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COMPUTATIONAL APPROACH TO SOLVING A LAYERED BEHAVIOUR DIFFERENTIAL EQUATION WITH LARGE DELAY USING QUADRATURE SCHEME

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This paper deals with the computational approach to solving the singularly perturbed differential equation with a large delay in the differentiated term using the two-point Gaussian quadrature. If the delay is bigger than the perturbed parameter, the layer behaviour of the solution is destroyed, and the solution becomes oscillatory. With the help of a special type mesh, a numerical scheme consisting of a fitting parameter is developed to minimize the error and to control the layer structure in the solution. The scheme is studied for convergence. Compared with other methods in the literature, the maximum defects in the approach are tabularized to validate the competency of the numerical approach. In the suggested technique, we additionally focused on the effect of a large delay on the layer structure or oscillatory behaviour of the solutions using a special form of mesh with and without a fitting parameter. The effect of the fitting parameter is demonstrated in graphs to show its impact on the layer of the solution.

Key words: singularly perturbed delay differential equation, layer behavior, fitting parameter, Gaussian quadrature.

1. Introduction

Differential equations with delay are ones in which the state variable's temporal evolution is arbitrary dependent on a specified history, i.e. the rate of change of physical systems depends not only on their present state but also on their past history [1]. These problems also occur in the modelling of many practical phenomena such as thermo-elasticity [2], hybrid optical system [3], in population dynamics [4], in models for physiological processes [5], red blood cell system [6], predator-prey models [7], optimal control theory [8] and in the potential in nerve cells by random synaptic inputs in dendrites [9].

The works of Doolan *et al.* [10], Driver [11], El'sgol'ts and Norkin [12], Glizer [13], Miller *et al.* [14], and Smith [15] are only a few of the books and publications available for additional research on the mathematical elements of the model class and singular perturbation problems.

For a class of singularly perturbed differential-difference equations, Lange and Miura [16, 17] presented an asymptotic method. The authors of [16] looked at a mathematical model for calculating the time it takes for random synaptic inputs in dendrites to generate potential activity in nerve cells. In [17], the authors demonstrated the issues with solutions that exhibit fast oscillations. A thorough numerical study [18, 19] based on finite difference methods has been started by Kadalbajoo and his team. In [20], Ravikanth and Murali devised a numerical technique for solving singularly perturbed differential-differential equations via tension splines that only includes a negative shift in the differentiated term. A computational method is devised in [21] for linear and nonlinear singularly perturbed delay problems on piecewise uniform.

2. Numerical scheme

Consider the layer behavior delay differential equation

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$$\varepsilon\omega''(\phi)+A(\phi)\omega'(\phi-\delta)+B(\phi)\omega(\phi)=F(\phi) \tag{2.1}$$

with the boundary conditions

$$\omega(\phi)=\theta(\phi)\ , \quad -\delta\le\phi\le 0 \quad \text{and} \quad y(1)=\beta \tag{2.2}$$

where $\varepsilon(0<\varepsilon<<1)$ is a small positive parameter, δ is the delay arguments the functions $A(\phi),B(\phi),F(\phi)$ and $\theta(\phi)$ are smooth and the β is a positive constant. When ε is small, depending upon the coefficient of convection term, Eq. (1) exhibits layers and turning points. For $\delta\neq 0$ and δ is of $o(\varepsilon)$, the layer behavior of the problem under consideration is maintained. However, when the delay is greater than the perturbation, i.e., δ is of $O(\varepsilon)$, the solution's layer behavior is no longer preserved, and the solution oscillates. Here, we address the problem with small and large delay.

2.1. Left end boundary layer problem

When $A(\phi)\ge M>0$ on $\phi_i\le\phi\le\phi_{i+1}$, M being a positive constant, the solution of the problem Eq. (2.1), with Eq. (2.2) displays boundary layer behavior at the left side of the interval $\phi_i\le\phi\le\phi_{i+1}$.
Using Taylor series of the retarded term, we have

$$\omega'(\phi-\varepsilon)=\omega'(\phi)-\varepsilon\omega''(\phi) \quad \text{gives} \quad \varepsilon\omega''(\phi)=\omega'(\phi)-\omega'(\phi-\varepsilon) \tag{2.3}$$

using Eq. (2.3), Eq. (2.1) is replaced by the first order delay differential equation

$$\omega'(\phi)=\omega'(\phi-\varepsilon)-A(\phi)y'(\phi-\delta)-B(\phi)\omega(\phi)+F(\phi). \tag{2.4}$$

The region [0, 1] is subdivided into N equal subregions of mesh size $h=\frac{1}{N}$ so that $\phi_i=ih, i=0,1,\dots,N$ are the grid points. The delay term is handled by choosing the mesh size as $h=\frac{\delta}{m}$, where $m=pq$, p is a positive integer and q is the mantissa of δ so that the term $\omega'(\phi-\delta)$ becomes the grid point.
Integrating Eq. (2.4) with respect to x from ϕ_i to ϕ_{i+1}, we get

$$\int_{\phi_i}^{\phi_{i+1}}\omega'(\phi)d\phi=\int_{\phi_i}^{\phi_{i+1}}((\omega'(\phi-\varepsilon)-A(\phi)\omega'(\phi-mh)-B(\phi)\omega(\phi)+F(\phi))d\phi,$$

$$\omega_{i+1}-\omega_i=\int_{\phi_i}^{\phi_{i+1}}\left(\omega'(\phi-\varepsilon)-A(\phi)\omega'(\phi-mh)-B(\phi)\omega(\phi)+F(\phi)\right)d\phi. \tag{2.5}$$

Now, to approximate the values of the integrations in Eq. (2.5), we use the Gaussian two-point quadrature scheme given by

$$\int_{-1}^{1}\mathcal{G}(\phi)\,d\phi=\mathcal{G}\left(\frac{1}{\sqrt{3}}\right)+\mathcal{G}\left(\frac{-1}{\sqrt{3}}\right).$$

For any differentiable function $\mathcal{G}(\phi)$ in an arbitrary region $[\phi_i,\phi_{i+1}]$, the two-point Gaussian quadrature scheme is given by

$$\int_{\phi_i}^{\phi_{i+1}}\mathcal{G}(\phi)\,d\phi=\frac{h}{2}\left(\mathcal{G}(\phi_i+r)+\mathcal{G}(\phi_{i+1}-r)\right) \tag{2.6}$$

where $r=\dfrac{h\left(1-\frac{1}{\sqrt{3}}\right)}{2}$. Using Eq. (2.6) in Eq. (2.5), we get

$$\begin{aligned}&\omega_{i+1}-\omega_i=\omega(\phi_{i+1}-\varepsilon)-\omega(\phi_i-\varepsilon)(\phi_{i+1})\omega(\phi_{i+1-m})-\mathrm{A}(\phi_i)\omega(\phi_{i-m})+\\&+\frac{h}{2}\left[\mathrm{A}'(\phi_{i+1}-k)\omega(\phi_{i+1-m}-r)+\mathrm{A}'(\phi_i+k)\omega(\phi_{i-m}+r)\right]+\\&-\frac{h}{2}\left[\mathrm{B}(\phi_{i+1}-r)\omega(\phi_{i+1}-r)+\mathrm{B}(\phi_i+k)\omega(\phi_i+r)\right]+\frac{h}{2}\left[F(\phi_{i+1}-r)+F(\phi_i+r)\right].\end{aligned} \tag{2.7}$$

Using the linear interpolation for $\omega(\phi_{i+1}-\varepsilon),\omega(\phi_i-\varepsilon),\omega(\phi_{i-m}+r)$, $\omega(\phi_{i+1-m}-r)$, Eq.(2.7) reduces to

$$\begin{aligned}&\left\{\frac{\varepsilon}{h}+\mathrm{B}(\phi_i+r)\left(\frac{r}{2}\right)\right\}\omega_{i-1}+\left\{\frac{-2\varepsilon}{h}+\mathrm{B}(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)+\mathrm{B}(\phi_i-r)\left(\frac{h+r}{2}\right)\right\}\omega_i+\\&+\left\{\frac{\varepsilon}{h}+\mathrm{B}(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i+1}+\left\{\mathrm{A}(\phi_{i+1})-\mathrm{A}'(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i+1-m}+\\&+\left\{-\mathrm{A}(\phi_i)-\mathrm{A}'(\phi_{i+1}-r)\left(\frac{k}{2}\right)-\mathrm{A}'(\phi_i+r)\left(\frac{h+r}{2}\right)\right\}\omega_{i-m}+\left\{\mathrm{A}'(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i-1-m}=\\&=\frac{h}{2}\left[F(\phi_{i+1}-r)+F(\phi_i+r)\right].\end{aligned} \tag{2.8}$$

