Int. J. of Applied Mechanics and Engineering, 2022, vol.27, No.4, pp.179-193
DOI: 10.2478/ijame-2022-0058

A METHOD FOR COMPARISON OF LARGE DEFLECTION IN BEAMS

Yasser TAGHIPOUR* and Sara DARFARIN
Department of Mechanical Engineering, Sirjan University of Technology, Sirjan, IRAN.
P.O.B. 7816418097 Sirjan, IRAN
E-mail: ytaghipour@sirjantech.ac.ir

The deflection analysis of beams has been recently an active area of research. The large deflection of beams refers to deflections occurring due to large displacements and small strains. This type of deflection has been one of the areas of interest in the development of beam deformation methods. The wide diversity of beam deformation methods highlights the importance of their comparison to further elucidate the properties and features of each method and determine their benefits and limitations. In this study, a new comparison model is introduced which involves three steps, instead of only comparing final results for verification in common studies. In the first step, a complete comparison is made based on the assumptions and approximations of each method of the kinematics of deformation, displacement, and strain fields. After selecting the most accurate method in the first step, the displacement functions are determined by polynomial approximation under different loading and support conditions based on the selected method. In the third step, the displacement functions are used to calculate the strains in each method. The conclusion is based on comparing the strains. This comparative model can be used as a benchmark to compare different theories of deformation analysis.

Keywords: comparison method, beam, large deflection, displacement function, strain.

1. Introduction

In recent years, the deflection analysis of beams has been one of the hottest topics in solid-state mechanical engineering. The large deflection of beams refers to the deflection due to large displacements and small strains. This type of deflection has been one of the areas of interest in the development of beam deformation analysis methods. Considering the variety of beam deformation methods, a comparison of these methods is of great importance.

Ohtsuki [1] described the large deflection of a simply supported beam under symmetrical three-point bending. An experiment was also performed to validate the applicability of the proposed model.

Beléndez *et al*. [2] analyzed the large deflection of a cantilever beam under a uniformly distributed load and an external vertical concentrated load at the free end, both experimentally and numerically. They compared the numerical results with the experimental findings to find the modulus of elasticity of the beam and verify the numerical results.

Nanakorn and Vu [3] proposed a new 2D Euler-Bernoulli beam element for the large displacement analysis using the total Lagrangian formulation. They confirmed the validity and efficiency of the proposed element by comparing various numerical results found in the literature.

Xiao [4] investigated the large deflection of prismatic cantilever beams under a distributed load. An approximate analytical solution was obtained using the homotopy analysis method. The solution was validated through comparison with the nonlinear shooting method.

Mohyeddin and Fereidoon [5] analyzed large deflections of a simply supported beam exposed to a point load in the middle. The results were compared with available experimental data and those obtained for the Euler-Bernoulli beam.

* To whom correspondence should be addressed

Li and Li [6] studied the three-point bending of a beam based on the Timoshenko theory. They finally compared large deflections and large rotations with their classical equivalents and experimental results.

Taghipour and Baradaran [7] proposed a large deflection model by considering surface effects for nanowires. This model was verified by experimental data and used for interpretation of some other experiments.

Abu-Alshaikh *et al*. [8] analyzed large deflections of a prismatic cantilever beam under concentrated force and moment. A new numerical solution was found by an elliptic integral construction. The results were obtained based on the Euler-Bernoulli beam theory, verified by comparing them with available numerical data.

Taghipour and Baradaran [9] developed a large deflection model based on the nonlocal theory for nanowires. The numerical results were verified with the existence of numerical data for small deflections of the Timoshenko nanobeams.

Bouadjadja *et al*. [10] investigated the large deflection response of composite cantilever beam experimentally and analytically. The analytical model was derived based on the classical Euler–Bernoulli beam theory with both symmetric and antisymmetric laminated beams and validated by those obtained analytically and experimentally in the literature.

Zeng *et al*. [11] proposed a numerical approach for the cantilever beam under a force pointing at a fixed point with large deflections. They verified the analysis model with a commercial finite element analysis.

Estabragh and Baradaran [12] investigated a finite element model for large deflection analysis of nanobeams based on the modified couple stress theory. The results were validated with the numerical data in the literature.

Li *et al*. [13] improved the homotopy analysis method to solve the strongly nonlinear differential equation, for example for a cantilever beam subjected to point a load at the free end. The results were validated with the traditional homotopy method.

Most of the mentioned studies and other comparative investigations compared the final results of different methods with each other for verification without following a specific comparison strategy. Because their main goal of comparisons was to evaluate the methods. This allows a fair and insightful comparison only under the same conditions. In such cases, important effective factors such as deformation assumptions, the displacement field, and strain field are often neglected.

In this study, a new comparison model is proposed in a three-step strategy, as opposed to most studies that compared their results by considering the points mentioned overhead. The different methods were compared by classifying large analytical deflection models (proposed for beams) in terms of the type of kinematics of deformation, and the simplifications and assumptions of the displacement and strain fields in the first step. The values were parametrically compared in this step. The displacement functions were extracted as polynomial functions based on the most accurate method, in the second step. Finally, in the third step, different methods of analyzing the deflection of beams were compared based on different modes of support, loading, thickness, and deflection. Assuming the displacement field, the strain field was calculated by different introduced methods. Then, the values of the calculated strains were compared with the reference values (the most accurate strain values in this study i.e., strain values based on the Timoshenko method, without any simplifying hypotheses).

