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BILATERAL POLYNOMIAL EQUATIONS WITH UNIMODULAR RIGHT-HAND-SIDE MATRICES

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Necessary and sufficient conditions are established for the existence of a solution to some bilateral polynomial matrix equations with unimodular right-hand-side matrices. A procedure for the computation of the solution is derived and illustrated by a numerical example. Two examples of applications of bilateral polynomial matrix equations are presented.

Keywords: bilateral, polynomial equation, unimodular procedure, solution

1. Introduction

The polynomial equation approach to linear control systems has been considered in many papers and books (Kučera, 1972; 1981; 1979; Kaczorek, 2002c; 1993; 2003). In (Kaczorek, 2003) the solvability problem of polynomial matrix equations and its relationship with the infinite eigenvalue assignment by state feedback was considered. The infinite eigenvalue assignment is the crucial issue in the design of perfect observers (Kaczorek, 2000a; 2002b; 2002c, Dai, 1989). Necessary and sufficient conditions for the infinite eigenvalue assignment by state feedback in linear systems were established in (Chu and Ho, 1999; Kaczorek, 2003).

In this paper necessary and sufficient conditions will be established for the existence of a solution to some bilateral polynomial matrix equations with unimodular right-hand-side matrices and a procedure for the computation of the solutions will be given. Some examples of applications of this type of bilateral polynomial matrix equations will also be given.

2. Problem Formulation

Let $\mathbb{R}^{n\times m}$ ($\mathbb{R}^n := \mathbb{R}^{n\times 1}$) be the set of $n \times m$ matrices with the entries from the field $\mathbb{R}$ of real numbers and $\mathbb{R}^{n\times m}[s]$ be the set of $n \times m$ polynomial matrices with real coefficients in the variable s. Consider the polynomial matrix equation

$$[Es - A]X + BYC = U(s), \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $U(s) \in \mathbb{R}^{n\times n}[s]$ is a unimodular matrix with $\det U(s) = \alpha$ (α is a scalar independent of s). It is assumed that $\operatorname{rank} B = m$ and $\operatorname{rank} C = p$. The problem can be formulated as follows: Given matrices E, A, B, C and $U(s)$, find a solution X, Y of Eqn. (1) satisfying the conditions

$$X = I_n, \quad Y \in \mathbb{R}^{m\times p}, \tag{2}$$

where I_n stands for the $n \times n$ identity matrix.

3. Problem Solution

Theorem 1. *The problem has a solution only if*

$$\operatorname{rank}[Es - A, B] = \operatorname{rank}\begin{bmatrix} Es - A \\ C \end{bmatrix} = n \tag{3}$$

for all finite $s \in \mathbb{C}$ (the field of complex numbers) and

$$D = Es - U(s) \in \mathbb{R}^{n\times n} \tag{4}$$

is a real matrix independent of s.

Proof. Since

$$Es - A + BYC = [Es - A, B]\begin{bmatrix} I_n \\ YC \end{bmatrix} = [I_n, BY]\begin{bmatrix} Es - A \\ C \end{bmatrix} \tag{5}$$

and $\det U(s) = \alpha$, it follows that (1) and (2) imply (3).

From (1) and (2) we have

$$Es - U(s) = A - BYC = D \in \mathbb{R}^{n\times n}. \tag{6}$$

Therefore, Eqn. (1) has a solution (2) only if (3) and (4) are satisfied. ■

Let $P, Q \in \mathbb{R}^{n\times n}$ be nonsingular matrices of elementary row and column operations (Kaczorek, 1993; 2003) such that

$$PB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad CQ = [C_1 \;\; 0], \tag{7}$$

where $B_1 \in \mathbb{R}^{m\times m}$ and $C_1 \in \mathbb{R}^{p\times p}$ are nonsingular matrices owing to the assumption $\operatorname{rank} B = m$ and $\operatorname{rank} C = p$. Equation (1) for $X = I_n$ can be rewritten as

$$BYC = A - D, \tag{8}$$

where D is defined by (4).

Premultiplying (8) by P, postmultiplying the result by Q and using (7) we obtain

$$\begin{bmatrix} B_1 \\ 0 \end{bmatrix} Y[C_1 0] = \left[\begin{array}{c:c} A_1 - D_1 & A_2 - D_2 \\ \hdashline \multicolumn{2}{c}{A_3 - D_3} \end{array}\right], \tag{9}$$

where

$$PAQ = \left[\begin{array}{c:c} A_1 & A_2 \\ \hdashline \multicolumn{2}{c}{A_3} \end{array}\right], \quad PDQ = \left[\begin{array}{c:c} D_1 & D_2 \\ \hdashline \multicolumn{2}{c}{D_3} \end{array}\right], \tag{10}$$

$A_1, D_1 \in \mathbb{R}^{m\times p}$, $A_2, D_2 \in \mathbb{R}^{m\times(n-p)}$, $A_3, D_3 \in \mathbb{R}^{(n-m)\times n}$.