When the delay is larger than the perturbation, the layer can alter its character and even be demolished, or the solution displays oscillatory behavior in this case. Reorganizing the above equation, we get

$$\begin{aligned}&\frac{\varepsilon}{h}\{\omega_{i-1}-2\omega_i+\omega_{i+1}\}+\left\{\mathrm{B}(\phi_i+r)\left(\frac{r}{2}\right)\right\}\omega_{i-1}+\left\{\mathrm{B}(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)+\mathrm{B}(\phi_i-r)\left(\frac{h+r}{2}\right)\right\}\omega_i+\\&+\left\{\mathrm{B}(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i+1}+\left\{\mathrm{A}(\phi_{i+1})-\mathrm{A}'(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i+1-m}+\\&+\left\{-\mathrm{A}(\phi_i)-\mathrm{A}'(\phi_{i+1}-r)\left(\frac{r}{2}\right)-\mathrm{A}'(\phi_i+r)\left(\frac{h+r}{2}\right)\right\}\omega_{i-m}+\left\{\mathrm{A}'(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i-1-m}=\\&=\frac{h}{2}\left[F(\phi_{i+1}-r)+F(\phi_i+r)\right].\end{aligned}$$

To handle this layer character, we tried introducing a fitting parameter in this scheme. The fitting factor is determined with the help of the theory of singular perturbations. Inserting the fitting factor $\sigma(\rho)$ into the scheme Eq. (2.8), we get

$$\begin{aligned}
&\sigma\rho\{\omega_{i-1}-2\omega_i+\omega_{i+1}\}+\left\{B(\phi_i+r)\left(\frac{r}{2}\right)\right\}\omega_{i-1}+\\
&+\left\{B(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)+B(\phi_i-r)\left(\frac{h+r}{2}\right)\right\}\omega_i+\\
&+\left\{B(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i+1}+\left\{A(\phi_{i+1})-A'(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i+1-m}+\\
&+\left\{-A(\phi_i)-A'(\phi_{i+1}-r)\left(\frac{r}{2}\right)-A'(\phi_i+r)\left(\frac{h+r}{2}\right)\right\}\omega_{i-m}+\left\{A'(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i-1-m}=\\
&=\frac{h}{2}\left[F(\phi_{i+1}-r)+F(\phi_i+r)\right].
\end{aligned} \tag{2.9}$$

By using the process given by Doolan *et al.* [4], we get the fitting parameter

$$\sigma=\rho A(0)\frac{e^{A(0)m\rho}\left[1-e^{-A(0)\rho}\right]}{4\left[\sinh\frac{A(0)\rho}{2}\right]} \quad \text{where} \quad \rho=\frac{\varepsilon}{h} \tag{2.10}$$

Rewriting Eq. (2.8) we get,

$$\begin{aligned}
&\left\{\frac{\sigma\varepsilon}{h}+B(\phi_i+r)\left(\frac{r}{2}\right)\right\}\omega_{i-1}+\left\{\frac{-2\sigma\varepsilon}{h}+B(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)+B(\phi_i-r)\left(\frac{h+r}{2}\right)\right\}\omega_i+\\
&+\left\{\frac{\sigma\varepsilon}{h}+B(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i+1}+\left\{A(\phi_{i+1})-A'(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i+1-m}+\\
&+\left\{-A(\phi_i)-A'(\phi_{i+1}-r)\left(\frac{r}{2}\right)-A'(\phi_i+r)\left(\frac{h+r}{2}\right)\right\}\omega_{i-m}+\left\{A'(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i-1-m}=\\
&=\frac{h}{2}\left[F(\phi_{i+1}-r)+F(\phi_i+r)\right].
\end{aligned}$$

$$P_i\omega_{i-1}+Q_i\omega_i+R_i\omega_{i+1}+U_i\omega_{i-m+1}+V_i\omega_{i-m}+\tilde{W}_i\omega_{i-m-1}=G_i,\ i=1,2,\ldots,N-1. \tag{2.11}$$

$$P_i=\left\{\frac{\sigma\varepsilon}{h}+B(\phi_i+r)\left(\frac{r}{2}\right)\right\},\quad Q_i=\left\{\frac{-2\sigma\varepsilon}{h}+B(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)+B(\phi_i-r)\left(\frac{h+r}{2}\right)\right\},$$

$$R_i=\left\{\frac{\sigma\varepsilon}{h}+B(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\},\quad U_i=\left\{A(\phi_{i+1})-A'(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\},$$

$$V_i=\left\{-A(\phi_i)-A'(\phi_{i+1}-r)\left(\frac{r}{2}\right)-A'(\phi_i+r)\left(\frac{h+r}{2}\right)\right\},\ \tilde{W}_i=\left\{A'(\phi_i-r)\left(\frac{r}{2}\right)\right\},$$

$$G_i = \frac{h}{2}\left[F\left(\phi_{i+1}-r\right)+F\left(\phi_i+r\right)\right].$$

The scheme of Eq. (2.11) can be expressed as

$$\begin{aligned}
&P_i\omega_{i-1}+Q_i\omega_i+R_i\omega_{i+1}=G_i-U_i\omega_{i-m+1}-V_i\omega_{i-m}-\tilde{W}_i\omega_{i-m-1} \quad \text{for} \quad 1\le i\le m-1,\\
&P_i\omega_{i-1}+Q_i\omega_i+R_i\omega_{i+1}+U_i\omega_{i-m+1}=G_i-V_i\omega_{i-m}-\tilde{W}_i\omega_{i-m-1} \quad \text{for} \quad i=m,\\
&P_i\omega_{i-1}+Q_i\omega_i+R_i\omega_{i+1}+U_i\omega_{i-m+1}+V_i\omega_{i-m}=G_i-\tilde{W}_i\omega_{i-m-1} \quad \text{for} \quad i=m+1,\\
&P_i\omega_{i-1}+Q_i\omega_i+R_i\omega_{i+1}+U_i\omega_{i-m+1}+V_i\omega_{i-m}+\tilde{W}_i\omega_{i-m-1}=G_i \quad \text{for} \quad m+2\le i\le N-1.
\end{aligned} \tag{2.12}$$

The Gauss elimination scheme with partial pivoting is used to solve the aforementioned system of equations.

2.2. Right-end boundary layer problem

In this case $A(\phi)\le -M<0$ on $\phi_{i-1}\le\phi\le\phi_i$, M being a negative constant. then the solution of the problem Eq. (2.1), Eq. (2.2) shows the boundary layer at the right side of the interval $\phi_{i-1}\le\phi\le\phi_i$.
Using Taylors series expression of $\omega'(\phi+\varepsilon)$ gives

$$\omega'(\phi+\varepsilon)=\omega'(\phi)+\varepsilon\omega''$$

and implies

$$\varepsilon\omega''(\phi)\approx\omega'(\phi+\varepsilon)-\omega'(\phi). \tag{2.13}$$

Consequently, Eq. (2.1) is replaced by an equivalent first order delay neutral type differential equation

$$\omega'(\phi)=\omega'(\phi+\varepsilon)+\mathrm{A}(\phi)\omega'(\phi-\delta)+\mathrm{B}(\phi)\omega(\phi)-F(\phi). \tag{2.14}$$

Integrating Eq. (2.14) on θ_{i-1} to θ_i on both sides, we get

$$\int_{\phi_{i-1}}^{\phi_i}\omega'(\phi)d\phi=\int_{\phi_{i-1}}^{\phi_i}\left(\omega'(\phi+\varepsilon)+\mathrm{A}(\phi)\omega'(\phi-mh)+\mathrm{B}(\phi)\omega(\phi)-F(\phi)\right)d\phi. \tag{2.15}$$

Using the two-point Gaussian quadrature formula for any differentiable function $\mathcal{G}(\theta)$ in an arbitrary region $[\phi_{i-1},\phi_i]$, we get

$$\int_{\phi_{i-1}}^{\phi_i}\mathcal{G}(\phi)d\phi=\frac{h}{2}\left(\mathcal{G}(\phi_{i-1}-r)+\mathcal{G}(\phi_i+r)\right). \tag{2.16}$$

Using Eq. (2.16), from Eq. (2.15) we get

$$\begin{aligned}&\omega_i-\omega_{i-1}=\omega(\phi_i+\varepsilon)-\omega(\phi_{i-1}+\varepsilon)+\mathrm{A}(\phi_i)\omega(\phi_{i-m})-\mathrm{A}(\phi_{i-1})\omega(\phi_{i-1-m})+\\&-\frac{h}{2}\left[\mathrm{A}'(\phi_i-r)\omega(\phi_{i-m}-r)+\mathrm{A}'(\phi_{i-1}+r)\omega(\phi_{i-1-m}+r)\right]+\\&+\frac{h}{2}\left[\mathrm{B}(\phi_i-r)\omega(\phi_i-r)+\mathrm{B}(\phi_{i-1}+r)\omega(\phi_{i-1}+r)\right]+\frac{h}{2}\left[F(\phi_{i+1}-r)+F(\phi_i+r)\right].\end{aligned}\tag{2.17}$$

