozvansky

ITERATIVE LEARNING CONTROL FOR OVER-DETERMINED, UNDER-DETERMINED, AND ILL–CONDITIONED SYSTEMS

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This paper studies iterative learning control (ILC) for under-determined and over-determined systems, i.e., systems for which the control action to produce the desired output is not unique, or for which exact tracking of the desired trajectory is not feasible. For both cases we recommend the use of the pseudoinverse or its approximation as a learning operator. The Tikhonov regularization technique is discussed for computing the pseudoinverse to handle numerical instability. It is shown that for over-determined systems, the minimum error is never reached by a repetition invariant learning controller unless one knows the system exactly. For discrete time uniquely determined systems it is indicated that the inverse is usually ill-conditioned, and hence an approximate inverse based on a pseudoinverse is appropriate, treating the system as over-determined. Using the structure of the system matrix, an enhanced Tikhonov regularization technique is developed which converges to zero tracking error. It is shown that the Tikhonov regularization is a form of linear quadratic ILC, and that the regularization approach solves the important practical problem of how to intelligently pick the weighting matrices in the quadratic cost. It is also shown how to use a modification of the Tikhonov-based quadratic cost in order to produce a frequency cutoff. This robustifies good learning transients, by reformulating the problem as an over-determined system.

Keywords: Iterative Learning Control, over-determined, under-determined, ill-conditioned systems, pseudoinverse

1. Introduction and Problem Formulation

In this paper we study general Iterative Learning Control (ILC) laws for over-determined and under-determined systems as well as the regularization technique for ill-conditioned discrete time systems. Over- (under-) determined systems are systems where the desired output cannot be exactly achieved by any control, or systems in which several input control values can be chosen to obtain the desired output. There are many applications of iterative learning control: robotic manipulators, hard disk drives, chemical processing, etc. (Arimoto *et al.*, 1984; Longman, 1998; Moore, 1993; 1997; Owens *et al.*, 1995; Pervozvansky, 1995b; Rogers and Owens, 1992) Some of them, such as robotic manipulators, can clearly be either under-determined or over-determined systems.

Iterative learning control is designed to improve the performance of cyclical systems, and its basic idea is to use the information from the previous cycles to improve the system performance on the current cycle (Arimoto *et al.*, 1984; Longman, 1998; Moore, 1997; Rogers and Owens, 1992).

As in (Avrachenkov, 1998; Avrachenkov *et al.*, 1999), let us consider a controlled system represented by the operator F acting on Hilbert spaces U and Y, that is $F : U \to Y$, where U is the control space and Y is the space of observations or system outputs. Unless noted explicitly, the vector norm is given by the scalar product ($\|x\| = \sqrt{\langle x, x\rangle}$) and the operator norm is induced by this scalar product norm. Since we consider a cyclical system, the system evolution is given by

$$y_k = F(u_k), \quad k = 0, 1, \dots, \tag{1}$$

where $u_k \in U$ and $y_k \in Y$ are the control input and the observed output of the system at the k-th cycle, respectively. Let y_d be the desired output of the system.

Our aim is to find a control u_d which is a solution of the operator equation

$$y_d = F(u). \tag{2}$$

Of course, we suppose that we do not know exactly the operator F, otherwise Eqn. (2) can be solved by classical methods (Dennis and Schnabel, 1983) provided the computation is not too ill-conditioned. Since we are dealing with cyclically operating systems, iterative learning control can be used to solve the operator equation (2) *on line*. Here we consider the linear iterative learning control scheme (Arimoto *et al.*, 1984; Longman, 1998; Owens *et al.*, 1995; Pervozvansky, 1995a; Rogers and Owens, 1992):

$$u_{k+1} = u_k - Lz_k, \quad k = 0, 1, \dots \tag{3}$$

where $z_k = y_k - y_d$ is the error at the k-th cycle and L is a linear *learning* operator.

Now let us give a strict definition of over-determined and under-determined systems. We say that a system is *under-determined* if for some outputs there is no unique control action, and we say that a system is *over-determined* if there exist some values of the output that cannot be achieved, i.e., the range of operator F is a strict subset of the observation space Y. In the case of under-determined systems we are interested in the solution of Eqn. (2) which minimizes the norm $\|u\|$, whereas in the case of over-determined systems we are interested in the minimization of $\|y_d - F(u)\|$ (Longman *et al.*, 1989). Of course, in general, the minimum of $\|y_d - F(u)\|$ is not zero. Moreover, we show that for over-determined systems one cannot achieve this minimum using the learning procedure (3). This is a sharp contrast to the systems which are in one-to-one correspondence with the control actions and outputs. In the latter case the minimum of $\|y_d - F(u)\|$ is equal to zero and can be achieved by the appropriate choice of the learning operator.