2. The first step of the problem-solving approach

Depending on the type of kinematics of deformation, the analysis of the large deflection methods can be divided into two general categories. The first category refers to methods giving up from the shear strain effect in deformation, known as the Euler-Bernoulli beam theory. The second class includes methods that assume the shear strain at the cross-sectional area of a beam is constant (i.e., Timoshenko's beam theory). Finally, the most accurate method was selected.

2.1. Methods of large deflection analysis of beams based on the Euler-Bernoulli theory

Based on the Euler-Bernoulli beam theory, the beam displacement field will be as depicted in Fig.1. [3]:

$$u(X,Y,Z)=u_0(X)-Y\sin(\theta(X)), \tag{2.1}$$

$$v(X,Y,Z)=v_0(X)-Y\{1-\cos(\theta(X))\}, \tag{2.2}$$

$$w(X,Y,Z)=0 \tag{2.3}$$

where $u(X,Y,Z)$, $v(X,Y,Z)$ and $w(X,Y,Z)$ are the displacements of a point on the beam cross section, in the X, Y, and Z directions, $u_0(X)$ and $v_0(X)$ are the displacements of the centroid of the beam cross-section in the X and Y directions, respectively. Moreover, θ is the slope of the neutral axis of the beam and also the rotation of the beam cross section.
The strain components are based on the Green-Lagrange strain relationship as follows [14]:

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}\right)+\frac{1}{2}\left(\frac{\partial u_m}{\partial X_i}\frac{\partial u_m}{\partial X_j}\right), \quad i,j=1,2,3 \tag{2.4}$$

where $u_1=u$, $u_2=v$, $u_3=w$, $X_1=X$, $X_2=Y$ and $X_3=Z$.
Based on the Euler-Bernoulli theory, the only non-zero strain is the normal component, ε_{xx}:

$$\varepsilon_{xx}=\left(\frac{du_0}{dX}-Y\cos\theta\frac{d\theta}{dX}\right)+\frac{1}{2}\left(\frac{du_0}{dX}-Y\cos\theta\frac{d\theta}{dX}\right)^2+\frac{1}{2}\left(\frac{dv_0}{dX}-Y\sin\theta\frac{d\theta}{dX}\right)^2. \tag{2.5}$$

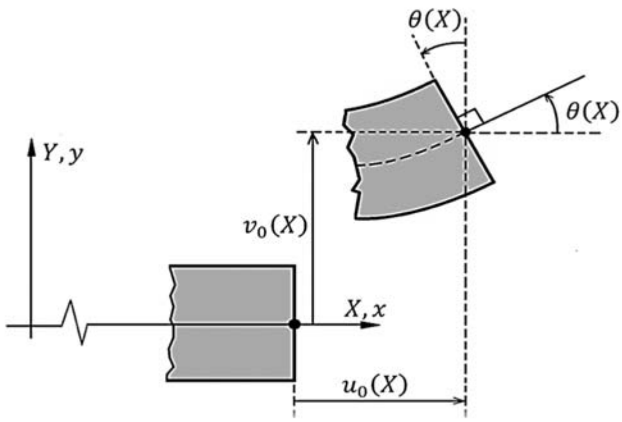

Fig.1. Beam deformation based on the Euler-Bernoulli beam theory.

2.1.1. The first method

In the first approximation of the Euler-Bernoulli beam theory [1, 5, 8, 10, 11, 13, 15-27], the following simplifying assumptions are considered in calculating the displacement field:

- Failure to change the axial length, i.e., $u_0=0$.
- $\cos\theta\approx 1$,

- $\varepsilon_{ij} \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_i}{\partial X_j}\right)$.

Finally, the strain of ε_{xx} can be obtained as follows [28]:

$$\varepsilon_{xx} = \frac{du}{dX} = -Y\cos\theta\frac{d\theta}{dX} = -Y\frac{dx}{ds}\frac{d\theta}{dx} = -Y\frac{d\theta}{dS} = -\frac{Y}{\rho}. \tag{2.6}$$

where, ρ is the radius of curvature of the neutral axis of the beam.
In other words, in this method, the normal strain is approximated by the engineering strain resulting from the pure bending:

$$\varepsilon_{xx} \approx \frac{\delta}{l_0} = \frac{l - l_0}{l_0} = -\frac{Y}{\rho}. \tag{2.7}$$

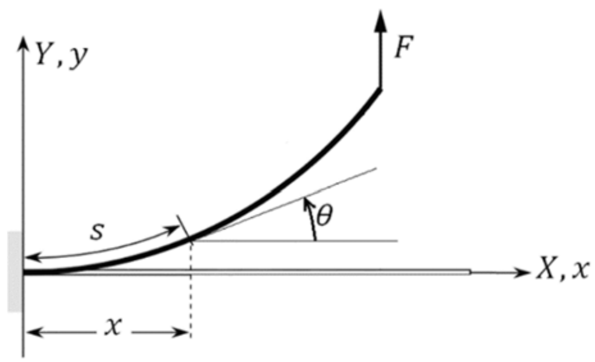

Fig.2. Display $(S - \theta)$ on a cantilever beam.

2.1.2. The second method

In the second approximation of the Euler-Bernoulli beam theory [29-31] the following simplifying assumptions are considered in calculating the displacement field:

- $\sin\theta \approx \tan\theta \approx \frac{dv_0}{dX}$,
- $\cos\theta \approx 1$,
- Consideration of the von Karman strains.