Using linear interpolation of the terms $\omega(\phi_{i-1}+\varepsilon)$, $\omega(\phi_i+\varepsilon)$, $\omega(\phi_{i-1-m}+r)$, $\omega(\phi_{i-m}-r)$ and $\omega(\phi_i-r)$, $\omega(\phi_{i-1}+r)$, we get

$$\begin{aligned}&\left\{\frac{\varepsilon}{h}-\mathrm{B}(\phi_{i-1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i-1}+\left\{\frac{-2\varepsilon}{h}+\mathrm{B}(\phi_i-r)\left(\frac{h+r}{2}\right)+\mathrm{B}(\phi_{i-1}+r)\left(\frac{r}{2}\right)\right\}\omega_i+\\&+\left\{\frac{\varepsilon}{h}-\mathrm{B}(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i+1}+\left\{-\mathrm{A}(\phi_{i-1})-\mathrm{A}'(\phi_{i-1}+r)\left(\frac{h-r}{2}\right)\right\}\omega_{i-1-m}+\\&+\left\{\mathrm{A}(\phi_i)-\mathrm{A}'(\phi_i-r)\left(\frac{h+r}{2}\right)-\mathrm{A}'(\phi_{i-1}+r)\left(\frac{r}{2}\right)\right\}\omega_{i-m}+\left\{\mathrm{A}'(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i+1-m}=\\&=\frac{h}{2}\left[F(\phi_{i+1}-r)+F(\phi_i+r)\right].\end{aligned}\tag{2.18}$$

When the shift parameter is larger than the perturbation, the layer can alter its character and even be demolished or the solution displays oscillatory behaviour in this case. Rearranging Eq. (2.18), we have

$$\begin{aligned}&\frac{\varepsilon}{h}\{\omega_{i-1}-2\omega_i+\omega_{i+1}\}+\left\{\mathrm{B}(\phi_{i-1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i-1}+\left\{\mathrm{B}(\phi_i-r)\left(\frac{h+r}{2}\right)+\right.\\&\left.+r)\left(\frac{r}{2}\right)\right\}\omega_i+\left\{\mathrm{B}(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i+1}+\left\{-\mathrm{A}(\phi_{i-1})-\mathrm{A}'(\phi_{i-1}+r)\left(\frac{h-r}{2}\right)\right\}\omega_{i-1-m}+\\&+\left\{\mathrm{A}(\phi_i)-\mathrm{A}'(\phi_i-r)\left(\frac{h+r}{2}\right)-\mathrm{A}'(\phi_{i-1}+r)\left(\frac{r}{2}\right)\right\}\omega_{i-m}+\left\{\mathrm{A}'(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i+1-m}=\\&=\frac{h}{2}\left[F(\phi_{i+1}-r)+F(\phi_i+r)\right].\end{aligned}\tag{2.19}$$

To handle these oscillations, we have made an effort by introducing a fitting parameter in this scheme with the special type quadrature method. The fitting factor is found out with the help of the theory of singular perturbations. Now here we insert the fitting factor to control the oscillations $\sigma(\rho)$ in the scheme Eq.(2.19), obtaining

$$\begin{aligned}&\sigma\rho\{\omega_{i-1}-2\omega_i+\omega_{i+1}\}+\left\{\mathrm{B}(\phi_{i-1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i-1}+\left\{\mathrm{B}(\phi_i-r)\left(\frac{h+r}{2}\right)+\mathrm{B}(\phi_{i-1}+r)\left(\frac{r}{2}\right)\right\}\omega_i+\\&+\left\{\mathrm{B}(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i+1}+\left\{-\mathrm{A}(\phi_{i-1})-\mathrm{A}'(\phi_{i-1}+r)\left(\frac{h-r}{2}\right)\right\}\omega_{i-1-m}+\left\{\mathrm{A}(\phi_i)-\mathrm{A}'(\phi_i-r)\left(\frac{h+r}{2}\right)+\right.\\&\left.+-\mathrm{A}'(\phi_{i-1}+r)\left(\frac{r}{2}\right)\right\}\omega_{i-m}+\left\{\mathrm{A}'(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i+1-m}==\frac{h}{2}\left[F(\phi_{i+1}-r)+F(\phi_i+r)\right].\end{aligned}$$

Using the process given in Doolan *et. al.* [4], we derived the fitting factor as

$$\sigma = \rho A(N+1)\frac{e^{A(N+1)m\rho}\left[1-e^{A(N+1)\rho}\right]}{4\left[\sinh\frac{A(N+1)\rho}{2}\right]}. \tag{2.20}$$

Equation (2.17) can be rewritten as

$$\left\{\frac{\sigma\varepsilon}{h} - B(\phi_{i-1}-r)\left(\frac{h-r}{2}\right)\right\}\omega_{i-1} + \left\{\frac{-2\sigma\varepsilon}{h} + B(\phi_i - r)\left(\frac{h+r}{2}\right) + B(\phi_{i-1}+r)\left(\frac{r}{2}\right)\right\}\omega_i +$$
$$+\left\{\frac{\sigma\varepsilon}{h} - B(\phi_i - r)\left(\frac{r}{2}\right)\right\}\omega_{i+1} + \left\{-A(\phi_{i-1}) - A'(\phi_{i-1}+r)\left(\frac{h-r}{2}\right)\right\}\omega_{i-1-m} +$$
$$+\left\{A(\phi_i) - A'(\phi_i - r)\left(\frac{h+r}{2}\right) - A'(\phi_{i-1}+r)\left(\frac{r}{2}\right)\right\}\omega_{i-m} + \left\{A'(\phi_i - r)\left(\frac{r}{2}\right)\right\}\omega_{i+1-m} =$$
$$= \frac{h}{2}\left[F(\phi_{i+1}-r) + F(\phi_i + r)\right].$$

$$P_i\omega_{i-1} + Q_i\omega_i + R_i\omega_{i+1} + U_i\omega_{i-m+1} + V_i\omega_{i-m} + \tilde{W}_i\omega_{i-m-1} = G_i,\ i = 1,2,\dots,N-1 \tag{2.21}$$

$$P_i = \left\{\frac{\sigma\varepsilon}{h} - B(\phi_{i-1}-r)\left(\frac{h-r}{2}\right)\right\}, \qquad Q_i = \left\{\frac{-2\sigma\varepsilon}{h} + B(\phi_i - r)\left(\frac{h+r}{2}\right) + B(\phi_{i-1}+r)\left(\frac{r}{2}\right)\right\},$$

$$R_i = \left\{\frac{\sigma\varepsilon}{h} - B(\phi_i - r)\left(\frac{r}{2}\right)\right\}, \qquad U_i = \left\{-A(\phi_{i-1}) - A'(\phi_{i-1}+r)\left(\frac{h-r}{2}\right)\right\},$$

$$V_i = \left\{A(\phi_i) - A'(\phi_i - r)\left(\frac{h+r}{2}\right) - A'(\phi_{i-1}+r)\left(\frac{r}{2}\right)\right\}, \qquad \tilde{W}_i = \left\{A'(\phi_i - r)\left(\frac{r}{2}\right)\right\},$$

$$G_i = \frac{h}{2}\left[F(\phi_{i+1}-r) + F(\phi_i + r)\right].$$

The scheme of Eq.(2.21) can be expressed as

$$\begin{aligned}
&P_i\omega_{i-1} + Q_i\omega_i + R_i\omega_{i+1} = G_i - U_i\omega_{i-m+1} - V_i\omega_{i-m} - \tilde{W}_i\omega_{i-m-1} \quad \text{for} \quad 1 \le i \le m-1,\\
&P_i\omega_{i-1} + Q_i\omega_i + R_i\omega_{i+1} + U_i\omega_{i-m+1} = G_i - V_i\omega_{i-m} - \tilde{W}_i\omega_{i-m-1} \quad \text{for} \quad i = m,\\
&P_i\omega_{i-1} + Q_i\omega_i + R_i\omega_{i+1} + U_i\omega_{i-m+1} + V_i\omega_{i-m} = G_i - \tilde{W}_i\omega_{i-m-1} \quad \text{for} \quad i = m+1,\\
&P_i\omega_{i-1} + Q_i\omega_i + R_i\omega_{i+1} + U_i\omega_{i-m+1} + V_i\omega_{i-m} + \tilde{W}_i\omega_{i-m-1} = G_i \quad \text{for} \quad m+2 \le i \le N-1.
\end{aligned} \tag{2.22}$$

The system Eq. (22) is solved efficiently using Gauss elimination.

3. Analytic results

Lemma 1: (Discrete Minimum Principal). Suppose $\omega_0 \geq 0$ and $\omega_N \geq 0$. Then $L^N \omega_i \leq 0,\ \forall\ i = 0,1,2,...N-1$ given that $\omega_i \geq 0, \forall\ \ i = 0,1,2,...N$.