In the next section we study linear systems. The linear theory can be easily understood and it clearly demonstrates the essence of the problems as well as possible approaches to their solution. Then, in the third section, we discuss the regularization technique as well as generalization to non-linear systems. Finally, in the last section we show that the pseudoinverse method and regularization techniques can also be applied to discrete time systems with ill-conditioned operators.

2. Linear Systems

In this section we study the case of a general linear system described by a bounded linear operator F. We assume that the range $\mathcal{R}(F)$ of the system operator is closed. This ensures the existence of a pseudoinverse (or Moore-Penrose generalized inverse) operator $F^\dagger$ (Ding and Huang, 1996). In this paper we make an extensive use of the Moore-Penrose generalized inverse. A comprehensive study of the Moore-Penrose generalized inverse can be found in the book of Campbell and Meyer (1979).

By straightforward calculations one can obtain from (1) and (3) the following recursion formula (Longman, 1998; Moore, 1997; Rogers and Owens, 1992):

$$z_{k+1} = [I - FL]z_k, \quad k = 0, 1, \dots . \tag{4}$$

Note that if F is a one-to-one operator, then the condition

$$\|I - FL\| < 1 \tag{5}$$

is satisfied for a large class of learning operators. In particular, this condition implies that the sequence $\{z_k\}$ converges to zero in norm.

Now let us consider over-determined systems. The recurrent equation (4) still holds, but the condition $\|I - FL\| < 1$ cannot be satisfied by choosing any L. Let us show this in the case of finite-dimensional spaces U and Y. It is known (Beklemishev, 1983) that if we take the Euclidean norm, i.e. $\|A\|_E = (\sum_{j=1}^n \sum_{i=1}^n |a_{ij}|^2)^{1/2}$, then $\|I - FL\|_E$ achieves its minimum at $L = F^\dagger$, where $F^\dagger$ is the pseudoinverse (or the Moore-Penrose generalized inverse) of the operator F. That is,

$$\|I - FF^\dagger\|_E \le \|I - FL\|_E \tag{6}$$

for any operator L. Note that $FF^\dagger$ is an orthogonal projection mapping Y onto $\mathcal{R}(F)$. Therefore $I - FF^\dagger$ is equal to zero only for systems described by a one-to-one operator or for under-determined systems. Next we consider over-determined systems for which $I - FF^\dagger \neq 0$. We note that $\|A\|_2 \le \|A\|_E$, where $\|\cdot\|_2$ is the norm induced by the scalar product (Dennis and Schnabel, 1983). Next we recall that $I - FF^\dagger$ is an orthogonal projection and hence $\|I - FF^\dagger\|_2 = 1$. Thus

$$1 = \|I - FF^\dagger\|_2 \le \|I - FF^\dagger\|_E \le \|I - FL\|_E$$

and, consequently, the condition (5) cannot be satisfied for over-determined systems. Also, we conclude from inequality (6) that probably the best strategy for over-determined systems is to choose the learning operator L as close to the pseudoinverse $F^\dagger$ as possible.

An intuitive explanation of the fact that $\|I - FL\| \ge 1$ for over-determined systems is clear: There exists such values of the output which cannot be achieved by the system using any control action and hence the sequence $\{z_k\}$ cannot converge to zero with any L. However, if $y_d \notin \mathcal{R}(F)$, we are interested in the minimization of $\|y_d - Fu\|$. That is, ideally we want to choose a learning operator such that the sequence $\{z_k\}$ converges to $z_d := Fu_d - y_d \neq 0$, where $u_d = \arg\min \|y_d - Fu\|$. Recall that $u_d = F^\dagger y_d$ provides a solution to the optimization problem $\min_u \|y_d - Fu\|$ (Beklemishev, 1983; Campbell and Meyer, 1979). Thus

$$z_d = Fu_d - y_d = FF^\dagger y_d - y_d = [FF^\dagger - I]y_d. \quad (7)$$

In the next theorem we give convergence conditions of the general iterative learning procedure (3) for linear over-determined systems. In particular, we will show that unless one knows the system exactly, the limit of the sequence $\{z_k\}$ does not equal z_d. Namely, by using the learning algorithm (3) we are not able to minimize $\|y_d - Fu\|$. Nevertheless, as the theorem shows, we are able to find a good approximation of the optimal solution.