The strain of ε_{xx} is as follows:

$$\varepsilon_{xx} = \left(\frac{du_0}{dX} - Y\frac{d^2v_0}{dX^2}\right) + \frac{1}{2}\left\{\left(\frac{du_0}{dX} - Y\frac{d^2v_0}{dX^2}\right)^2 + \left(\frac{dv_0}{dX}\right)^2\right\}. \tag{2.8}$$

Assuming the von Karman strains, the terms $\left(\frac{du_0}{dX} - Y\frac{d^2v_0}{dX^2}\right)^2$ can be omitted from the rest of the expressions, such that:

$$\varepsilon_{xx} \approx \left(\frac{du_0}{dX} - Y\frac{d^2v_0}{dX^2}\right) + \frac{1}{2}\left(\frac{dv_0}{dX}\right)^2 . \tag{2.9}$$

2.2. Methods of large deflection analysis of beams based on Timoshenko's theory

According to Fig.3., the most general displacement field in this analysis method [7, 9, 32, 33] is:

$$u(X,Y,Z) = u_0(X) - Y\sin(\varphi(X)), \tag{2.10}$$

$$v(X,Y,Z) = v_0(X) - Y\{1 - \cos(\varphi(X))\}, \tag{2.11}$$

$$w(X,Y,Z) = 0, \tag{2.12}$$

where $u(X,Y,Z)$, $v(X,Y,Z)$ and $w(X,Y,Z)$ are the displacements of a point on the beam cross section, in the X, Y, and Z directions. In addition, $u_0(X)$ and $v_0(X)$ are the displacements of centroid of the beam cross-section, in the X and Y directions. Moreover, θ and φ denote the slope of the neutral axis of the beam and the rotation of the beam cross-section, respectively. Based on the Timoshenko beam theory, $\gamma = \theta - \varphi$ is the mean shear strain in the beam cross-section.

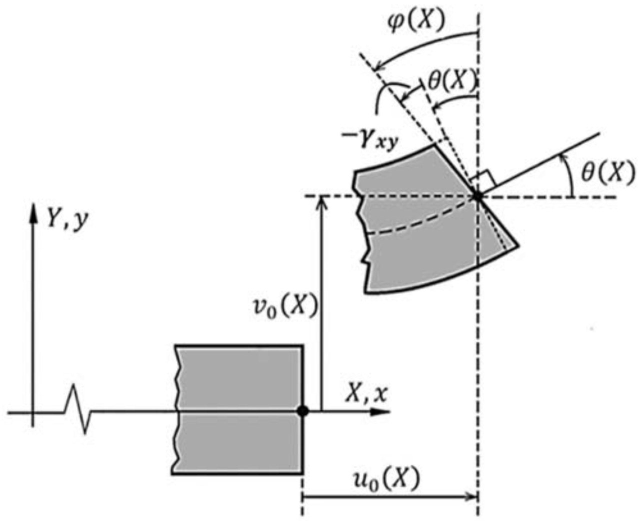

Fig.3. Beam deformation based on the Timoshenko beam theory.

The strain components can be determined based on the Green-Lagrange strain relationship and the only non-zero strains based on the Timoshenko theory are the shear strain $\gamma_{xy} = 2\varepsilon_{xy}$ and the vertical strain ε_{xx} :

$$\varepsilon_{xx} = \left(\frac{du_0}{dX} - Y\cos\varphi\frac{d\varphi}{dX}\right) + \frac{1}{2}\left(\frac{du_0}{dX} - Y\cos\varphi\frac{d\varphi}{dX}\right)^2 + \frac{1}{2}\left(\frac{dv_0}{dX} - Y\sin\varphi\frac{d\varphi}{dX}\right)^2, \tag{2.13}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = -\sin\varphi\left(1 + \frac{du_0}{dX}\right) + \frac{dv_0}{dX}\cos\varphi. \tag{2.14}$$

2.2.1. The third method

In the first approximation of the Timoshenko beam theory [5, 32, 33], the simplifying assumptions for calculating the displacement field are:

- $\sin\theta \approx \tan\theta \approx \varphi$,
- $\cos\theta \approx 1$,
- Using the von Karman strains.

Finally, the strains of ε_{xx} and γ_{xy} can be calculated as follows:

$$\varepsilon_{xx} = \left(\frac{du_0}{dX} - Y\frac{d\varphi}{dX}\right) + \frac{1}{2}\left\{\left(\frac{du_0}{dX} - Y\frac{d\varphi}{dX}\right)^2 + \left(\frac{dv_0}{dX}\right)^2\right\}, \tag{2.15}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \left(\frac{dv_0}{dX} - \varphi\right) - \varphi\left(\frac{du_0}{dX} - Y\frac{d\varphi}{dX}\right). \tag{2.16}$$

Assuming the von Karman strains, the terms $\left(\frac{du_0}{dX} - Y\frac{d^2v_0}{dX^2}\right)^2$ and $\varphi\left(\frac{du_0}{dX} - Y\frac{d\varphi}{dX}\right)$ can be eliminated from the rest of the expressions, so:

$$\varepsilon_{xx} \approx \left(\frac{du_0}{dX} - Y\frac{d\varphi}{dX}\right) + \frac{1}{2}\left(\frac{dv_0}{dX}\right)^2, \tag{2.17}$$

$$\gamma_{xy} = 2\varepsilon_{xy} \approx \frac{dv_0}{dX} - \varphi. \tag{2.18}$$

