Int. J. of Applied Mechanics and Engineering, 2022, vol.27, No.3, pp.49-62
DOI: 10.2478/ijame-2022-0035

APPLICATION OF REANALYSIS METHODS IN STRUCTURAL MECHANICS

I.DELYOVÁ, P.FRANKOVSKÝ*, J.BOCKO, P.SIVÁK and R.KURIMSKÝ
Department of Applied Mechanics and Mechanical Engineering, Faculty of Mechanical Engineering
Technical University of Košice, Letná 9, 042 00 Košice, SLOVAKIA
E-mail: peter.frankovsky@tuke.sk

When designing structures, it is often necessary to re-analyse a structure that is different in some parts from the original one. As real structures are often complex, their analysis is therefore very challenging. In such cases, reanalysis methods are advantageously used. The aim of this paper is to approach the problem of solving the constructions using reanalysis method in which the time taken in solving algebraic equations is reduced. In particular, the purpose of this work is to demonstrate on a chosen system the time savings and the advantages of the chosen direct efficient reanalysis method for a given design problem. A basic condition for meeting these criteria is the modernization of computational procedures in the mechanics of compliant solids.

Key words: structural reanalysis, algebraic equations, computational efficiency, change in the design variables

1. Introduction

Nowadays, the development of industrial production requires that all fundamental changes be made on the basis of objective analyses and optimization considerations. It is therefore necessary to know the procedures for the design of machines and equipment, but also to know the conditions for using efficient production methods with the least possible deficiencies and with the aim of perfect use of parameters for the operation of complex equipment.

Reanalysis methods are among the computational procedures that are used in situations where the design of a structure requires an analysis of a structure that is little different from the original one, or different only in some parts. Real structures are usually complex, therefore in such cases it is advantageous to use reanalysis methods, which by using the information from the previous analysis allow a significant reduction of the computational time. The most significant development of reanalysis methods was observed in the years 1960-1970, when the need for fast and efficient optimization increased due to technical and industrial development (Kołakowski *et al.* [1], Kirsch [2]).

Reanalysis is basically about solving a system of linear algebraic equations, the part that is more time consuming when solving using the finite element method. When applied to specific selected problems, the direct efficient method of reanalysis is used, it is suitable for use in solving problems of a given type where computational time is reduced.

In general, structural responses cannot be expressed explicitly in terms of the properties of the structure itself, whereas structural analysis involves solving a set of simultaneous equations. Reanalysis methods are primarily designed to analyze structures that are effectively modified as a result of various changes in their properties. The goal of reanalysis is primarily to evaluate the structural solutions (e.g., displacements, forces, and stresses) for these changes without having to solve the entire set of modified simultaneous equations, and these solution procedures usually correspond to the original structure.

Reanalysis methods can be applied to some common design problems such as:

---

* To whom correspondence should be addressed

- In design optimization, iterative methods consisting of several iterative analyses are used in the solution. The high computational cost in this reanalysis method poses a significant problem and is one of the main obstacles in the actual solution process.
- When analysing damaged structures or structures, it is necessary to analyse the structure considering the possibility of various changes due to deterioration of working conditions, damage, accident or lack of maintenance. In general, many hypothetical scenarios describing different types of damage can be envisaged and need to be considered and respected. Numerous analyses are needed to assess and evaluate different hypothetical damages for different types of damage.
- When designing the different design stages of complex structures, it may be necessary to repeatedly analyze the structure itself, which will be gradually modified during the actual design process. The modified structure is thus subjected to different loading conditions and modes.
- Non-linear analysis of structures is usually performed using iterative methods. The actual solution can be performed in a variety of ways, but in general, a set of updated linear equations must be solved iteratively. An example of a solution can be found in the field of vibration analysis.
- However, reanalysis methods can also be useful in other applications such as probabilistic structural analysis or conceptual design problems (Kirsch [2]).

One of the fundamental issues that may arise is the real need for an efficient reanalysis method given the significant increase in demands on computer processing, i.e., the power, memory, and hard disk storage space of the computer itself. In this respect, it has been found that, despite the rapid developments in computing technology, the demands associated with computational costs and time constraints relating to the use of resources in the field of structural analysis do not preclude them. This situation arises due to the constant growth in the requirements for accuracy and therefore complexity of analytical models (Bocko [3], Delyová *et al.* [4]).

The two main drivers of the complexity of the problem relate mainly to the complexity of the model analysis and the analysis procedure itself. Model complexity is a function of various parameters such as the number of degrees of freedom of the model elements and the topology of the structure (determines the bandwidth of the stiffness matrix).