Proof. Let $j \in \{0,1,2,3,...,N\}$ be such that $= \underset{0\leq i\leq N}{min}\ \omega_i$ and assume that $\omega_j < 0$. Clearly, $j \notin \{0,N\}$. Then we have $\omega_i - \omega_{i-1} \leq 0,\ \omega_i - \omega_{i-1} \geq 0$ and for $1 \leq j < 0$, we have

$$
\begin{aligned}
L^N \omega_K &= \sigma\varepsilon\left\{\frac{\omega_{i-1} - 2\omega_i + \omega_{i+1}}{h^2}\right\} + \left\{\mathrm{B}(\phi_i + r)\left(\frac{r}{2}\right)\right\}\omega_{i-1} + \\
&+\left\{\mathrm{B}(\phi_{i+1} - r)\left(\frac{h-r}{2}\right) + \mathrm{B}(\phi_i + r)\left(\frac{h+r}{2}\right)\right\}\omega_i + \left\{\mathrm{B}(\phi_{i+1} - r)\left(\frac{h-r}{2}\right)\right\}\omega_{i+1} + \\
&+\left\{\mathrm{A}(\phi_{i+1}) - \mathrm{A}'(\phi_{i+1} - r)\left(\frac{h-r}{2}\right)\right\}\omega_{i+1-m} + \\
&+\left\{\mathrm{A}(\phi_i) - \mathrm{A}'(\phi_{i+1} - r)\left(\frac{r}{2}\right) - \mathrm{A}'(\phi_i + r)\left(\frac{h+r}{2}\right)\right\}\omega_{i-m} + \\
&+\left\{\mathrm{A}'(\phi_i + r)\left(\frac{r}{2}\right)\right\}\omega_{i-1-m} - \left\{\frac{F(\phi_{i+1} - r) + F(\phi_i + r)}{2}\right\} = \\
&= \sigma\varepsilon\left\{\frac{(\omega_{i+1} - \omega_i) - (\omega_i - \omega_{i-1})}{h^2}\right\} + \left\{\mathrm{B}(\phi_i + r)\left(\frac{r}{2}\right)\right\}\omega_{i-1} + \\
&+\left\{\mathrm{A}(\phi_{i+1}) - \mathrm{A}'(\phi_{i+1} - r)\left(\frac{h}{2}\right) + \mathrm{A}'(\phi_{i+1} - r)\left(\frac{r}{2}\right)\right\}\omega_{i+1-m} + \\
&+\left\{\mathrm{A}(\phi_i) - \left(\mathrm{A}'(\phi_{i+1} - r) + \mathrm{A}'(\phi_i + r)\right)\left(\frac{r}{2}\right) - \mathrm{A}'(\phi_i + r)\left(\frac{h}{2}\right)\right\}\omega_{i-m} + \\
&-\left\{\mathrm{A}'(\phi_i + r)\left(\frac{r}{2}\right)\right\} - \left\{\frac{F(\phi_{i+1} - r) + F(\phi_i + r)}{2}\right\}
\end{aligned}
$$

where $L^N \omega_K > 0$, since $(\omega_{i+1} - \omega_i) \geq 0,\ (\omega_i - \omega_{i-1}) \leq 0,\ \mathrm{A} > 0,\ \mathrm{B} > 0,\ \mathrm{A}'\langle 0, F\rangle 0$.
Thus, we have $L^N \omega_j > 0$ for $1 \leq j < m$ which contradicts the hypothesis that $L^N \omega_j \leq 0\ \ \forall i = 0,1,2,...N-1$ of the discrete minimum principle. Therefore, our assumption that $\omega_j < 0,$ is wrong. Hence $\omega_K \geq 0.$ but $j \in \{0,1,2,3,...,N-1\}$ is an arbitrary positive integer, so $\omega_K \geq 0.\ \ \forall i = 1,2,...N$.

Lemma 2. The solution ω of the discrete problem Eq. (2.1) with the boundary conditions Eq. (2.2) satisfies $\omega_0 \leq \frac{F_0}{\mathrm{B}_0} + \max\left(|\theta(0)|,|\beta|\right) \pm \omega(\phi)$.

Proof. Let us define $\Psi^{\pm}(x) = \frac{F_0}{\mathrm{B}_0} + \max\left(|\theta(-1)|,|\beta|\right) \pm \omega(\theta)$. Then we have

$$\Psi^{\pm}(0)=\frac{F_0}{B_0}+\max\left(\left|\theta(0)\right|,\left|\beta\right|\right)\pm\omega(0)=\frac{F_0}{B_0}+\max\left(\left|\theta(0)\right|,\left|\beta\right|\right)\pm\theta(0),$$

$$\Psi^{\pm}(1)=\frac{F_0}{B_0}+\max\left(\left|\theta(0)\right|,\left|\beta\right|\right)\pm\omega(1)=\frac{F_0}{B_0}+\max\left(\left|\theta(0)\right|,\left|\beta\right|\right)\pm\beta\geq 0,$$

$$\begin{aligned}L^N\omega_K&=\sigma\varepsilon\left\{\frac{\omega_{i-1}-2\omega_i+\omega_{i+1}}{h^2}\right\}+\left\{B(\phi_i+r)\left(\frac{h+2r}{2}\right)\right\}\left(\frac{\omega_{i-1}+\omega_i}{h}\right)+\\&+\left\{B(\phi_{i+1}-r)\left(\frac{r-h}{2}\right)\right\}\left(\frac{\omega_{i+1}-\omega_i}{h}\right)+A(\phi_{i+1})\omega_{i+1-m}-A(\phi_i)\omega_{i-m}+\\&+\left\{A'(\phi_{i+1}-r)\left(\frac{h}{2}\right)\right\}\left(\frac{\omega_{i+1-m}-\omega_{i-m}}{h}\right)+\left\{-A'(\phi_i+r)\left(\frac{h}{2}\right)\right\}\left(\frac{\omega_{i-m}-\omega_{i-1-m}}{h}\right)+\\&-\left\{\frac{F(\phi_{i+1}-r)+F(\phi_i+r)}{2}\right\},\end{aligned}$$

$$\begin{aligned}L^N\omega_K&=\sigma\varepsilon\left(\Psi^{\pm}(\phi)\right)''+\left\{B(\phi_i+r)\left(\frac{h+r}{2}\right)\right\}\left(\frac{\omega_{i-1}+\omega_i}{h}\right)+\\&+\left\{B(\phi_{i+1}-r)\left(\frac{r-h}{2}\right)\right\}\left(\frac{\omega_{i+1}-\omega_i}{h}\right)+A(\phi_{i+1})\omega_{i+1-m}-A(\phi_i)y_{i-m}+\\&+\left\{A'(\phi_{i+1}-r)\left(\frac{h}{2}\right)\right\}\left(\frac{\omega_{i+1-m}-\omega_{i-m}}{h}\right)+\left\{-A'(\phi_i+r)\left(\frac{h}{2}\right)\right\}\left(\frac{\omega_{i-m}-\omega_{i-1-m}}{h}\right)+\\&-\left\{\frac{F(\phi_{i+1}-r)+F(\phi_i+r)}{2}\right\},\end{aligned}$$

$$\begin{aligned}L_\varepsilon\Psi^{\pm}(\phi)&=\sigma\varepsilon\left(\Psi^{\pm}(\phi)\right)''+\left\{B(\phi_i+r)\left(\frac{r}{2}\right)\right\}\omega_{i-1}+\\&+\left\{\left(B(\phi_{i+1}-r)+B(\phi_i-r)\right)\left(\frac{h}{2}\right)+\left(B(\phi_{i+1}-r)+B(\phi_i-r)\right)\left(\frac{r}{2}\right)\right\}\omega_i+\\&+\left\{B(\phi_{i+1}-r)\left(\frac{h}{2}\right)-B(\phi_{i+1}-r)\left(\frac{r}{2}\right)\right\}\omega_{i+1}+\\&+\left\{A(\phi_{i+1})-A'(\phi_{i+1}-r)\left(\frac{h}{2}\right)+A'(\phi_{i+1}-r)\left(\frac{r}{2}\right)\right\}\omega_{i+1-m}+\\&+\left\{A(\phi_i)-\left(A'(\phi_{i+1}-r)+A'(\phi_i+r)\right)\left(\frac{r}{2}\right)-A'(\phi_i+r)\left(\frac{h}{2}\right)\right\}\omega_{i-m}+\\&-\left\{A'(\phi_i-r)\left(\frac{r}{2}\right)\right\}\omega_{i-1-m}-\left\{\frac{F(\phi_{i+1}-r)+F(\phi_i+r)}{2}\right\},\end{aligned}$$

$$L_\varepsilon\Psi^{\pm}(\phi)\geq 0.$$

4. Convergence analysis

The matrix equation for the left-end boundary layer problem, including the specified boundary conditions in Eq. (2.12), is as follows:

$$\mathcal{A}\nu+\mathcal{B}+T_i(h)=0 \tag{4.23}$$

$$\mathcal{A}=\begin{bmatrix} Q_1 & R_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ P_2 & Q_2 & R_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ 0 & P_3 & Q_3 & R_3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ U_m & 0 & \cdots & P_m & Q_m & R_m & 0 & \cdots & \cdots & \cdots & 0\\ V_{m+1} & U_{m+1} & 0 & \cdots & P_{m+1} & Q_{m+1} & R_{m+1} & 0 & \cdots & \cdots & 0\\ W_{m+2} & V_{m+2} & U_{m+2} & 0 & \cdots & P_{m+2} & Q_{m+2} & R_{m+2} & 0 & \cdots & 0\\ 0 & W_{m+3} & V_{m+3} & U_{m+3} & 0 & \cdots & P_{m+3} & Q_{m+3} & R_{m+3} & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0\\ \cdots & \cdots & 0 & W_{N-2} & V_{N-2} & U_{N-2} & 0 & \cdots & P_{N-2} & Q_{N-2} & R_{N-2}\\ \cdots & \cdots & \cdots & 0 & W_{N-1} & V_{N-1} & U_{N-1} & 0 & \cdots & P_{N-1} & Q_{N-1} \end{bmatrix}$$

and $\mathcal{B}=[\Upsilon_1,\Upsilon_2,\Upsilon,\ldots\Upsilon_m,\Upsilon_{m+1},\Upsilon_{m+2},\ldots,\Upsilon_{N-2},\Upsilon_{N-1}]$

where
$$\Upsilon_i=\begin{cases} G_i-U_iw_{i-m+1}-V_iw_{i-m}-\tilde{W}_iw_{i-m-1}-P_1w_0 & \text{for } 1\le i\le m-1,\\ G_i-V_iw_{i-m}-\tilde{W}_iw_{i-m-1} & \text{for } i=m,\\ G_i-\tilde{W}_iw_{i-m-1} & \text{for } i=m+1,\\ G_i-R_{N-1}w_N & \text{for } m+2\le i\le N-1. \end{cases}$$

$$P_i=\left\{\frac{\sigma\varepsilon}{h}+B(\phi_i+r)\left(\frac{r}{2}\right)\right\},\quad Q_i=\left\{\frac{-2\sigma\varepsilon}{h}+B(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)+B(\phi_i-r)\left(\frac{h+r}{2}\right)\right\},$$

$$R_i=\left\{\frac{\sigma\varepsilon}{h}+B(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\},\quad U_i=\left\{A(\phi_{i+1})-A'(\phi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\},$$

$$V_i=\left\{-A(\phi_i)-A'(\phi_{i+1}-r)\left(\frac{r}{2}\right)-A'(\phi_i+r)\left(\frac{h+r}{2}\right)\right\},\quad \tilde{W}_i=\left\{A'(\phi_i-r)\left(\frac{r}{2}\right)\right\}$$

$$G_i=\frac{h}{2}\left[F(\phi_{i+1}-r)+F(\phi_i+r)\right],\ \text{for}\ i=1,2,\ldots,N-1,$$

and

$$\left|\tau_i^N\right|\le O(h^2),\ \nu=[\nu_1,\nu_2,\ldots,\nu_{N-1}]^T,\ T_i(h)=[T_1,T_2,\ldots,T_{N-1}]^T,\ O=[0,0,\ldots,0]^T$$

are related vectors of Eq. (15). Let $\tilde{\nu}=\left[\tilde{\nu}_1,\tilde{\nu}_2,\ldots,\tilde{\nu}_{N-1}\right]^T \cong \nu$ satisfy the equation

$$\mathcal{A}\tilde{\nu}+\mathcal{B}=0. \tag{4.24}$$

Let $e_i=\nu_i-\nu_i, i=1(1)N-1$ be the error so that $E=\left[e_1,e_2,\ldots,e_{N-1}\right]^T=u-U$.
Subtracting Eq. (4.23) from Eq. (4.24), we obtain the error equation

$$\mathcal{A}E=T_i(h). \tag{4.25}$$

Let $\tilde{\mathcal{S}}_i$ be the sum of the i^{th} row elements of the matrix A. Then we have

$$\tilde{\mathcal{S}}_i=h\left[\mathrm{B}(\phi_{i+1}-r)+\frac{\mathrm{B}(\phi_i+r)}{2}\right]+r\left[\mathrm{B}(\phi_i+r)-\mathrm{B}(\phi_{i+1}-r)\right] \quad \text{for} \quad 1\le i\le m-1,$$

$$\tilde{\mathcal{S}}_i=\mathrm{A}(\phi_{i+1})+h\left[\mathrm{B}(\phi_{i+1}-r)+\frac{\mathrm{B}(\phi_i+r)-\mathrm{A}'(\phi_{i+1}-r)}{2}\right]+$$
$$+k\left[\mathrm{B}(\phi_i+r)-\mathrm{B}(\phi_{i+1}-r)+\frac{\mathrm{A}'(\phi_{i+1}-r)}{2}\right] \quad \text{for} \quad i=m,$$

$$\tilde{\mathcal{S}}_i=\mathrm{A}(\phi_{i+1})-\mathrm{A}(\phi_i)+h\left[\mathrm{B}(\phi_{i+1}-r)+\frac{\mathrm{B}(\phi_i+r)-\mathrm{A}'(\phi_{i+1}-r)-\mathrm{A}'(\phi_i+r)}{2}\right]+$$
$$+k\left[\mathrm{B}(\phi_i+r)-\mathrm{B}(\phi_{i+1}-r)-\frac{\mathrm{A}'(\phi_i+r)}{2}\right] \quad \text{for} \quad i=m+1,$$

$$\tilde{\mathcal{S}}_i=\mathrm{A}(\phi_{i+1})-\mathrm{A}(\phi_i)+h\left[\mathrm{B}(\phi_{i+1}-r)+\frac{\mathrm{B}(\phi_i+r)-\mathrm{A}'(\phi_{i+1}-r)-\mathrm{A}'(\phi_i+r)}{2}\right]+$$
$$+k\left[\mathrm{B}(\phi_i+k)-\mathrm{B}(\phi_{i+1}-r)\right] \quad \text{for} \quad m+2\le i\le N-1.$$

Let $\zeta_{1^*}=\min\limits_{1\le i\le N}\left|\mathrm{A}(t_i)\right|$ and $\zeta_1^*=\max\limits_{1\le i\le N}\left|\mathrm{A}(t_i)\right|$, $\zeta_{2^*}=\min\limits_{1\le i\le N}\left|\mathrm{B}(t_i)\right|$ and $\zeta_2^*=\max\limits_{1\le i\le N}\left|\mathrm{B}(t_i)\right|$. Since $0<\varepsilon\ll 1$ and $\varepsilon\propto O(h)$, with sufficiently small h, it is verified that $\mathcal{A}$ is irreducible and monotone. Therefore $\mathcal{A}^{-1}$ exists and $\mathcal{A}^{-1}\ge 0$. So, we can deduce from Eq. (4.25) that

$$E\le\mathcal{A}^{-1}T. \tag{4.26}$$

For small h, we have

$$\tilde{\mathcal{S}}_i\ge h\zeta_{2^*}, \quad \text{for} \quad 1\le i\le m-1,$$

$$\tilde{\mathcal{S}}_i \geq h\left[\mathrm{B}\left(\phi_{i+1}-r\right)+\frac{\mathrm{B}\left(\phi_i+r\right)-\mathrm{A}'\left(\phi_{i+1}-r\right)}{2}\right] \geq hC, \quad \text{for} \quad i=m,$$

$$\tilde{\mathcal{S}}_i \geq h\left[\mathrm{B}\left(\phi_{i+1}-r\right)+\frac{\mathrm{B}\left(\phi_i+r\right)-\mathrm{A}'\left(\phi_{i+1}-r\right)-\mathrm{A}'\left(\phi_i+r\right)}{2}\right] \geq hC, \quad \text{for} \quad i=m+1,$$

$$\tilde{\mathcal{S}}_i \geq h\left[\mathrm{B}\left(\phi_{i+1}-r\right)+\frac{\mathrm{B}\left(\phi_i+r\right)-\mathrm{A}'\left(\phi_{i+1}-r\right)-\mathrm{A}'\left(\phi_i+r\right)}{2}\right] \geq hC \text{ for } m+2 \leq i \leq N-1.$$

Let $\mathcal{A}_{i,k}^{-1}$ be the $(i,k)^{th}$ element of $\mathcal{A}^{-1}$ and we define $\mathcal{A}^{-1} = \max_{1\leq i\leq N-1} \sum_{k=1}^{N-1} \mathcal{A}_{i,k}^{-1}$ and $T(h) = \max_{1\leq i\leq N-1} \left|T_i(h)\right|$.

Since $\mathcal{A}_{i,k}^{-1} \geq 0$ and $\sum_{k=1}^{N-1} \mathcal{A}_{i,k}^{-1}.\tilde{\mathcal{S}}_k = 1$ for $i = 12,3,...,N-1$.