2.2.2. The fourth method

The second approximation of the Timoshenko beam theory [5] does not involve any simplistic assumptions in calculating the displacement field. However, the nonlinear part of the strain was removed from ε_{xx}. Finally, the strains are defined as follows:

$$\varepsilon_{xx} \approx \frac{du_0}{dX} - Y\cos\varphi\frac{d\varphi}{dX}, \tag{2.19}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = -\sin\varphi\left(1+\frac{du_0}{dX}\right)+\frac{dv_0}{dX}\cos\varphi . \tag{2.20}$$

2.2.3. The fifth method

In the third approximation of the Timoshenko beam theory [7, 9, 33], the displacement field is as simple as possible without simplifying assumptions; only the strain ε_{xx} is simplified using the consistent linearization method as follows:

$$\varepsilon_{xx} = \left(1+\frac{du_0}{dX}\right)\cos\varphi+\frac{dv_0}{dX}\sin\varphi - Y\frac{d\varphi}{dX} - 1 , \tag{2.21}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = -\sin\varphi\left(1+\frac{du_0}{dX}\right)+\frac{dv_0}{dX}\cos\varphi . \tag{2.22}$$

2.3. Selection of the most accurate method

In this research, the most accurate method is the fifth method. A code was used based on the finite element components to solve the equations of the fifth method as mentioned by Felippa [33]. The results of the finite element program were compared with experimental results [7]. The comparison results are depicted in Fig.4. for emphasis and reminder.

The assumed cantilever beam is made of steel. It is subjected to a uniformly distributed load along the length (the weight of the beam, *0.758 N/m*) and a vertical concentrated force *F* at the free end. The width, height, and length of the cross-sectional area of the rectangular beam were *0.025*, *0.0004*, and *0.4 m*, respectively [2].

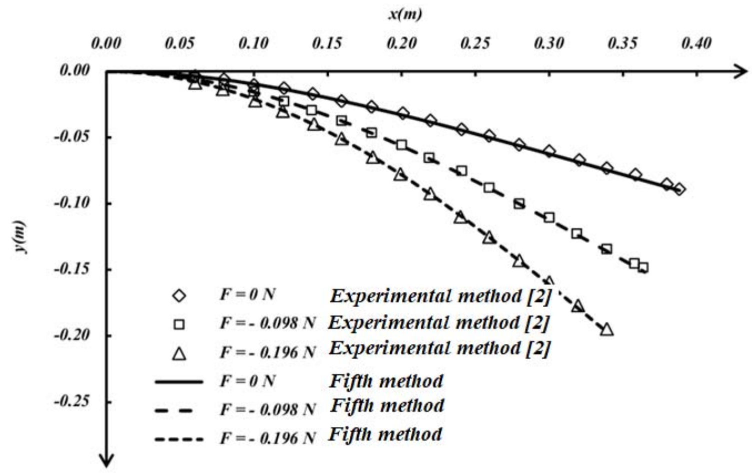

Fig.4. Comparison of the results of the fifth method for large deflections of a steel cantilever beam under distributed and concentrated force with experimental data [7].

As mentioned before, it was decided to obtain the functions of the displacement field from the most accurate analytical methods, i.e., the fifth method.

3. The second step of the problem-solving approach for a sample

The steel cantilever beam was under a uniformly distributed load along the length (weight of the beam, *0.758 N/m*) and a vertical concentrated force at the free end (*0.196 N*). The width, height, and length of the cross-sectional area of the rectangular beam were *0.025*, *0.0004*, and *0.4 m*, respectively [2].

The final position functions of the nodes were obtained from the results of the fifth method. The results of the fifth method included the values of displacement in the horizontal, X, and vertical, Y, directions. The rotation of the cross-sectional area of the beam also involved u, v, and φ, components.

The following high order polynomials were used to obtain $y(x)$, $u(X)$, $v(X)$, and $\varphi(X)$ functions:

$$y(x) = -29\cdot 319x^6 + 35\cdot 342x^5 - 13\cdot 099x^4 + \\ +3\cdot 3901x^3 - 2\cdot 3459x^2 + 0\cdot 0004x, \tag{3.1}$$

$$u(X) = +8\cdot 2195X^6 - 14\cdot 895X^5 + 11\cdot 052X^4 + \\ -3\cdot 5053X^3 - 0\cdot 0102X^2 + 0\cdot 0003X, \tag{3.2}$$

$$v(X) = +3\cdot 2648X^6 - 2\cdot 4104X^5 - 1\cdot 9777X^4 + \\ +3\cdot 8114X^3 - 2\cdot 364X^2 + 0\cdot 0007X, \tag{3.3}$$

$$\varphi(X) = -17\cdot 814X^6 + 23\cdot 575X^5 - 8\cdot 2908X^4 - 6\cdot 7071X^2 - 4\cdot 6591X\ . \tag{3.4}$$

According to Fig.4., it is not possible to determine the functions of the displacement field from the values of the experimental method with a smooth function. In experimental measurements – as can be seen in Fig.4.– some deflections were slightly under-reported or over-reported while some were reported accurately. In addition, it is not easy or maybe impossible to obtain experimental results under various conditions. Therefore, the function obtained from analytical measurements of an analysis method with acceptable accuracy will be much smoother, more realistic and computable for different conditions.

4. The third step of the problem-solving approach

The strain was calculated using the simplification hypotheses described in the previous section and the polynomial functions obtained from adaptation to the displacement values. In strain calculations, the *Y*-coordinate of the beam was *0.0002 m*; that is, the vertical strain at the top of the neutral level. The maximum vertical strain at each point of the beam should be examined.

The strain charts obtained by means of the described method were based on simplifying assumptions for the displacement field in Figs 5-7.