Theorem 1. *Let the learning operator satisfy the condition*

$$\|F^\dagger - L\| < \frac{1}{\|F\|}. \quad (8)$$

Then the iterative learning procedure (3) converges, i.e.

$$\lim_{k\to\infty} z_k = z_\infty = \left[I - F(F^\dagger - L)\right]^{-1} z_d, \quad (9)$$

where z_d *is given by (7). Furthermore, if* $y_d \in \mathcal{R}(F)$,

$$\lim_{k\to\infty} z_k = 0.$$

Proof. Let us consider the recursion formula (4) and rewrite the operator $I - FL$ as follows:

$$\begin{aligned} I - FL &= I - FF^\dagger + FF^\dagger - FL \\ &= I - FF^\dagger + F(F^\dagger - L). \end{aligned}$$

Next we consider the product

$$[I - FF^\dagger]F(F^\dagger - L) = [F - FF^\dagger F](F^\dagger - L).$$

Since the pseudoinverse operator $F^\dagger$ satisfies the equation $FF^\dagger F = F$ (Campbell and Meyer, 1979), we have

$$[I - FF^\dagger]F(F^\dagger - L) = 0.$$

Next, using the above property and the fact that $I - FF^\dagger$ is an orthogonal projection, we can give the following expression of the powers of $I - FL$:

$$\begin{aligned} [I - FL]^k &= \left[(I - FF^\dagger) + F(F^\dagger - L)\right]^k \\ &= (I - FF^\dagger) + F(F^\dagger - L)(I - FF^\dagger) \\ &\quad + \cdots + \left[F(F^\dagger - L)\right]^{k-1}(I - FF^\dagger) \\ &\quad + \left[F(F^\dagger - L)\right]^k \\ &= (I + F(F^\dagger - L) \\ &\quad + \cdots + \left[F(F^\dagger - L)\right]^{k-1})(I - FF^\dagger) \\ &\quad + \left[F(F^\dagger - L)\right]^k. \end{aligned}$$

From the condition (8) we conclude that the series $I + F(F^\dagger - L) + [F(F^\dagger - L)]^2 + \cdots$ is absolutely convergent and, consequently

$$[I - FL]^k \to \left[I - F(F^\dagger - L)\right]^{-1}(I - FF^\dagger),$$

as $k \to \infty$. Then we show that $(I - FF^\dagger)z_0 = z_d$:

$$\begin{aligned} (I - FF^\dagger)z_0 &= (I - FF^\dagger)(Fu_0 - y_d) \\ &= (F - FF^\dagger F)u_0 + [FF^\dagger - I]y_d \\ &= [FF^\dagger - I]y_d = z_d. \end{aligned}$$

Finally, if $y_d \in \mathcal{R}(F)$, $z_d = FF^\dagger y_d - y_d = y_d - y_d = 0$, since $FF^\dagger$ is an orthogonal projection on $\mathcal{R}(F)$, and hence $z_\infty = 0$. This completes the proof. ■

The following straightforward norm estimation of the limiting error can be immediately obtained.

Corollary 1. *The upper bound on the norm of the error in the limit in the case of an over-determined system is given by*

$$\|z_\infty\| \le \frac{1}{1 - \|F\|\|F^\dagger - L\|}\|z_d\|.$$

There are several useful conclusions that can be drawn from the results of Theorem 1. First, the expression (9) for the limiting error suggests that the learning operator L should be chosen as close to the pseudoinverse $F^\dagger$ of the system operator as possible. The later guarantees that we will find a good approximation of the solution to the minimization problem $\min \|y_d - Fu\|$. Second, the convergence of (3) takes place if the condition (8) is satisfied. This condition can be easily satisfied in the case when the operators L and $F^\dagger$ act on the same subspaces. However, it is not the case in the presence of singular perturbations (Avrachenkov and Pervozvansky, 1998a; 1998b; Pervozvansky and Avrachenkov, 1997). The updating ILC procedures discussed below could be a solution to this problem.

Finally, one can see that when $y_d \notin \mathcal{R}(F)$, unless one knows exactly the system operator F, it is not possible to achieve the minimum of $\|y_d - Fu\|$ using the learning procedure (3) with any choice of L. Therefore, ILC procedures with the *updating* learning operator (Avrachenkov *et al.*, 1999; Beigi, 1997; Longman *et al.*, 1989) can be very useful for over-determined systems. Namely, the updating methods (Avrachenkov *et al.*, 1999; Beigi, 1997; Longman *et al.*, 1989) construct the sequence of learning operators $\{L_k\}$ which converge to $F^\dagger$, thus one may expect that the updating learning procedure will provide an exact solution to the optimization problem $\min_u \|y_d - Fu\|$.

As was mentioned at the beginning of the section, for systems described by a one-to-one operator the learning procedure (3) converges (under certain conditions on L) to a solution of the operator equation (2). The next theorem demonstrates that the application of the learning procedure (3) to under-determined systems produces similar results.