Hence $\sum_{k=1}^{m-1} \mathcal{A}_{i,k}^{-1} \leq \dfrac{1}{\min\limits_{1\leq k\leq m-1} \tilde{\mathcal{S}}_k} < \dfrac{1}{h\zeta_{2^*}}$, $\quad i=1,2,3,\ldots,m-1.$ (4.27)

$$\mathcal{A}_{i,k}^{-1} \leq \frac{1}{\tilde{\mathcal{S}}_m} < \frac{1}{hC}, \quad k=m,\ m+1, \tag{4.28}$$

Furthermore $\sum_{k=m+2}^{N-1} \mathcal{A}_{i,k}^{-1} \leq \dfrac{1}{\min\limits_{m+2\leq k\leq N-1} \tilde{\mathcal{S}}_k} < \dfrac{1}{hC}$, $\quad i=m+2,\ m+3,...,N-1.$ (4.29)

Using Eqs. (4.27) – (4.29) and Eq. (4.26), we get

$$E \leq O(h). \tag{4.30}$$

As a result, it demonstrates that the proposed numerical integration scheme is first-order convergent. The same procedure can also be used to examine the convergence analysis for the right end boundary layer.

5. Numerical illustrations

To illustrate the comparative performance of the suggested method, it is implemented on numerical experiments with left-end and right-end layer for the small and large values of δ. The maximum absolute errors in the solution of the examples are calculated using the double mesh principle $E^N = \max_{0\leq i\leq N}\left|\omega_i^N - W_{2i}^{2N}\right|$. The maximum absolute errors are tabulated in the form of tables 1- 8 for considered examples. The computed errors are compared with the results given in [9]. It has been observed that our method yields more precise results than the method suggested in [9].

Example 1. $\varepsilon\omega''(\phi)+\omega'(\phi-\delta)-\omega(\phi)=0$ with $\omega(0)=1$ and $\omega(1)=1$.

Example 2. $\varepsilon\omega''(\phi)+\exp(\phi)\omega'(\phi-\delta)-\phi\omega(\phi)=0$ with $\omega(0)=1$ and $\omega(1)=1$.

Example 3. $\varepsilon\omega''(\phi)-\omega'(\phi-\delta)-\omega(\phi)=0$ with $\omega(0)=1$ and $\omega(1)=-1$.

Example 4. $\varepsilon\omega''(\phi)-e^{\phi}\omega'(\phi-\delta)-\omega(\phi)=0$, with $\omega(0)=1, \omega(1)=1$.

Example 5. $\varepsilon\omega''(\phi)+\omega'(\phi-\delta)+\omega(\phi)=0$ with $\omega(\phi)=1$ for $-\delta\le\phi\le 0$, $\omega(1)=1$.

Example 6. $\varepsilon\omega''(\phi)+0.25\omega'(\phi-\delta)-\omega(\phi)=0$; $\omega(\phi)=1$ for $-\delta\le\phi\le 0$, $\omega(1)=0$.

Table 1. The maximum absolute error in Example 1 for $\varepsilon=0.01$.

N	$\delta=1\times\varepsilon$	$\delta=2\times\varepsilon$	$\delta=3\times\varepsilon$	$\delta=4\times\varepsilon$	$\delta=5\times\varepsilon$
500	5.7171e-03	5.6433e-03	5.0893e-03	2.9055e-03	1.0635e-03
1000	2.8066e-03	2.7871e-03	2.4936e-03	1.3860e-03	5.0466e-04
1500	1.8596e-03	1.8490e-03	1.6509e-03	9.0957e-04	3.3066e-04
2000	1.3904e-03	1.3832e-03	1.2338e-03	6.7682e-04	2.4586e-04
2500	1.1103e-03	1.1051e-03	9.8498e-04	5.3891e-04	1.9567e-04
3000	9.2409e-04	9.2003e-04	8.1965e-04	4.4767e-04	1.6250e-04

Table 2. The maximum absolute error in Example 2 for $\varepsilon=0.01$.

N	$\delta=1\times\varepsilon$	$\delta=2\times\varepsilon$	$\delta=3\times\varepsilon$	$\delta=4\times\varepsilon$	$\delta=5\times\varepsilon$
500	1.4784e-03	1.3758e-03	1.1185e-03	9.0294e-04	4.5532e-04
1000	7.1992e-04	6.6958e-04	5.4822e-04	4.3872e-04	2.1754e-04
1500	4.7604e-04	4.4247e-04	3.6308e-04	2.8965e-04	1.4285e-04
2000	3.5561e-04	3.3041e-04	2.7141e-04	2.1618e-04	1.0633e-04
2500	2.8379e-04	2.6363e-04	2.1670e-04	1.7243e-04	8.4677e-05
3000	2.3611e-04	2.1931e-04	1.8035e-04	1.4341e-04	7.0351e-05

Table 3. The maximum absolute error in Example 3 for $\varepsilon=0.01$.

N	$\delta=1\times\varepsilon$	$\delta=2\times\varepsilon$	$\delta=3\times\varepsilon$	$\delta=4\times\varepsilon$	$\delta=5\times\varepsilon$
500	7.9989e-02	5.6699e-02	4.7088e-02	2.0363e-02	5.8039e-03
1000	3.7649e-02	2.7251e-02	2.2936e-02	9.5514e-03	2.7362e-03
1500	2.4601e-02	1.7936e-02	1.5152e-02	6.2342e-03	1.7890e-03
2000	1.8273e-02	1.3367e-02	1.1312e-02	4.6266e-03	1.3289e-03
2500	1.4532e-02	1.0653e-02	9.0243e-03	3.6779e-03	1.0569e-03
3000	1.2063e-02	8.8551e-03	7.5062e-03	3.0520e-03	8.7739e-04

Table 4. The maximum absolute error in Example 4 for $\varepsilon=0.01$.

N	$\delta=1\times\varepsilon$	$\delta=2\times\varepsilon$	$\delta=3\times\varepsilon$	$\delta=4\times\varepsilon$	$\delta=5\times\varepsilon$
500	4.9635e-02	3.3536e-02	2.3111e-02	1.5571e-02	5.9069e-03
1000	2.3235e-02	1.6030e-02	1.1198e-02	7.4637e-03	2.7959e-03
1500	1.5162e-02	1.0531e-02	7.3890e-03	4.9055e-03	1.8305e-03
2000	1.1251e-02	7.8407e-03	5.5134e-03	3.6529e-03	1.3605e-03
2500	8.9437e-03	6.2453e-03	4.3973e-03	2.9098e-03	1.0825e-03
3000	7.4216e-03	5.1893e-03	3.6569e-03	2.4179e-03	8.9884e-04

Table 5. The maximum absolute error in Example 5 for $\varepsilon = 0.01$.

N	$\delta = 1\times\varepsilon$	$\delta = 2\times\varepsilon$	$\delta = 3\times\varepsilon$	$\delta = 4\times\varepsilon$
500	*4.5602e-03*	*4.5309e-03*	*3.9614e-03*	*2.4622e-03*
1000	*2.2386e-03*	*2.2386e-03*	*1.9414e-03*	*1.1807e-03*
1500	*1.4859e-03*	*1.4833e-03*	*1.2855e-03*	*7.7614e-04*
2000	*9.6082e-04*	*5.7803e-04*	*1.1118e-03*	*1.1090e-03*
2500	*8.8818e-04*	*8.8560e-04*	*7.6707e-04*	*4.6047e-04*
3000	*7.3955e-04*	*7.3709e-04*	*6.3834e-04*	*3.8265e-04*

Table 6. The maximum absolute error in Example 6 for $\varepsilon = 0.01$.

N	$\delta = 1\times\varepsilon$	$\delta = 2\times\varepsilon$	$\delta = 3\times\varepsilon$	$\delta = 4\times\varepsilon$
500	*3.5351e-03*	*5.9970e-03*	*7.5423e-03*	*8.4194e-03*
1000	*1.7795e-03*	*3.0008e-03*	*3.7642e-03*	*4.1959e-03*
1500	*1.1891e-03*	*2.0013e-03*	*2.5082e-03*	*2.7942e-03*
2000	*8.9302e-04*	*1.5012e-03*	*1.8805e-03*	*2.0945e-03*
2500	*7.1493e-04*	*1.2011e-03*	*1.5042e-03*	*1.6750e-03*
3000	*5.9606e-04*	*1.0010e-03*	*1.2533e-03*	*1.3956e-03*

Table 7. The maximum absolute error in Example 5 for $\varepsilon = 0.1$.

$N\downarrow$	$\delta = 0.03$	$\delta = 0.05$	$\delta = 0.08$
Results with fitting parameter			
100	9.0504e-04	1.3784e-03	1.8354e-03
200	4.8897e-04	7.2077e-04	9.3765e-04
300	3.3391e-04	4.8719e-04	6.2932e-04
400	2.5349e-04	3.6786e-04	4.7360e-04
500	2.0418e-04	2.9546e-04	3.7963e-04
Results in [19]			
100	2.0790e-003	5.3420e-003	38340e-002
200	1.1020e-003	3.2160e-003	2.1830e-002
300	7.4900e-004	2.2860e-003	1.5200e-002
400	5.6800e-004	1.7700e-003	1.1650e-002
500	4.5700e-004	1.4440e-003	9.4430e-003

Table 8. The maximum absolute error for Example 6 for $\delta = 0.03$.