The shear strain was omitted in the first and second methods. Among the third, fourth, and fifth methods, only in the third one, the shear strain had a formulation different from the reference strain. Therefore, it was compared in Fig.7. The reference shear strain values ranged between *0* and *0.001*.

For a more accurate comparison of the results, the deflections of beams were assessed under different situations.

Uniformly distributed loads and concentrated forces (at the free end of the cantilever beam and centrally-loaded for the simply-supported beam) were applied and examined in separate cases.

In all cases, the length and width of the beam were *0.4* and *0.025 m*, respectively. But for analyzing thin to thick beams, heights or thicknesses were considered *0.0004*, *0.05*, and *0.2 m*.

For analyzing small to large deflections, the deflection ratio was defined by the following equation, considering deflection ratios of *0.000025*, *0.01*, *0.1*, and *0.5*:

$$Deflection\,ratio = \frac{Maximum\,beam\,deflection}{Length\,of\,beam} \, . \tag{4.1}$$

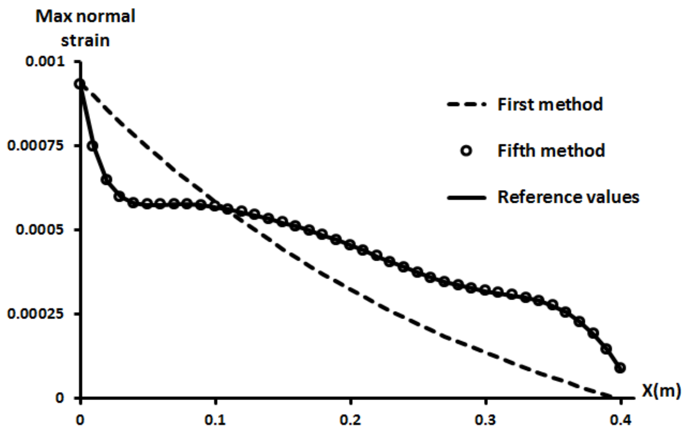

Fig.5. Comparison of maximum normal strains of the first and fifth method with maximum reference maximum normal strains along the length of a cantilever beam under uniformly distributed load (*0.758 N/m*) and concentrated force at the free end (*F=0.196 N*).

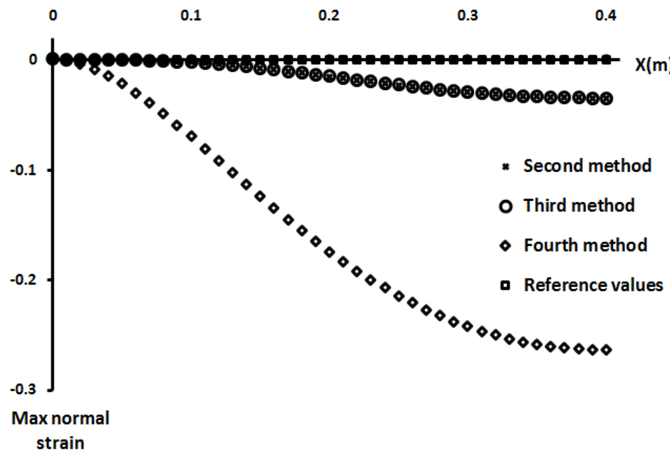

Fig.6. Comparison of maximum normal strains of the second, third, and fourth method with maximum reference normal strains along the length of a cantilever beam under a uniformly distributed load (*0.758 N/m*) and concentrated force at the free end (*F=0.196 N*).

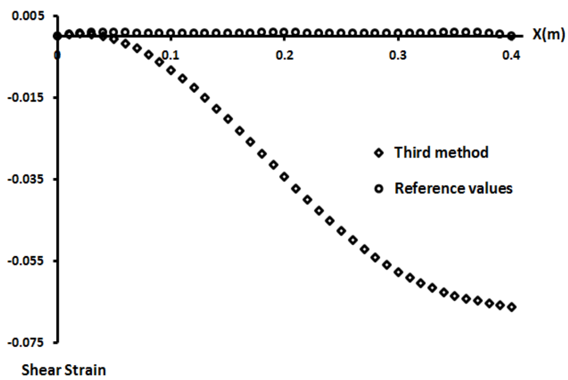

Fig.7. Comparison of shear strains of the third method with reference shear strains along the length of a cantilever beam under a uniformly distributed load (*0.758 N/m*) and concentrated force at the free end (*F=0.196 N*).

Calculations were performed for beams under different conditions, but the results are presented in this study. In addition to the previous example, the results of simple and clamped beams, with a thickness of *0.05 m* and deflection ratio of *0.1* are presented under uniformly distributed loads.

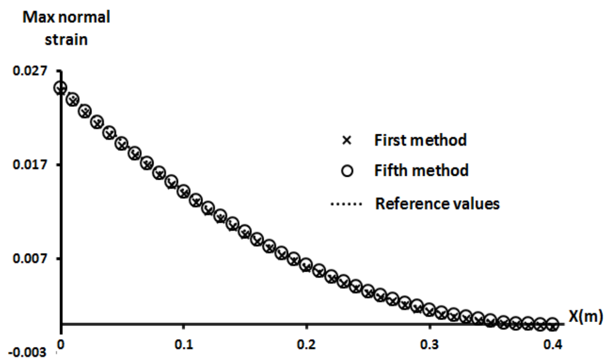

Fig.8. Comparison of maximum normal strains of the first and fifth method with maximum reference normal strains along the length of a cantilever beam with a thickness of *0.05 m* and deflection ratio of *0.1*, under a uniformly distributed load.