2. Reanalysis methods

Reanalysis methods hold great importance in the environment of development and computing. In the context of optimization of structures, mathematical optimization methods, which are based on the repeated solution of modified structures, have begun to be widely used (Bocko [3], Rong *et al.* [5]).

When designing structures, it is often necessary to reanalyse a structure that differs from the original only in certain parts. In these cases, it is advantageous to use reanalysis methods, which use the information from the previous analysis to reduce the computational time. By reanalysis, we basically mean solving a system of linear algebraic equations, the part that is more time consuming when solved using FEM.

These methods can be divided into the following groups

- direct elimination,
- approximation,
- iterative.

Direct methods can be further divided into:

- initial stress or displacement methods,
- modified displacement vector methods,
- modified matrix inversion methods,
- superposition methods.
- methods of modifying decomposed matrices

Direct reanalysis methods are effective in changing the rank in the stiffness matrix. In particular, these methods are useful in situations where a relatively small proportion of the change in structure and change in the stiffness matrix can be represented by smaller submatrices. Direct methods lose their effectiveness when the changes of the submatrices in the stiffness matrices are too large.

Most of the direct methods are just based on Sherman-Morrison formulas. Other direct methods such as the Virtual Distortion Method (VDM) and Theorem of Structural Variations (TSV) can be considered as variants of Woodbury's formulas, which require collinear loads applied to the modified terms to calculate the influence matrix. In VDM, the reduced set of equations is then solved by a set of scalar products of the influence vectors. In the case of TSV, the adjusted displacements and forces are expressed in terms of the values due to the unit load (Kołakowski *et al.* [1], Kirsch [2], Akgün *et al.* [6]).

3. Application of the superpositioned direct reanalysis method

To demonstrate and clarify the issue, we present a direct reanalysis method that is derived using the potential energy of the system. The effects caused by the new added element are introduced using Lagrange coefficients. This direct method of reanalysis of structures is related to the deformation variant of the finite element method and allows us to implement arbitrary changes in the structure by trying to avoid re-elimination of the stiffness matrix. This approach allows us to achieve considerable time savings.

Using the finite element method (FEM), we use a system of algebraic equations to describe the structure

$$[\boldsymbol{K_0}]\{\boldsymbol{x_0}\}=\{\boldsymbol{p_0}\}, \tag{3.1}$$

where $[\boldsymbol{K_0}]$ is the stiffness matrix of the structure, $\{\boldsymbol{x_0}\}$ is the displacement vector of nodal points, $\{\boldsymbol{p_0}\}$ is the load vector.

Using the superposition method, it is possible to detect the internal force quantities in the elements and the displacements of the nodal points of the structure when the cross-section of the material and elements are changed (Wu *et al.* [7], Cao *et al.* [8], Huang *et al.* [9]).

It can be shown that the internal force in member j of a membered structure in which the cross section of member *i* has changed will be

$$\boldsymbol{P}_j^* = \boldsymbol{P}_j + \boldsymbol{r}_i \boldsymbol{f}_{ji}, \tag{3.2}$$

where $i \neq j$, $\boldsymbol{P}_j^*$, $\boldsymbol{P}_j$ are the internal forces in element j of the original or modified structure, is the internal force in element j of the original structure from the unit load applied to the ends of element i, which is tensile and acts in the direction of the element, $\boldsymbol{r}_i$ is the coefficient of variation, which is determined from the equilibrium conditions.

In the modified element, the internal force is then defined as follows

$$\boldsymbol{P}_j^* = \boldsymbol{P}_j + \boldsymbol{r}_i\left(1 + d\boldsymbol{f}_{ii}\right), \tag{3.3}$$

where the change in element i is defined as

$$d = \frac{d\boldsymbol{A}_i}{\boldsymbol{A}_i}, \tag{3.4}$$

$\boldsymbol{A}_i$ is the cross-sectional area of the element.

The equilibrium conditions at the ends of the modified element give the equation

$$\boldsymbol{r}_i = \frac{-d\boldsymbol{P}_i}{\left(1 + d\boldsymbol{f}_{ii}\right)} . \tag{3.5}$$

The displacements of the modified structure $\delta*$ are calculated by superposition

$$\delta^* = \delta + \boldsymbol{r}_i \delta_i \tag{3.6}$$

where δ, δ_i are the displacements of the nodes of the original structure from the unit loads and the external load, respectively. This procedure can be used to make a change in only one element, but can be extended to multiple elements.