Theorem 2. *Let the controlled system be under-determined, i.e.,* $\mathcal{R}(F) = Y$, *and let the following condition be satisfied:*

$$\|F^\dagger - L\| < \frac{1}{\|F\|}.$$

Then the learning procedure (3) converges to a solution of (2) and the following next norm bound holds:

$$\|z_{k+1}\| \le \|F\|\|F^\dagger - L\|\|z_k\|, \quad k = 0, 1, \ldots. \tag{10}$$

Proof. Recall that $FF^\dagger$ is an orthogonal projection onto $\mathcal{R}(F)$, the range of the system operator. Since the system is under-determined, $\mathcal{R}(F) = Y$ and hence $FF^\dagger = I$. Thus we can write

$$I - FL = FF^\dagger - FL = F[F^\dagger - L].$$

This immediately implies (10). Next, the norm inequality (10) ensures convergence if $\|F^\dagger - L\| < 1/\|F\|$. ■

Note that the convergence condition in Theorem 2 is the same as the condition (8) in Theorem 1. This shows that in the cases of both over-determined and under-determined systems, one should try to choose the learning operator as close to the pseudoinverse $F^\dagger$ as possible.

3. Regularization Methods and Extension to Non-Linear Models

As was pointed out in the previous section, if the system operator is not invertible, we should take a good approximation of the pseudoinverse operator of the system as the learning operator. Namely, let $\tilde{F}$ be some approximation (or known part) of the controlled system. Then, in theory, we must take $L = \tilde{F}^\dagger$. However, in practice, the computation of $\tilde{F}^\dagger$ is quite demanding and often numerically unstable (Campbell and Meyer, 1979). This is especially the case for singularly perturbed systems (Avrachenkov and Pervozvansky, 1998a; 1998b; Pervozvansky and Avrachenkov, 1997). To overcome this problem, we suggest to use the Tikhonov regularization (Beklemishev, 1983; Tikhonov and Arsenin, 1974). Namely, it is known that the pseudoinverse operator can be expressed as the following limit (Beklemishev, 1983; Campbell and Meyer, 1979; Tikhonov and Arsenin, 1974):

$$F^\dagger = \lim_{\alpha \to 0} (F^* F + \alpha^2 I)^{-1} F^*,$$

where F^* is the adjoint operator to F. Thus, in practice one has to solve the next system for $\Delta_{k+1} u = u_{k+1} - u_k$:

$$(\tilde{F}^* \tilde{F} + \alpha^2 I)\Delta_{k+1} u = -\tilde{F}^* z_k, \tag{11}$$

where the parameter α is chosen in such a way that the matrix $\tilde{F}^* \tilde{F} + \alpha^2 I$ is well conditioned. A regularization for ILC similar to (11) was first proposed in the paper of Pervozvansky (1995a). Note that (11) is the stationary point equation for the minimization problem (Beklemishev, 1983; Tikhonov and Arsenin, 1974)

$$\min_{\Delta_{k+1} u} \left\{ \|z_k + \tilde{F}\Delta_{k+1} u\|^2 + \alpha^2 \|\Delta_{k+1} u\|^2 \right\}. \tag{12}$$

The last formulation allows us to generalize the method for the case of non-linear systems. That is, the control for the next iteration is calculated by $u_{k+1} = u_k + \Delta_{k+1} u$, where $\Delta_{k+1} u$ is a solution of the following optimization problem:

$$\min_{\Delta_{k+1} u} \left\{ \|z_k + D\tilde{F}(u_k)\Delta_{k+1} u\|^2 + \alpha^2 \|\Delta_{k+1} u\|^2 \right\},$$

where $D\tilde{F}(u_k)$ is the Fréchet derivative of the system operator $F(u)$ at $u = u_k$. One can also construct a learning procedure which solves *on line* the following non-linear (and not quadratic) optimization problem:

$$\min_{u} \left\{ \|y_d - F(u)\|^2 + \alpha^2 \|u\|^2 \right\}$$

instead of the operator equation (2). The convergence of these *regularization* learning procedures for non-linear systems will be investigated in future publications.

4. Creating an Enhanced Tikhonov Regularization for Ill-Conditioned Systems

The previous sections treated both under-determined and over-determined systems, and established the importance of using a pseudoinverse matrix F, as well the possible need for regularization. Now we turn to uniquely determined systems, neither under- nor over-determined, with the same number of inputs as outputs. Linear discrete time system models are considered. First we discuss the fact that the matrix F is nearly always ill-conditioned in discrete time linear systems. The design of ILC laws needs techniques to handle this difficulty. Here we discuss and develop several approaches. The first one gets around the ill-conditioned nature of the problem by replacing it with an over-determined system. Then the results of the previous sections apply again. The second approach directly confronts the ill-conditioning and applies the Tikhonov regularization. However, we have considerable insight into the structure of the problem. As a result, an enhanced form of the Tikhonov regularization is developed that quickly learns what it can learn quickly, and uses regularization only for that part of the system that is hard to invert. This forms a new, well tuned and promising ILC law.