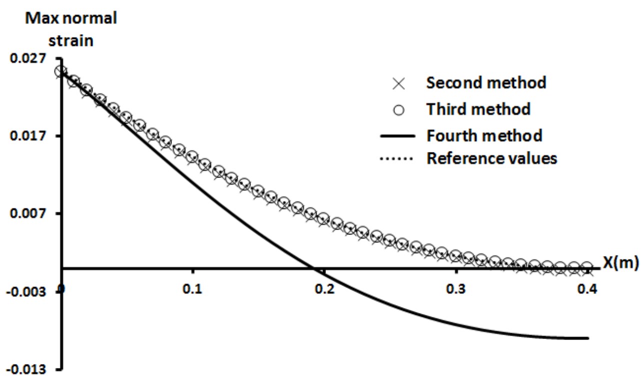

Fig.9. Comparison of the maximum normal strain of the second, third, and fourth method with maximum reference normal strain along the length of a cantilever beam with a thickness of *0.05 m* and deflection ratio of *0.1*, under a uniformly distributed load.

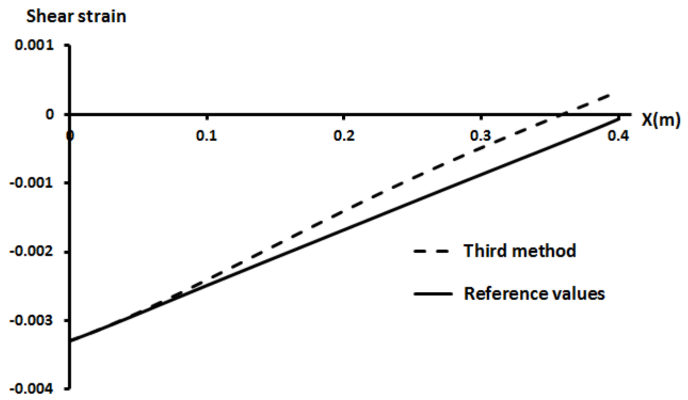

Fig.10. Comparison of shear strains of the third method with reference shear strains along the length of a cantilever beam with a thickness of *0.05 m* and deflection ratio of *0.1*, under a uniformly distributed load.

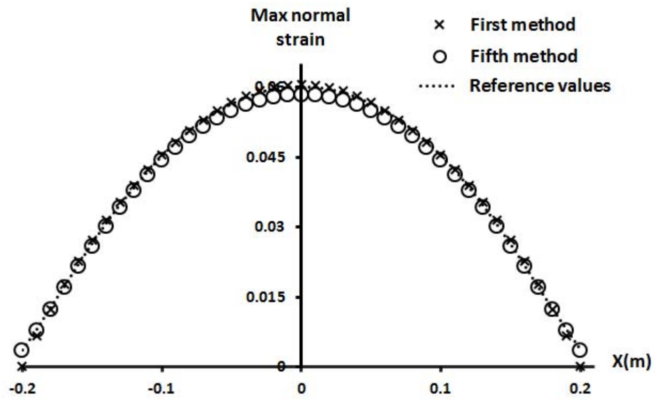

Fig.11. Comparison of maximum normal strains of the first and fifth method with maximum reference normal strains along the length of the simply-supported beam with a thickness of *0.05* m and deflection ratio of *0.1*, under a uniformly distributed load.

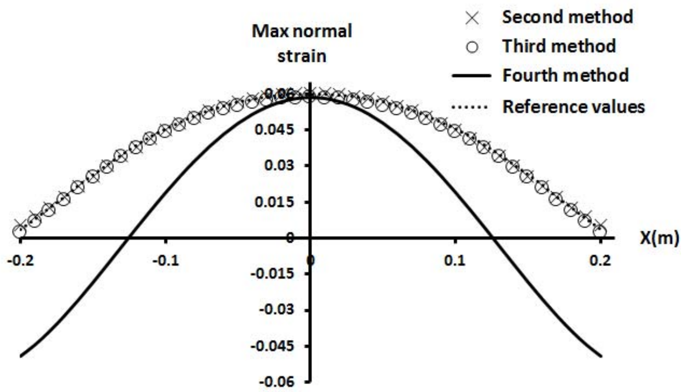

Fig.12. Comparison of the maximum normal strain of the second, third, and fourth method with maximum reference normal strain along the length of the simply-supported beam with a thickness of *0.05 m* and deflection ratio of *0.1*, under a uniformly distributed load.

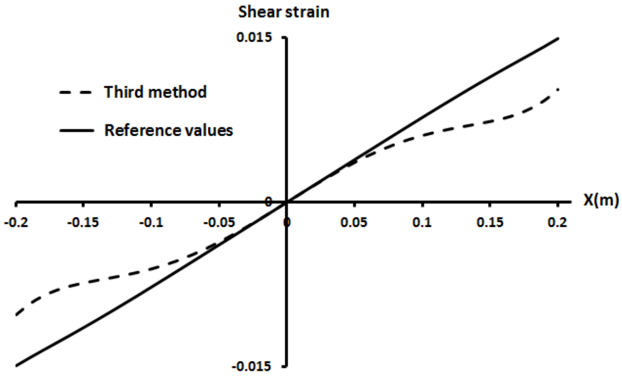

Fig.13. Comparison of shear strains of the third method with reference shear strains along the length of the simply-supported beam with a thickness of *0.05 m* and deflection ratio of *0.1*, under a uniformly distributed load.