$\varepsilon\downarrow$	$N = 100$	$N = 200$	$N = 400$
Results with fitting parameter			
2^{-2}	5.8020e-05	1.7321e-05	1.2480e-05
2^{-3}	2.1893e-04	1.3941e-04	7.7020e-05
2^{-4}	1.4693e-03	7.9136e-04	4.0921e-04
2^{-5}	4.9989e-03	2.6121e-03	1.3329e-03
2^{-6}	1.8743e-02	9.4585e-03	4.7544e-03
2^{-7}	5.4492e-02	2.6598e-02	1.3169e-02

Cont.Table 8. The maximum absolute error for Example 6 for $\delta = 0.03$.

Results in [9]			
2^{-2}	6.1200e-004	3.0700e-004	1.5400e-004
2^{-3}	1.6340e-003	8.2600e-004	4.1500e-004
2^{-4}	4.2470e-003	2.1770e-003	1.1020e-003
2^{-5}	1.1675e-002	6.1590e-003	3.1660e-003
2^{-6}	3.3680e-002	1.8811e-002	9.9520e-003
2^{-7}	9.7727e-002	5.9836e-002	3.3166e-002

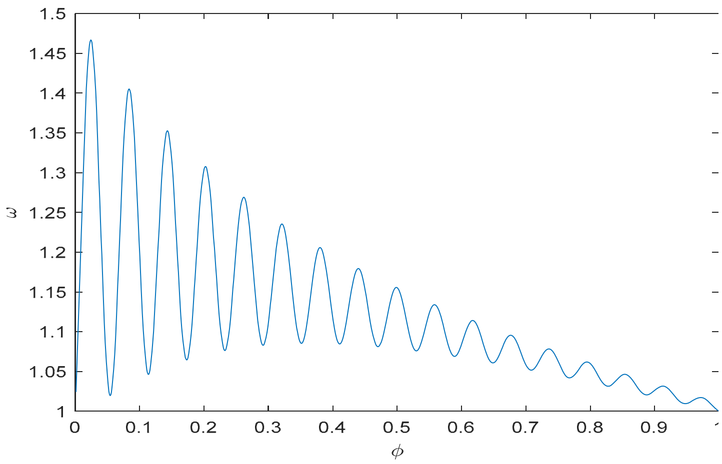

Fig.1. Solution profile in Example 1 for $\varepsilon = 0.01$ and $\delta = 1.5 \times \varepsilon$ without fitting parameter.

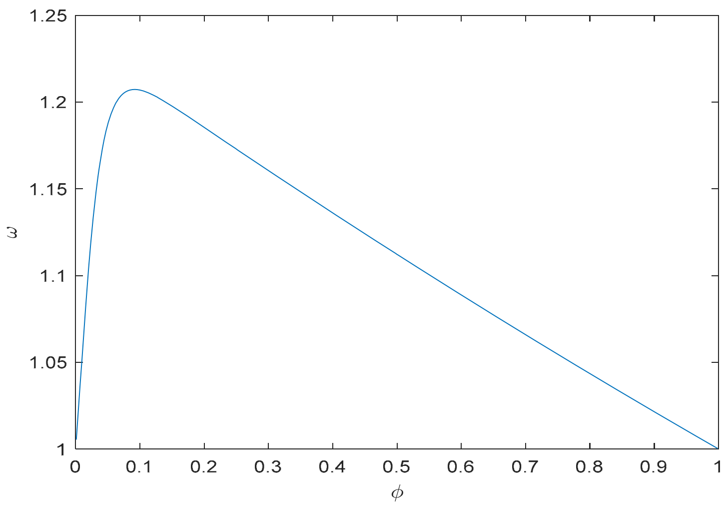

Fig.2. Numerical solution of Example 1 for $\varepsilon = 0.01$ and $\delta = 1.5 \times \varepsilon$ with fitting parameter.

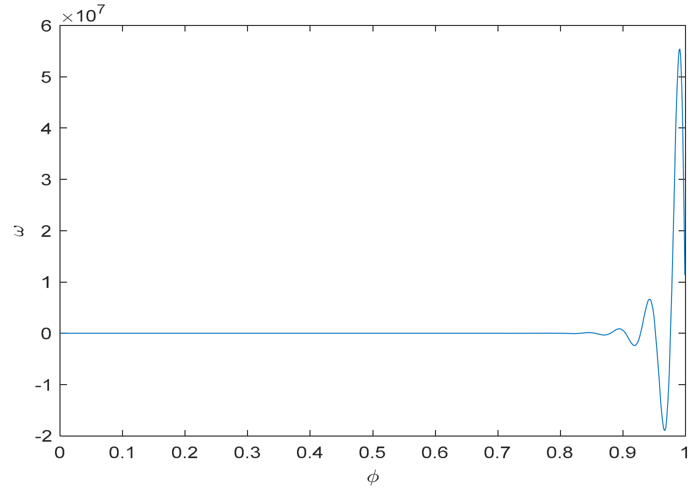

Fig.3. Solution profile in Example 2 for $\varepsilon = 0.01$ and $\delta = 1.5 \times \varepsilon$ without fitting parameter.

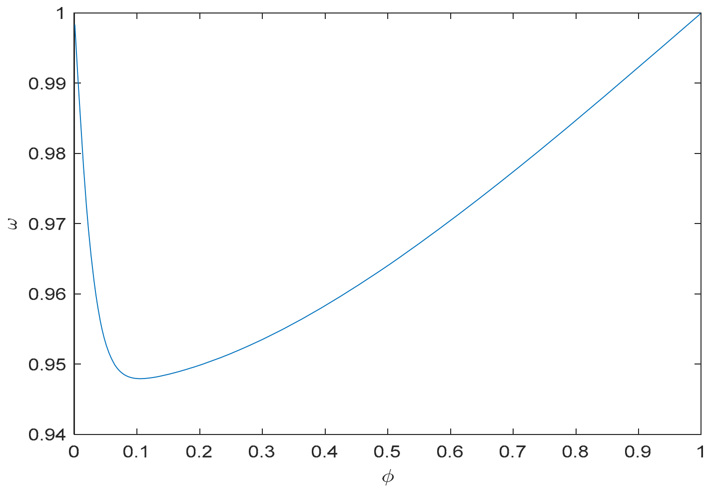

Fig.4. Solution profile in Example 2 for $\varepsilon = 0.01$ and $\delta = 1.5 \times \varepsilon$ with fitting parameter.

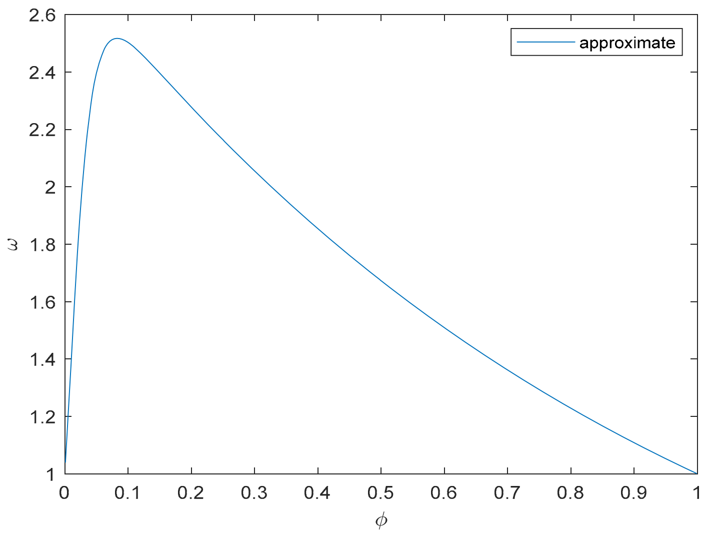

Fig.5. Solution profile in Example 3 for $\varepsilon = 0.01$ and $\delta = 1.5 \times \varepsilon$ without fitting parameter.

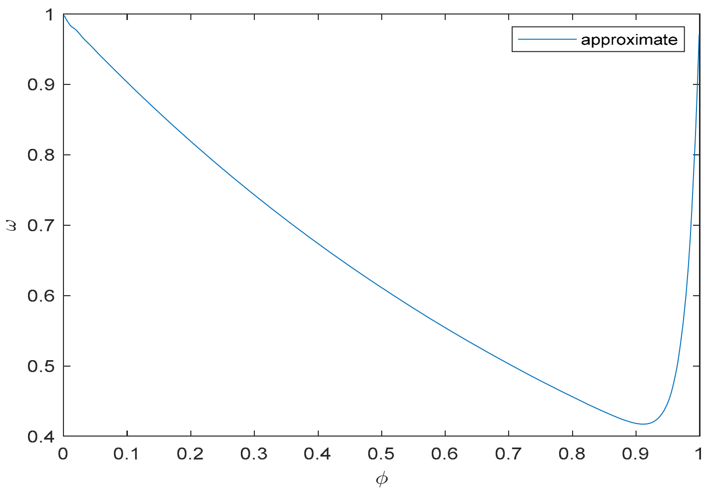

Fig.6. Numerical solution of Example 3 for $\varepsilon = 0.01$ and $\delta = 1.5 \times \varepsilon$ with fitting parameter.