4.1. Ill-Conditioning of Inverse Control in Discrete Time Systems

When a system governed by a linear differential equation is fed by a zero order hold as in typical digital control, it is possible to describe the input-output relationship by a difference equation, and to do so without approximation. When the pole excess of the continuous time transfer function is greater than one, this discretization process will generically introduce enough zeros in the z-transfer function that it will have a pole excess of 1. Asymptotically, as the sample time gets large, when one new zero is introduced, it will appear at -1; when two zeros are introduced one will be inside the unit circle and the other will be outside of it; when three zeros are introduced one will be inside, one outside and the third at -1, etc. (Astrom *et al.*, 1980). This means that in the majority of practical applications, there are zeros in the discrete time model that are outside the unit circle. If one wants to invert the system to determine the control needed to produce a chosen output, one substitutes the desired solution into the output terms of the difference equation, and then must solve it for the input that can make the sum of the input terms match the sum of the output terms. But with a zero outside the unit circle, this is the solution of an unstable difference equation, which usually precludes the use of inverse control. In a matrix formulation, the same effect is manifested as the ill-conditioning of a matrix inverse computation. The linear operator F in this discrete time problem is a matrix of Markov parameters giving the output history of the system as the product of F times the input history (plus an initial condition term that is the same every repetition). Various methods are available for finding Markov parameters, including one sometimes described as a subspace method and called OKID, or observer Kalman filter identification (Juang *et al.*, 1993). We assume in this paper that one is able to find reasonable entries for this matrix, and that the matrix is not so large as to preclude finding an inverse, except when it is ill-conditioned. It is normally of the full rank—all that is required is the first Markov parameter to be non zero (or in the multi-input, multi-output case, the first Markov block element to be of the full rank). Then all singular values of F are non zero. However, every unstable zero of the system z-transfer function is associated with a small singular value that makes the matrix ill-conditioned. For typical trajectory lengths with typical sample rates, the matrix F is large and sufficiently ill-conditioned so that the computation of the inverse control become impossible. It is perhaps worth pointing out that in spite of the ill-conditioning of the matrix, in typical learning control applications this does not imply that there may be large control actions needed. One normally asks for a desired trajectory that is clearly feasible for the system, and hence a true inverse of the ill-conditioned matrix produces reasonable control magnitudes.

4.2. Treating Ill-Conditioned Systems as Over-Determined Systems

Since there will normally be only a few singular values that are particularly small, and which prevent one from taking the inverse of F, it is natural to consider ignoring this part of the input space. The dimension of the input space is the number of inputs to the system model times the number of time steps in the desired trajectory. This number could easily be of the order of 1000. The number of particularly small singular values is very roughly one half the pole excess of the original continuous time system, for a sufficiently fast sample rate. For a third order system with no zeros, the number of small singular values will be one. Ignoring one singular value out of 1000 seems reasonable, and this would produce an over determined problem with an input space of 999 values and an output space of the dimension 1000.

At the end of Section 2, it was discussed that in over-determined cases one should pick the learning operator as close as possible to the pseudoinverse of F. Here we are treating a limiting case of this, picking the learning operator as the best approximation to the true inverse that we can find, a pseudoinverse that inverts all parts of the space except those related to the particularly small singular values.

This approach can be written in mathematical detail as follows. Let the singular value decomposition of F be $F = U\Sigma V^* = [U_1\ U_2]\,\mathrm{diag}(\Sigma_1, \Sigma_2)[V_1\ V_2]^*$, where diag represents the square block diagonal matrix with the diagonal blocks as indicated. Σ_1 and Σ_2 are diagonal matrices of singular values, with Σ_2 containing the small singular values that cause the ill-conditioning. An approximation to F is given by $\underline{F} = [U_1\ U_2]\,\mathrm{diag}(\Sigma_1, 0)[V_1\ V_2]^* = U_1\Sigma_1 V_1^*$, obtained by setting Σ_2 to zero. Setting it to zero eliminates the ill-conditioning, creating a singular matrix instead. Next we pick the learning gain matrix L to be the Moore-Penrose pseudoinverse of $\underline{F}$, i.e., $L = V_1\Sigma_1^{-1}U_1^*$ ((Oh *et al.*, 1997) uses this approach to create local learning strategies). Then (4) can be rewritten as follows:

$$[U_1\ U_2]^* z_{k+1} = \mathrm{diag}(0, I)[U_1\ U_2]^* z_k, \qquad (13)$$

where y_k, u_k, z_k are column matrices of the output, input, and error histories, respectively. This equation shows that provided the model is perfect, the iterations converge in one repetition, making the error zero in the space spanned by the orthonormal columns of U_1 and leaving the error in the space spanned by U_2 unaltered. The steady state error is $z_\infty = U_2 U_2^* z_0$. To make Theorem 1 applicable to this problem, replace $\tilde{F}$ by $\underline{\tilde{F}}$ in (7), (8) and (9). This approach is a reasonable way to eliminate the problem of ill-conditioning, and to produce a substantial decrease in the tracking error. However, if we want a method that also eliminates the error in the space spanned by U_2, some form of regularization is needed, as discussed in the next section.