5. Conclusion

A large beam deflection was validated and with new comparison method. Most of the large deflection beam approaches were studied in this research. To provide useful and practical results, in this study we considered about *50* situations, from small to large deflections, with various height-to-length ratios (from thin to thick) for the cantilever and simple boundary conditions of the beams under concentrated forces and uniformly distributed loads.

Functions of the displacement field can be obtained from analytical measurements, since experimental measurements are not available for all situations, as they are usually under- or over-estimated.

The comparisons, make it possible to understand the scope of the applied modeling methods. The fifth method was employed to determine the functions of the displacement fields as it exhibited high precision in the analysis of the beam deflection.

The results indicated that the strains of the fifth method were not significantly different from the reference values - which are the most accurate strain values in the analysis of beam deformation, i.e., strain values based on the Timoshenko method, without any simplifying hypotheses. This implies the proper accuracy of the fifth method. Therefore, it can be used to examine small to large deflections of beams under different boundary conditions and loadings.

Similar to the second method, the third method applied von Karman's strain theory. Concerning the shear strain in the calculations, it offered more accurate results compared to the second method for various thicknesses. On the other hand, the third method considered only the case of shear strain different from the reference values. At higher thicknesses, a comparison of shear strain exhibited more appropriate behavior.

The fourth method does not have simplifying assumptions for the displacement field while assuming an approximation in calculations of the strain by removing the nonlinear part compared to the reference strain. Finally, the results can be presented as follows:

1. The fifth method was very accurate in analyzing the beams; thus, it is suitable for determining the functions of the displacement field.
2. For small to large deflections under different loading and boundary conditions and height-to-length ratios, the fifth method showed the highest compatibility with the reference strain.
3. The fifth method can model beams, under different loading and boundary conditions.
4. In small deflections, the third, fourth, and fifth method had sufficient accuracy.
5. Large deflection assumption (i.e., deflection due to large displacement and small strains) is the basic assumption in these results. For the beam with a thickness of *0.2 m*, the difference between the results of the fifth method and the reference strain values was significant which can be due to the big deflection with great strain values.

The present comparative model and the method of determining the displacement field functions can be used for comparing different theories of analysis and simulation of deformations without fully understanding all the methods used for solving the equations.

Prediction of the mechanical response of beams is very important in designing mechanical systems and structures.

Acknowledgments

My special thanks are extended to the Department of Mechanical Engineering of the Sirjan University of Technology for their support.

Nomenclature

d – differentiation
F – concentrated force
l – current length of the beam
l_0 – initial length of the beam
S – arc length of the beam between the fixed end and the desired point
u – displacement in the x co-ordinate direction
u_0 – displacement of the neutral axis of the beam in x co-ordinate direction
v – displacement in the y co-ordinate direction
v_0 – displacement of the neutral axis of the beam in y co-ordinate direction
w – displacement in the z co-ordinate direction
w_0 – displacement of the neutral axis of the beam in z co-ordinate direction
x – current x co-ordinate
X – initial x co-ordinate
y – current y co-ordinate
Y – initial y co-ordinate
z – current z co-ordinate
Z – initial z co-ordinate
γ_{xy} – shear strain component
δ – displacement in the y co-ordinate direction
ε_{ij} – strain component
ε_{xx} – normal strain component
ε_{xy} – shear strain component
θ – slope of the beam axis
ρ – radius of curvature of the neutral axis of the beam
∂ – partial differentiation
φ – rotation of the cross-sectional area of the beam

References

[1] Ohtsuki A. (1986): *An analysis of large deflection in a symmetrical three-point bending of the beam.*– Bulletin of JSME, vol.29, No.253, pp.1988-1995, DOI .org/10.1299/jsme1958.29.1988.

[2] Beléndez T., Neipp C. and Beléndez A. (2003): *Numerical and experimental analysis of a cantilever beam: a laboratory project to introduce geometric nonlinearity in mechanics of materials*.– Int. J. Eng. Educ., vol.19, No.6, pp.885-892.

[3] Nanakorn P. and Vu L. (2006): *A 2D field-consistent beam element for large displacement analysis using the total Lagrangian formulation.*– Finite Elem. Anal. Des., vol.42, No.14, pp.1240-1247, DOI.org/10.1016/j.finel.2006.06.002.

[4] Xiao Y. (2011): *Large deflection of cantilever beam with uniformly distributed load using homotopy analysis method.*– Adv. Mat. Res., Trans. Tech. Publ. Ltd., vol.250, pp.1222-1225, DOI.org/10.4028/www.scientific.net/AMR.250-253.1222.

[5] Mohyeddin A. and Fereidoon A. (2014): *An analytical solution for the large deflection problem of Timoshenko beams under three-point bending.*– Int. J. Mech. Sci., vol.78, pp.135-139, DOI.org/10.1016/j.ijmecsci.2013.11.005.

[6] Li D.K. and Li X.F. (2016): *Large deflection, and rotation of Timoshenko beams with frictional end support under three-point bending*.– Comptes Rendus Mécanique, vol.344, No.8, pp.556-568, DOI.org/10.1016/j.crme.2016.01.007.

[7] Taghipour Y. and Baradaran G.H. (2016): *A finite element modeling for large deflection analysis of uniform and tapered nanowires with good interpretation of experimental results.*– Int. J. Mech. Sci., vol.114, pp.111-119, DOI.org/10.1016/j.ijmecsci.2016.05.006.

[8] Abu-Alshaikh I., Alkhaldi1 H.S. and Beithou N. (2018): *Large deflection of prismatic cantilever beam exposed to combination of end inclined force and tip moment.*– Mod. Appl. Sci. vol.12, No.1, pp.98-111, DOI.org/10.5539/mas.v12n1p98.