The displacements of the modified structure are determined using the relation

$$\delta^* = \delta + \sum_{j=1}^{n} \boldsymbol{r}_j \delta_j . \tag{3.7}$$

The cross-sectional area of the modified members is then

$$\overline{\boldsymbol{a}}_1 = \overline{\boldsymbol{a}}_1 + \delta \overline{\boldsymbol{a}}_1 , \tag{3.8}$$

where the bar indicates that it is a modified design, 1 indicates that it is a modified member. For a subset $\overline{\boldsymbol{p}}_1$ of the internal forces of the modified structure

$$\overline{\boldsymbol{p}}_1 = \left(\boldsymbol{I} + \overline{\boldsymbol{C}}_1\right)\left(\boldsymbol{p}_1 + \boldsymbol{S}_{11} \boldsymbol{r}_1\right), \tag{3.9}$$

$\boldsymbol{I}$ is the unit matrix, $\boldsymbol{C}$ is the diagonal matrix containing the terms $\frac{\delta \boldsymbol{a}_i}{\boldsymbol{a}_i}$, $\boldsymbol{S}_{11}$ is the sensitivity submatrix defined by the relation

$$\begin{Bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{Bmatrix}, \tag{3.10}$$

$\boldsymbol{r}$ is the vector of coefficients of variation and for the modified member

$$\boldsymbol{r}_1 = \boldsymbol{C}_1 \boldsymbol{S}_{11} \boldsymbol{r}_1 + \boldsymbol{C}_1 \boldsymbol{p}_1 = 0. \tag{3.11}$$

Forces in unmodified elements are

$$\overline{\boldsymbol{p}}_2 = \boldsymbol{p}_2 + \boldsymbol{S}_{21} \boldsymbol{r}_1 , \tag{3.12}$$

where $\boldsymbol{S}_{21}$ is the sensitivity matrix of the forces in the unmodified members to changes in the modified members.

By modifying the relations, we obtain the equation of the internal forces of the modified structure

$$\overline{\boldsymbol{p}}_1 = \left(\boldsymbol{I} + \boldsymbol{C}_1^{-1}\right) \boldsymbol{r}_1 . \tag{3.13}$$

The resulting relationship is less computationally intensive and can greatly speed up the calculation in the case of a large number of modified elements.

Consider any change to the structure that may be caused by a change to an element in the structure, either by its removal or addition, or by a change in material characteristics, etc. In fact, this can generally be thought of as adding an element to the structure, since any of these changes will also cause a change in the stiffness matrix of the original structure of $[\Delta\boldsymbol{K}_0]$, i.e., an increment in the stiffness matrix of the structure. Such an addition of an element is equivalent to introducing unknown force quantities $\{\boldsymbol{f}\}$ into the original structure. These forces are applied by the new element to the other parts of the structure. The structure thus modified is then represented by an equation which takes the form

$$[\Delta\boldsymbol{K}_0]\{\boldsymbol{x}\}=\{\boldsymbol{p}\}+\{\boldsymbol{f}\}, \tag{3.14}$$

Where $\{\boldsymbol{x}\}$ is the vector of displacements of the nodal points of the modified structure, $\{\boldsymbol{p}\}$ is the vector of external forces that have changed for various reasons.

In the case that there is no change in the external load

$$\{\boldsymbol{p}\}=\{\boldsymbol{p}_0\}. \tag{3.15}$$

Using Eq.(3.14), it is also possible to write an equation at the level of the added element, which takes the form:

$$[\Delta\boldsymbol{K}_0]\{\boldsymbol{y}\}+\{\boldsymbol{f}\}=\{\boldsymbol{p}_v\}, \tag{3.16}$$

where $\{\boldsymbol{y}\}$ is a vector that is exactly the part of the displacement vector $\{\boldsymbol{x}\}$ that corresponds to the degrees of freedom at the edge of the element, or at their joints themselves, $\{\boldsymbol{p}_v\}$ is a vector of external forces acting on the new modified element.

The dependence between the vectors $\{\boldsymbol{x}\}$ and $\{\boldsymbol{y}\}$ is defined by the relation

$$\{\boldsymbol{y}\}=[\boldsymbol{C}]\{\boldsymbol{x}\} \tag{3.17}$$

where $[\boldsymbol{C}]$ is a Boolean matrix selecting the appropriate terms from the vector $\{\boldsymbol{x}\}$. The previous Eq.(3.7) ensures the compatibility of the displacements of the nodes of the added element with the displacements of the nodes in the rest of the structure itself. Thus, this equation expresses the connections between Eqs. (3.14) and (3.16).