4.3. Development of an Enhanced Tikhonov Regularization for Uniquely Determined Systems

Consider the learning law (11) based on the Tikhonov regularization and write it in terms of singular value decomposition. Substitute F for $\tilde{F}$ in (11), presuming we have an accurate model. Then the learning law (11) gives L in the form

$$L = V(\Sigma^2 + \alpha^2 I)^{-1}\Sigma U^*, \tag{14}$$

and the error as a function of repetitions satisfies

$$\begin{aligned} U^* z_{k+1} &= \big[I - \Sigma(\Sigma^2 + \alpha^2 I)^{-1}\Sigma\big]U^* z_k \\ &= \operatorname{diag}\left(\frac{\alpha^2}{\sigma_i^2 + \alpha^2}\right) U^* z_k. \end{aligned} \tag{15}$$

This converges to zero tracking error provided $|\alpha^2/(\sigma_i^2 + \alpha^2)| < 1$ for all i.

We can create a more sophisticated regularization law by using a different weight α_i for each singular value of F. In place of (11) use

$$\big(F^*F + V\operatorname{diag}(\alpha_i^2)V^*\big)\Delta_{k+1}u = -F^* z_k, \tag{16}$$

which produces the learning law $L = V[\Sigma^2 + \operatorname{diag}(\alpha_i^2)]^{-1}\Sigma U^*$. Then the error as a function of repetitions satisfies

$$U^* z_{k+1} = \operatorname{diag}\big(\alpha_i^2/(\sigma_i^2 + \alpha_i^2)\big)U^* z_k. \tag{17}$$

The extra freedom of having many α_i to choose is helpful.

Now consider how one might use this more sophisticated regularization. The first singular values are normally well determined and there is no need for regularization with regard to this part of the system. The use of a nonzero α_i for these singular values unnecessarily slows down the convergence of this part of the error. Regularization becomes necessary for the particularly small singular values in Σ_2, the singular values that prevent an easy inversion of the full matrix F. This way of thinking suggests that we simply increase those small singular values by an amount that makes the matrix invertible, and leave the other singular values alone. Create a modified F as $\overline{F} = U\operatorname{diag}(\Sigma_1, \Sigma_2 + \operatorname{diag}(\alpha_i))V^*$, which adds an α_i to each σ_i on the diagonal of Σ_2, picking the α_i such that the matrix is no longer ill-conditioned. Then choose the learning matrix as $L = \overline{F}^{-1}$. Paralleling the development of Section 4.2 produces

$$U^* z_{k+1} = \operatorname{diag}\Big(0, \Sigma_2\big(\Sigma_2 + \operatorname{diag}(\alpha_i)\big)^{-1}\Big)U^* z_k. \tag{18}$$

As in Section 4.2, the components of the error in the space spanned by U_1 are eliminated in the first repetition, but this time the error for the i-th column of U, a column appearing in U_2, is multiplied by $\alpha_i/(\sigma_i + \alpha_i)$ every repetition. Hence, provided the model is correct, the learning control law converges monotonically to zero tracking error when the α_i's are chosen to satisfy

$$\left|\frac{\alpha_i}{\sigma_i + \alpha_i}\right| < 1, \tag{19}$$

for all positive α_i.

Note what has been accomplished by this more detailed and sophisticated Tikhonov regularization. The learning law immediately eliminates the error in that part of the space for which we can obtain an inverse. Then the iterations of the learning control process converge to the inverse of the rest of the matrix, eventually eliminating the entire error. Equation (2) is finally solved, in spite of the fact that the matrix F is too ill-conditioned to invert.

5. Relationship between the Enhanced Tikhonov Regularization and the Existing ILC Laws

In this section we consider again uniquely determined systems, and show that there is a close connection between the enhanced Tikhonov regularization developed above and several existing iterative learning control laws. First we show that the contraction mapping ILC of (Jang and Longman, 1994) is a limiting case of the Tikhonov regularization. Then we consider linear quadratic theory for ILC. There are many works that suggest the use of a quadratic cost functional in repetitions of learning control, making a trade-off between the amount of change that is made in the control action from one repetition to the next, and the tracking error. This is a natural extension to the iterative learning control of linear quadratic theory for linear optimal control. Optimal control theory asks one to state an optimality criterion, and then develops the corresponding optimal control law. This works well when there is a criterion that is well defined based on physical objectives, such as time optimal or fuel optimal control. But when using the usual quadratic cost functional, there are two weighting matrices that must be specified, and one normally does not know how to specify them. One makes a choice, observes the resulting behavior, and repeatedly adjusts the choice until satisfied. It will be shown here that the enhanced Tikhonov regularization creates a specific quadratic cost functional and hence it picks these weights in an optimized manner based on the properties of the system. This gives considerable insight into the function of quadratic cost ILC, and represents a contribution to the LQ theory for ILC by telling the designer how to make wise choices for the gains. Finally, we make a connection between the enhanced Tikhonov regularization ILC design method of Section 4.3 and low pass filtering used to robustify good learning transients against singular perturbations.