[9] Taghipour Y. and Baradaran G.H. (2017): *Large deflection analysis of nanowires based on nonlocal theory using total Lagrangian finite element method.*– Acta Mech., vol.228, pp.2429-2442, DOI.org/10.1007/s00707-017-1837-0.

[10] Bouadjadja S., Tati A. and Guerira B. (2020): *Analytical and experimental investigations on large deflection analysis of composite cantilever beams.*– Mech. Adv. Mater. Struct., pp.1-9, DOI.org/10.1080/15376494.2020.1751901.

[11] Zeng W., Yan J., Hong Y. and Cheng S.S. (2021): *Numerical analysis of large deflection of the cantilever beam subjected to a force pointing at a fixed point.*– Appl. Math. Model., vol.92, pp.719-730, DOI.org/10.1016/j.apm.2020.11.023.

[12] Estabragh E.R. and Baradaran G.H. (2021): *Analysis of large deflection of nanobeams based on the modified couple stress theory by using finite element method.*– Arch. Appl. Mech., vol.91, pp.4717-4734, DOI.org/10.1007/s00419-021-02029-6.

[13] Li Y., Li X., Huo S. and Xie C. (2022): *Explicit solutions to large deformation of cantilever beams by improved homotopy analysis method i: rotation angle.*– Appl. Sci., vol.12, No.13, pp.1-24, DOI.org/10.3390/app12136400.

[14] Lai W.M., Rubin D. and Krempl E. (2009): *Introduction to continuum mechanics.*– Butterworth-Heinemann.

[15] Bisshopp K. and Drucker D. (1945): *Large deflection of cantilever beams.*– Q. Appl. Math., vol.3, No.3, pp.272-275.

[16] Frisch-Fay R. (1962): *Flexible Bars.*– Butterworths, London, p.220.

[17] Khwaja I. (1971): *Large deflexions of thin elastic bars by the electronic analogue computer.*– Int. J. Mech. Sci., vol.13, No.4, pp.323-330, DOI.org/10.1016/0020-7403(71)90057-9.

[18] Holden J. (1972): *On the finite deflections of thin beams.*– Int. J. Solids Struct., vol.8, No.8, pp.1051-1055, DOI.org/10.1016/0020-7683(72)90069-8.

[19] Mattiasson K. (1981): *Numerical results from large deflection beam and frame problems analyzed by means of elliptic integrals.*– Int. J. Numer. Meth. Eng., vol.17, No.1, pp.145-153, DOI.org/10.1002/nme.1620170113.

[20] Ang M.H., Wei W. and Teck-Seng L. (1993): *On the estimation of the large deflection of a cantilever beam.*– Proceedings of IECON '93 - 19th Annual Conference of IEEE Industrial Electronics, DOI.org/10.1109/IECON.1993.339311.

[21] Beléndez T., Neipp C. and Beléndez A. (2002): *Large and small deflections of a cantilever beam.*– Eur. J. Phys., vol.23, No.3, p.371-379, DOI.org/10.1088/0143-0807/23/3/317.

[22] Fertis D.G. (2006): *Basic theories and principles of nonlinear beam deformations.*– Nonlinear Structural Engineering: With Unique Theories and Methods to Solve Effectively Complex Nonlinear Problems, pp.1-61, DOI.org/10.1007/978-3-540-32976-3_1.

[23] Chouery F.A. (2006): *Exact and numerical solutions for large deflection of elastic non-prismatic beams.*– FAC Systems INC., WA.

[24] Visner J.C. (2007): *Analytical and experimental analysis of the large deflection of a cantilever beam subjected to a constant, concentrated force, with a constant angle, applied at the free end*.– University of Akron.

[25] Abolfathi A., Brennan M.J. and Waters T.P. (2010): *Large deflection of a simply supported beam*.– University of Southampton.

[26] Xiao Y. (2011): *Large deflection of cantilever beam with uniformly distributed load using homotopy analysis method*.– Adv. Mat. Res., Trans. Tech. Publ., vol.250-253, pp.1222-1225, DOI.org/10.4028/www.scientific.net/AMR.250-253.1222.

[27] Maleki M., Tonekaboni S.A.M. and Abbasbandy S. (2014): *A homotopy analysis solution to large deformation of beams under static arbitrary distributed load*.– Appl. Math. Model., vol.38, No.1, pp.355-368, DOI.org/10.1016/j.apm.2013.06.039.

[28] Beer F., Johnston E. and DeWolf J. (2015): *Mechanics of Materials*.– McGraw-Hill, New York.

[29] Nishawala V. (2011): *A Study of Large Deflection of Beams and Plates*.– Rutgers University-Graduate School-New Brunswick.

[30] De Borst R., Crisfield M.A., Remmers J.J. and Verhoosel C.V. (2012): *Nonlinear Finite Element Analysis of Solids and Structures*.– John Wiley & Sons.

[31] Kien N.D. (2012): *A Timoshenko beam element for large displacement analysis of planar beams and frames.*– Int. J. Struct. Stab. Dyn., vol.12, No.06, p.1250048.

[32] Zienkiewicz O.C. and Taylor R.L. (2000): *The finite element method: solid mechanics*.– Butterworth-Heinemann, vol.2.

[33] Felippa C.A. (2014): *Nonlinear finite element methods.*– The University of Colorado, Boulder, Colorado, USA.

Received: November 17, 2021
Revised: September 6, 2022