Considering Eqs. (3.14) and (3.16) and the constraint condition Eq.(3.17), the potential energy of the modified structure can thus be expressed in a modified form:

$$\begin{aligned}\pi=&\frac{1}{2}\{\boldsymbol{x}\}^T[\boldsymbol{K}_0]\{\boldsymbol{x}\}-\{\boldsymbol{x}\}^T\{\boldsymbol{p}\}+\frac{1}{2}\{\boldsymbol{y}\}^T[\Delta\boldsymbol{K}_0]\{\boldsymbol{y}\}+\\&-\{\boldsymbol{y}\}^T\{\boldsymbol{p}_v\}+\left(\{\boldsymbol{y}\}^T-\{\boldsymbol{x}\}^T[\boldsymbol{C}]^T\right)\{\lambda\},\end{aligned} \tag{3.18}$$

where $\{\lambda\}$ is a vector of Lagrange coefficients.

Let us define a vector of generalized coordinates $\{\boldsymbol{s}\}$ of the form:

$$\{\boldsymbol{s}\} = \begin{Bmatrix} \{\boldsymbol{x}\} \\ \{\boldsymbol{y}\} \end{Bmatrix}, \tag{3.19}$$

then after minimizing the functional for the minimum potential energy, the following relation will hold

$$\frac{\partial \pi}{\partial \{\boldsymbol{s}\}} = \begin{bmatrix} [\boldsymbol{K}_0] & [0] & [-\boldsymbol{C}]^T \\ [0] & [\boldsymbol{K}_0] & [\boldsymbol{I}] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{x}\} \\ \{\boldsymbol{y}\} \\ \{\lambda\} \end{Bmatrix} - \begin{Bmatrix} \{\boldsymbol{p}\} \\ \{\boldsymbol{p}_v\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix}. \tag{3.20}$$

If we add the constraint (3.17) to the expression (3.20), we obtain a system of equations of the form

$$\begin{bmatrix} [\boldsymbol{K}_0] & [0] & [-\boldsymbol{C}]^T \\ [0] & [\Delta \boldsymbol{K}_0] & [\boldsymbol{I}] \\ [-\boldsymbol{C}] & [\boldsymbol{I}] & [0] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{x}\} \\ \{\boldsymbol{y}\} \\ \{\lambda\} \end{Bmatrix} - \begin{Bmatrix} \{\boldsymbol{p}\} \\ \{\boldsymbol{p}_v\} \\ \{\boldsymbol{p}_v\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix}. \tag{3.21}$$

Comparing the second line with Eq.(3.16), it can be seen that the vector of unknown forces is equal to the vector of Lagrange coefficients $\{\boldsymbol{f}\} = \{\lambda\}$. This indicates that the vector of Lagrange coefficients expresses the forces between the added element and the rest of the structure. If we perform a rearrangement of this system, we obtain the expression

$$\begin{bmatrix} [\boldsymbol{K}_0] & [-\boldsymbol{C}]^T & [0] \\ [-\boldsymbol{C}] & [0] & [\boldsymbol{I}] \\ [0] & [\boldsymbol{I}] & [\Delta \boldsymbol{K}_0] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{x}\} \\ \{\lambda\} \\ \{\boldsymbol{y}\} \end{Bmatrix} - \begin{Bmatrix} \{\boldsymbol{p}\} \\ \{0\} \\ \{\boldsymbol{p}_v\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix}. \tag{3.22}$$

Even though this matrix contains zeros on its diagonal, there is still no need for a pivot in the elimination. This can be proved by using Gaussian elimination at the submatrix level in Eq.(3.22). After elimination, we obtain

$$\begin{aligned} &[-\boldsymbol{C}] \rightarrow [-\boldsymbol{C}] + \left([\boldsymbol{C}][\boldsymbol{K}_0]^{-1}\right)[\boldsymbol{K}_0] = [0], \\ &[0] \rightarrow [0] - \left([\boldsymbol{C}][\boldsymbol{K}_0]^{-1}\right)[\boldsymbol{C}]^T = -\left[\bar{\boldsymbol{K}}\right]. \end{aligned} \tag{3.23}$$