5.1. Contraction Mapping ILC Law is a Limiting Case of Tikhonov Regularization

Jang and Longman (1994) suggest the use of the learning law $L = \gamma F^*$. This law can be interpreted as performing at each repetition a step along the steepest descent direction for minimizing the sum of the squares of the errors for all time steps after the current one. If F is known exactly, this produces a monotonic decay of the tracking error. Note that as the scalar α becomes large, and makes Σ negligible, the Tikhonov based learning law (14) approaches this contraction mapping law with the learning gain $\gamma = \alpha^{-2}$. Hence, the contraction mapping law is a special case of the Tikhanov regularization and thus a special case of quadratic cost ILC as discussed below.

5.2. Enhanced Tikhonov Regularization Creates a Special, Well-Designed Linear Quadratic ILC Law

In the development of the ILC field one naturally asked how the most basic form of state variable control theory, the linear quadratic optimal control result, might be useful for the learning control objective. This can take two forms. One is to ask for a learning control law that learns to converge to the trajectory generated by linear quadratic optimal control for a chosen quadratic cost function (see, e.g., (Longman *et al.*, 1989; Longman and Chang, 1990)). In this case, the quadratic cost is a trade-off between the actual control effort and the tracking error. This is one solution to problems in which the desired trajectory is not feasible due to actuator saturation constraints. By contrast, the non-feasibility addressed by the over-determined problem in the previous sections can be classified as a geometric non-feasibility. The second use is to create a cost functional of the form

$$J(z_{k+1}, \Delta_{k+1}u) = z_{k+1}^T Q z_{k+1} + \Delta_{k+1}u^T R \Delta_{k+1}u, \tag{20}$$

where Q and R are symmetric positive-definite matrices. This time the penalty is not on the actual control, but on the change in control from the last repetition, and hence the purpose of the cost functional is to control the transients during the learning process by preventing too large a change in control from one repetition to the next. The iterations aim to converge to zero tracking error, not to the control action generated by a normal quadratic cost optimal control problem.

Now consider how the standard Tikhonov regularization is related to this linear quadratic ILC law. Note that the term $z_k + F\Delta_{k+1}u$ in (12) is equal to z_{k+1}, so that (12) can be rewritten as

$$\min_{\Delta_{k+1}u} \{\|z_{k+1}\|^2 + \alpha^2\|\Delta_{k+1}u\|^2\}. \tag{21}$$

We conclude that the traditional Tikhonov regularization for this problem produced a special case of the quadratic cost ILC control problem (20), with the Q and R matrices chosen as the identity and α^2 times the identity, respectively (Frueh and Phan, 2003).

Now consider the more sophisticated regularization of (17) (with (18) as a special case). Analogous developments show that this is associated with a quadratic cost of the form

$$\min_{\Delta_{k+1}u} \left\{z_{k+1}^T z_{k+1} + \Delta_{k+1}u^T \left[V \operatorname{diag}(\alpha_i^2) V^T\right] \Delta_{k+1}u\right\}. \tag{22}$$

The weighting matrix Q is still the identity, but by merging the enhanced Tikhonov regularization with the linear quadratic theory, we see that we could modify Q to emphasize parts of the trajectory where good tracking early in the learning process is important. This time the control weighting matrix R has a very specialized structure making use of the knowledge of the system.

The enhanced Tikhonov regularization of Section 4.3 addresses one of the weakest aspects of LQ theory, i.e. that one has little guidance as to how to pick the Q and R matrices, other than trial and error. Regularization makes use of the structure of the problem to come up with an R matrix with a very special structure specifically designed to address the source of the ill-conditioning in the ILC problem. This structure is not that which one could simply settle on while adjusting the weights according to the observed behavior, and hence it represents a contribution to the use of linear quadratic theory in ILC.

5.3. Achieving the Robustness of Good Learning Transients to Model Uncertainties by Modifying the Enhanced Tikhonov Regularization

An important issue in designing ILC laws is producing the robustness of good learning transients to singular perturbations (Avrachenkov and Pervozvansky, 1998a; 1998b) or, equivalently, to phase errors of the system model at high frequencies (Longman and Huang, 2003). Unlike normal feedback control, parasitic poles at high frequencies, far above the bandwidth of the controller, can easily destabilize a learning control system. In (Avrachenkov and Pervozvansky, 1998a; 1998b; Longman, 1998; 2000; Pervozvansky and Avrachenkov, 1997) stability and good transient robustness are obtained for such unmodeled high frequency dynamics by cutting off the learning above the frequency for which one no longer has confidence in the model (this cutoff can also be tuned based on observations of the frequency content of the response of the learning process). In what follows, we discuss how the frequency response, and hence a frequency cutoff, is related to singular values, and how to interpret the enhanced Tikhonov regularization in terms of a frequency weighted cost functional. Then we show how to modify the enhanced Tikhonov quadratic cost to create a frequency cutoff. In the process, we see that a frequency cutoff is equivalent to producing an over-determined ILC problem.