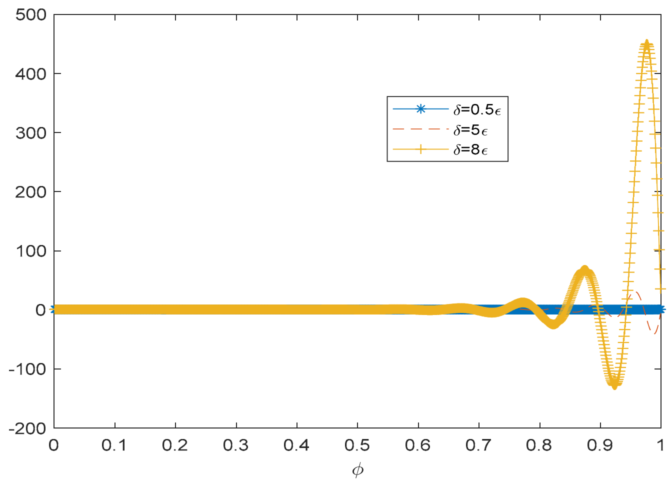

Fig.7. Solution profile in Example 4 for $\varepsilon = 0.01$ and $\delta = 1.5 \times \varepsilon$ without fitting parameter.

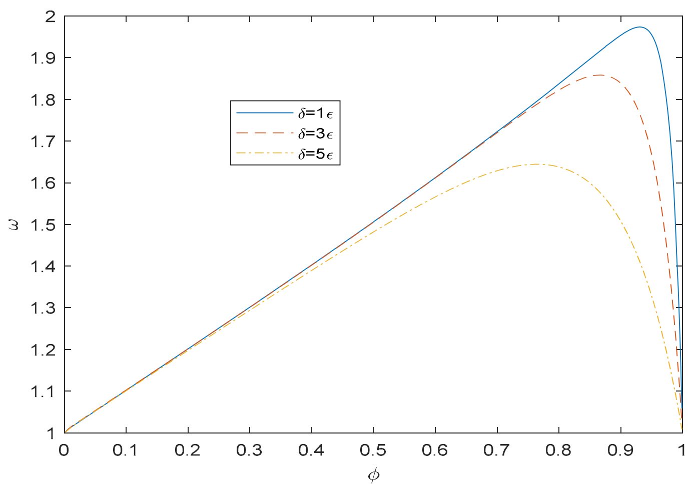

Fig.8. Solution profile in Example 4 for $\varepsilon = 0.01$ and $\delta = 1.5 \times \varepsilon$ with fitting parameter.

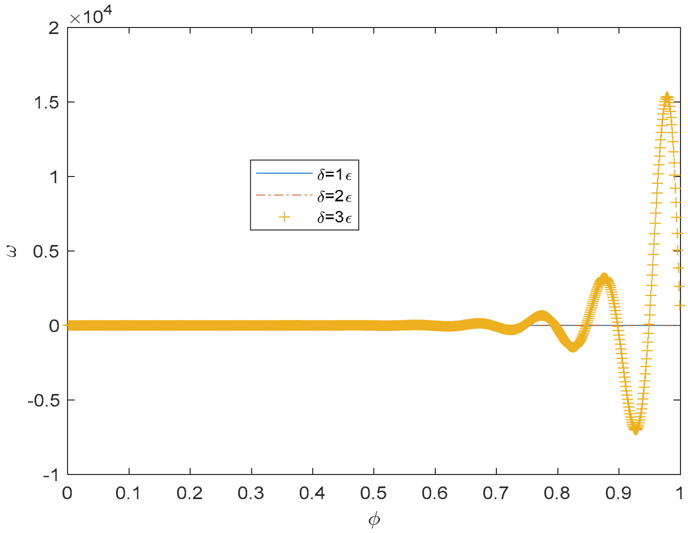

Fig.9. Numerical solution in Example 5 for $\varepsilon = 0.01$ without fitting parameter.

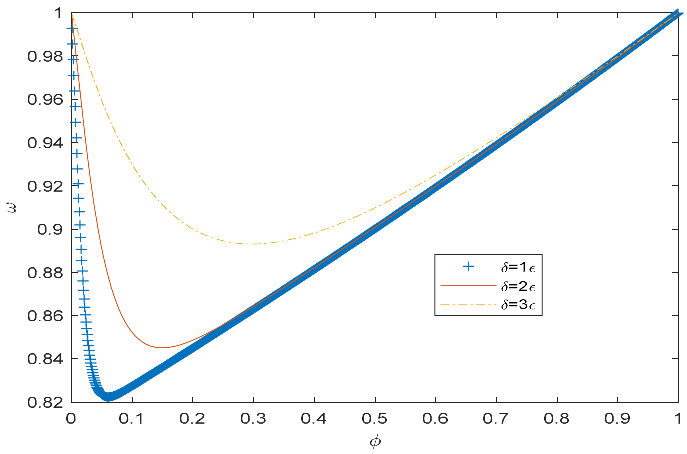

Fig.10. Solution profile in Example 5 for $\varepsilon = 0.01$ with fitting parameter.

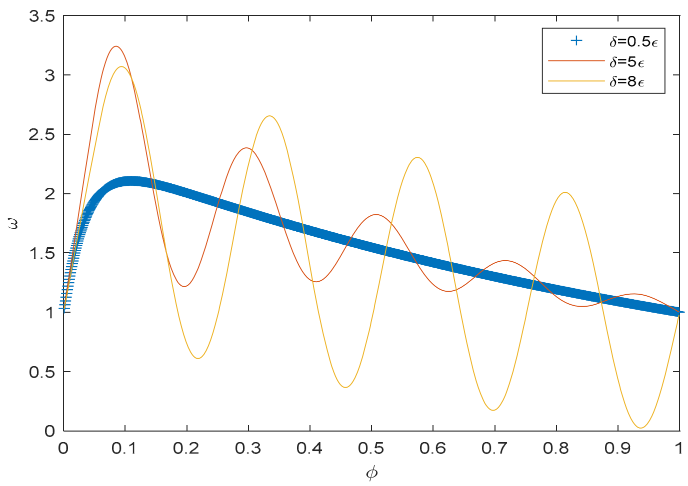

Fig.11. Numerical solution in Example 6 for $\varepsilon = 0.01$ without fitting parameter.

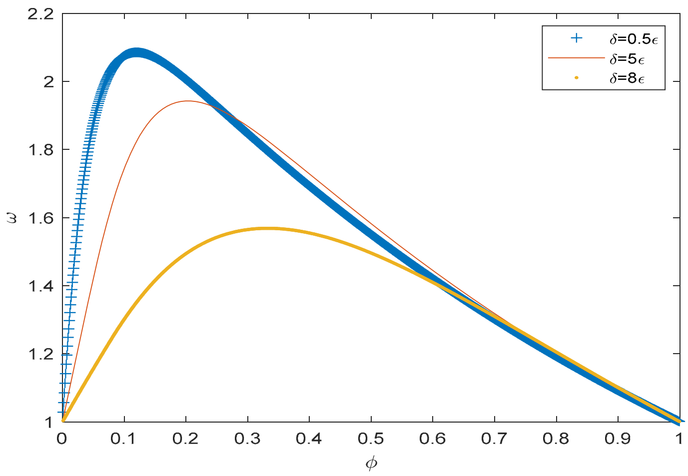

Fig.12. Solution profile in Example 6 for $\varepsilon = 0.01$ and with fitting parameter.

6. Conclusions

In this paper, a computational approach to solving the singularly perturbed differential equation with large delay is derived. Using a special type of mesh, a numerical scheme consisting of a fitting parameter is developed to minimize the error and to control the layer structure in the solution. Six examples are solved and numerical results with large delay are shown in Tabs 1-6. We have noticed that as the mesh size decreases maximum error also decreases. We have examined the method for small delay also and the numerical results are displayed in Tabs 7-8 with a comparison to the method given in [19]. Using these results, it is observed that the proposed scheme gives good results with the fitting parameter. We also focused on the influence of large delay on the layer structure or oscillatory behaviour of the solutions with and without the fitting parameter in the proposed scheme. The layer behaviour in the solution of the examples with and without the fitting parameter is shown in Figs 1-12. We have clearly noticed that the fitting parameter controls the oscillations in the layer for the large delay values. The proposed method is simple and it works very well with small delay as well as large delay.

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Nomenclature

$A(\phi), B(\phi), F(\phi)$ and $\theta(\phi)$ – smooth functions

C – positive constant

e_i – error

E – error matrix

E^N – absolute error

$\mathcal{G}$ – differentiable function

h – mesh size

L^N – differential operator with mesh size N

M – positive constant

N – number of sub intervals
p – positive integer
q – mantissa of δ
$T_i(h)$ – truncation error
β – constant
δ – delay parameter
ρ – mesh ration
σ – fitting parameter
ϕ –independent variable
ϕ_i – mesh points
ε – perturbation parameter
ω – solution

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