Since $\left[\bar{\boldsymbol{K}}\right]$ is a diagonal submatrix of the positively definite matrix $[\boldsymbol{K}_0]^{-1}$, the matrix $\left[\bar{\boldsymbol{K}}\right]$ is also positively definite and thus no pivot is needed. If we modify the last line of the system of Eq.(3.22), we obtain:

$$\begin{aligned} &[\boldsymbol{I}] \rightarrow [\boldsymbol{I}] + \left[\bar{\boldsymbol{K}}\right]^{-1}[\boldsymbol{K}] = [0], \\ &[\Delta \boldsymbol{K}_0] \rightarrow [\Delta \boldsymbol{K}_0] + \left[\bar{\boldsymbol{K}}\right]^{-1}[\boldsymbol{I}] = [\Delta \boldsymbol{K}_0] + \left[\bar{\boldsymbol{K}}\right]^{-1}. \end{aligned} \tag{3.24}$$

The matrix $\left[\bar{\boldsymbol{K}}\right]^{-1}$ is positive definite since the matrix $\left[\bar{\boldsymbol{K}}\right]$ is also positive definite, and the sum of the singular matrix $\left[\Delta\boldsymbol{K}_0\right]$ and the matrix $\left[\bar{\boldsymbol{K}}\right]^{-1}$ is actually a matrix in which pivoting is not necessary. Hence, for the elimination in the system of Eq.(3.22), no pivot is needed and we can use Crout's method for the solution (Bocko, [3], Mo *et al.*[10], Materna *et al.*[11]).

In the system of Eq.(3.22), the vector $\{\boldsymbol{x}\}$ represents the displacements of the nodes of the modified structure and can be decomposed as follows

$$\{\boldsymbol{x}\}=\{\boldsymbol{x}_0\}+\{\Delta\boldsymbol{x}_0\}, \tag{3.25}$$

$\{\Delta\boldsymbol{x}_0\}$ is the increment of the displacement vector.

According to the action of external forces on the modified structure, two types of load can be determined in this way:

1. no external forces are modified, i.e., $\{\boldsymbol{p}\}=\{\boldsymbol{p}_0\}$.
2. the external forces change by the increment $\{\Delta\boldsymbol{p}_0\}$, which implies

$$\{\boldsymbol{p}\}=\{\boldsymbol{p}_0\}+\{\Delta\boldsymbol{p}_0\}, \tag{3.26}$$

when substituted into expression (3.22) we get

$$\begin{bmatrix} [\boldsymbol{K}_0] & [-\boldsymbol{C}]^T & [0] \\ [-\boldsymbol{C}] & [0] & [\boldsymbol{I}] \\ [0] & [\boldsymbol{I}] & [\Delta\boldsymbol{K}_0] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{x}_0\}+\{\Delta\boldsymbol{x}_0\} \\ \{\lambda\} \\ \{\boldsymbol{y}\} \end{Bmatrix} = \begin{Bmatrix} \{\boldsymbol{K}_0\}\{\boldsymbol{x}_0\} \\ \{0\} \\ \{\boldsymbol{p}_{0,v}\} \end{Bmatrix} \tag{3.27}$$

and after simplification

$$\begin{bmatrix} [\boldsymbol{K}_0] & [-\boldsymbol{C}]^T & [0] \\ [-\boldsymbol{C}] & [0] & [\boldsymbol{I}] \\ [0] & [\boldsymbol{I}] & [\Delta\boldsymbol{K}_0] \end{bmatrix} \begin{Bmatrix} \{\Delta\boldsymbol{x}_0\} \\ \{\lambda\} \\ \{\boldsymbol{y}\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{\boldsymbol{C}\}\{\boldsymbol{x}_0\} \\ \{\boldsymbol{p}_{0,v}\} \end{Bmatrix}. \tag{3.28}$$

Note that there are zeros at the top of the vector on the right-hand side. Hence, it is not necessary to perform elimination for this part.

An adjustment similar to the previous one can be applied in case of a change in the external load. In this case, Eq.(3.26) will hold, where for the vector $\{\Delta\boldsymbol{p}_0\}$

$$\{\Delta\boldsymbol{p}_0\}=[\boldsymbol{C}]^T\{\boldsymbol{q}\}, \tag{3.29}$$

$\{\boldsymbol{q}\}$ is a vector representing the forces introduced together with the added element and acting on the edge of this element.