Jang and Longman (1996) discuss the relationship between the singular values of F and the magnitude of the frequency response of the system at discrete frequencies. As the trajectory becomes long, the singular values converge to the magnitude response. In other words, (22) is a finite time, time domain version of a frequency weighted cost functional, with the α_i^2's being weights for different frequencies. There is a natural connection between the quadratic cost in the time domain and the frequency domain using Parseval's theorem.

In the quadratic cost Tikhonov regularization of (22) this kind of robustness can be accomplished as a limiting case by letting the α_i weights associated with the singular values above the cutoff frequency tend to infinity. Then no control action is taken in this part of the space. To get to the limit in this process, one can transform the control variables to new coordinates $\bar{u} = V^* u$, then delete those control variables associated with singular values that relate to the frequencies above the cutoff, and then formulate the quadratic cost for the remaining control coordinates. The truncation of singular values is a finite time version of the cliff filtering in (Plotnik and Longman, 1999). (For related developments for continuous time systems see also (Avrachenkov and Pervozvansky, 1998a; 1998b; Pervozvansky and Avrachenkov, 1997).) This limiting case produces an over-determined problem.

In terms of the new control variables, we can create a new rectangular matrix F, and then the learning law, for example the one above Eqn. (17), becomes $L = V[\Sigma^T \Sigma + \mathrm{diag}(\alpha_i^2)]^{-1} \Sigma^* U^*$. This L generates changes in the control action $\bar{u}$ from those of the previous repetition, making use of the singular value decomposition of this new rectangular F. A procedure of this kind is also appropriate when the dimension of an F for which one can numerically obtain a singular value decomposition is less than the full dimension of the problem.

As a regularization procedure, the Tikhonov regularization aims to produce robustness. It naturally produces robustness to errors in the knowledge of the frequency magnitude response of the system, i.e., singular values, and addresses the issues of ill-conditioning. But no statements are made here relative to producing robustness to errors in one's knowledge of the singular vectors in U and V. These vectors contain the phase information, and the stability of the ILC iterations is sensitive to phase errors. As the trajectory gets long compared to the time constants of the system, the column vectors in U and V start to look rather like sine and cosine waves, or two sines with different phases. The $V^* z_k$ finds the projection of the error onto these sinusoids. Then the singular values influence the amplitude changes, and multiplying by the columns of U puts in the output sinusoid with the appropriate phase change included for the associated discrete frequency. It is important to have robustness to errors in the phase information in the model F, and this advocates the use of a frequency cutoff.

Hence, it is recommended here that the enhanced regularization or LQ ILC law be used (Eqns. (16) and (22)) and that it be combined with a singular value cutoff as described above. This approach to learning control benefits from the use of a system model, it involves regularization that solves the ill-conditioned model inversion problem by iterations, the regularization produces robustness to singular value or magnitude frequency response errors of the model, and then the truncation produces robustness to errors in the phase information at high frequency.

6. Conclusions

This paper studies first the application of iterative learning control to linear over-determined and under-determined systems. It is shown that in the case of either over-determined or under-determined linear systems the learning operator has to be chosen as close to the pseudoinverse operator of the system as possible. Conditions are provided for the convergence of the learning iterations. In the case of under-determined systems as well as in the case of an over-determined system with the condition that the desired trajectory is feasible ($y_d \in \mathcal{R}(F)$), the norm of error converges to zero as the iterations go to infinity. However, in the case of over-determined systems when $y_d \notin \mathcal{R}(F)$, it is not possible to obtain an exact solution for $\min \|y_d - Fu\|$ with any choice of the iteration invariant learning operator. The generalized inverse method is shown to be closely related to the Tikhonov-type regularization, which can be equally applied to non-linear systems. Then we turn to uniquely determined systems and show that often the methods for over-determined systems are relevant, and that, in addition, several ILC laws can be perceived as forms of the Tikhonov regularization. It is said that computing the inverse for discrete time linear dynamic systems is normally ill-conditioned. A generalized Tikhonov regularization technique is developed that in the first iteration produces the pseudoinverse solution of the part of the system that is easily invertible, and iterates to achieve the inverse for the rest of the system. It is shown that quadratic cost iterative learning control can be seen as a form of the Tikhonov regularization, and that the enhanced regularization approach solves the basic problem in quadratic cost approaches, i.e. how to select the weighting matrices. The approach produces an intelligent choice precisely tuned to the problem at hand, and one that we would not find by a normal adjustment. The need for a frequency cutoff for robustness to model errors is discussed, and it is shown how one can modify the Tikhonov-based ILC in order to produce this cutoff. The approach converts the problem into an over-determined one.

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