Theorem 2. *Let the conditions (3) and (4) be satisfied. Equation (1) has a solution (2) if and only if*

$$A_2 = D_2, \quad A_3 = D_3. \tag{11}$$

The desired solution is given by

$$Y = B_1^{-1}(A_1 - D_1)C_1^{-1}. \tag{12}$$

Proof. From (9) we have

$$A_1 Y C_1 = A_1 - D_1, \tag{13}$$

and the conditions (11). The matrices B_1 and C_1 are nonsingular and from (13) we obtain the solution (12). ■

Example 1. Find a solution (2) of Eqn. (1) with

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 2 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{14}$$

$$U(s) = \begin{bmatrix} 1 & -1 & s \\ 0 & -\alpha & 0 \\ 0 & s-2 & -1 \end{bmatrix}.$$

In this case the assumptions (3) and (4) are satisfied since

$$\operatorname{rank}[Es - A, B] = \operatorname{rank} \begin{bmatrix} 1 & -1 & s & 0 \\ -1 & -2 & 1 & 1 \\ 0 & s-2 & -1 & 0 \end{bmatrix} = 3,$$

$$\operatorname{rank} \begin{bmatrix} Es - A \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & -1 & s \\ -1 & -2 & 1 \\ 0 & s-2 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} = 3$$

for all finite $s \in \mathbb{C}$, and the matrix

$$D = Es - U(s) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 2 & 1 \end{bmatrix} \tag{15}$$

is real.

The matrices P and Q satisfying (7) have the forms

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

since

$$PB = \begin{bmatrix} 1 \\ \cdots \\ 0 \\ 0 \end{bmatrix}, \quad CQ = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

Using (14)–(16), we obtain

$$PAQ = \left[\begin{array}{c:c} A_1 & A_2 \\ \hdashline \multicolumn{2}{c}{A_3} \end{array}\right] = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ \hdashline -1 & 3 & -1 \\ 0 & 2 & 1 \end{array}\right],$$
$$PDQ = \left[\begin{array}{c:c} D_1 & D_2 \\ \hdashline \multicolumn{2}{c}{D_3} \end{array}\right] = \left[\begin{array}{cc:c} 0 & \alpha & 0 \\ \hdashline -1 & 3 & -1 \\ 0 & 2 & 1 \end{array}\right]. \tag{17}$$

From (17) it follows that the conditions (11) are satisfied and Eqn. (1) with (14) has a solution. Using (12), we obtain the desired solution

$$Y = B_1^{-1}(A_1 - D_1)C_1^{-1} = [1, -\alpha]. \tag{18}$$

It is easy to verify that (18) and $X = I_3$ satisfy the equation since

$$\begin{aligned} &[Es - A]X + BYC \\ &= \begin{bmatrix} 1 & -1 & s \\ -1 & -2 & 1 \\ 0 & s-2 & -1 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [1, -\alpha] \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & s \\ 0 & -\alpha & 0 \\ 0 & s-2 & -1 \end{bmatrix}. \end{aligned}$$

4. Applications

Consider the singular continuous-time linear system

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{19}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the semistate, input and output vectors, respectively, and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ with $\det E = 0$.

The infinite eigenvalue assignment problem for (19) can be formulated as follows: Given matrices E, A, B, C and a nonzero scalar α, find an output-feedback gain matrix $F \in \mathbb{R}^{m\times p}$ such that

$$\det[Es - A + BFC] = \alpha. \tag{20}$$

The equality (20) can be written down as

$$\det\left\{[Es - A, B]\begin{bmatrix} I_n \\ FC \end{bmatrix}\right\} = \det U(s) \tag{21}$$

and

$$[Es - A]X + BYC = U(s), \tag{22}$$

where

$$X = I_n, \quad Y = F \tag{23}$$

and $U(s) \in \mathbb{R}^{n\times n}[s]$ is a unimodular matrix with $\det U(s) = \alpha$ (Kaczorek, 2003).

The transfer matrix of (19) with the output-feedback $u = v - Fy = v - FCx$ ($v \in \mathbb{R}^m$ is a new input) is given by

$$T(s) = C[Es - A + BFC]^{-1}B. \tag{24}$$

If $Es - A + BFC = U(s)$ with $U(s)$ being unimodular, then the transfer matrix in (24) takes the form $T(s) = CU^{-1}(s)B$ being a polynomial matrix. Therefore, finding the solution (23) of (22) is equivalent to finding an output-feedback gain matrix F such that the closed-loop transfer matrix is polynomial.

5. Concluding Remarks

Necessary and sufficient conditions for the existence of the solution (2) to the polynomial matrix equation (1) have been established. A procedure for the computation of the solution was derived and illustrated by a numerical example. The studies presented in (Kaczorek, 2003) are a particular case of the ones given here for $C = I_n$. Two examples of applications of the equation were presented. An extension of the presented approach to two-dimensional matrix polynomial equations (Kaczorek, 1993) is possible but it is not trivial.

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