Then the system of equations takes the form

$$\begin{bmatrix} [\boldsymbol{K_0}] & [-\boldsymbol{C}]^T & [0] \\ [-\boldsymbol{C}] & [0] & [\boldsymbol{I}] \\ [0] & [\boldsymbol{I}] & [\Delta\boldsymbol{K_0}] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{x_0}\}+\{\Delta\boldsymbol{x_0}\} \\ \{\lambda\} \\ \{\boldsymbol{y}\} \end{Bmatrix} = \begin{Bmatrix} \{\boldsymbol{p_0}\}+[\boldsymbol{C}]^T\{\boldsymbol{q}\} \\ \{0\} \\ \{\boldsymbol{p_{0,v}}\} \end{Bmatrix} \tag{3.30}$$

and after the introduction of substitution

$$\{\lambda_1\}=\{\lambda\}+\{\boldsymbol{q}\} \tag{3.31}$$

we obtain a system of equations

$$\begin{bmatrix} [\boldsymbol{K_0}] & [-\boldsymbol{C}]^T & [0] \\ [-\boldsymbol{C}] & [0] & [\boldsymbol{I}] \\ [0] & [\boldsymbol{I}] & [\Delta\boldsymbol{K_0}] \end{bmatrix} \begin{Bmatrix} \{\Delta\boldsymbol{x_0}\} \\ \{\lambda_1\} \\ \{\boldsymbol{y}\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{\boldsymbol{C}\}\{\boldsymbol{x_0}\} \\ \{\boldsymbol{q}\}+\{\boldsymbol{p}_v\} \end{Bmatrix}. \tag{3.32}$$

For solving system Eq.(3.32), it is again convenient to use Crout's method, which allows elimination of the submatrix $[\boldsymbol{K_0}]$ during the reanalysis itself, and thus to proceed with the elimination of additional columns and rows, as indicated in the figure. It is by this procedure that a considerable time saving is obtained (Fig.1).

As in the system of Eq.(3.28), also in the system Eq.(3.32) there are zeros at the top of the right-hand side vector, so there is no need to perform elimination for this part (Wu *et al.* [12], Tertel *et al.* [13], Bittnar *et al.* [14]).

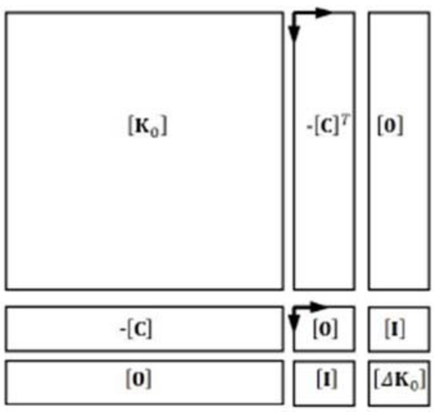

Fig.1. Resulting system of equations.

4. Application of the reanalysis method to a general planar member system

Consider the planar member system in Fig.2. In this case, we show the execution of an effective reanalysis with specific values for the geometric and material properties of the selected members. The tensile modulus of elasticity is $E=2\cdot 10^5\ MPa$, diameter of members is $d=40\,mm$, the external loading force of the system is $F=300\,N$ at an angle $\gamma=30°$.

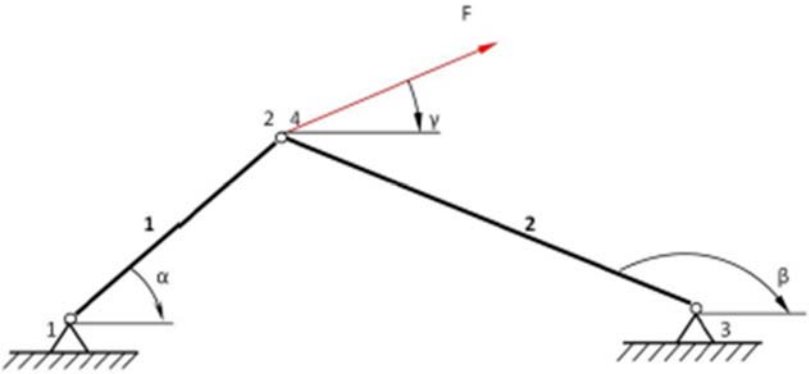

Fig.2. Designed member system.

Components of the applied force are as follows:

$$F_x = F\cos\gamma = 259.8\ N\,,$$
$$F_y = F\sin\gamma = 150\ N\,. \tag{4.1}$$

The direction cosine of the angles of the rods members can be determined from the relations

$$c_x = \frac{x_j - x_i}{L_c}\,,$$
$$c_y = \frac{y_j - y_i}{L_c}\,, \tag{4.2}$$

and the lengths of the members

$$L_c = \sqrt{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2}\,. \tag{4.3}$$

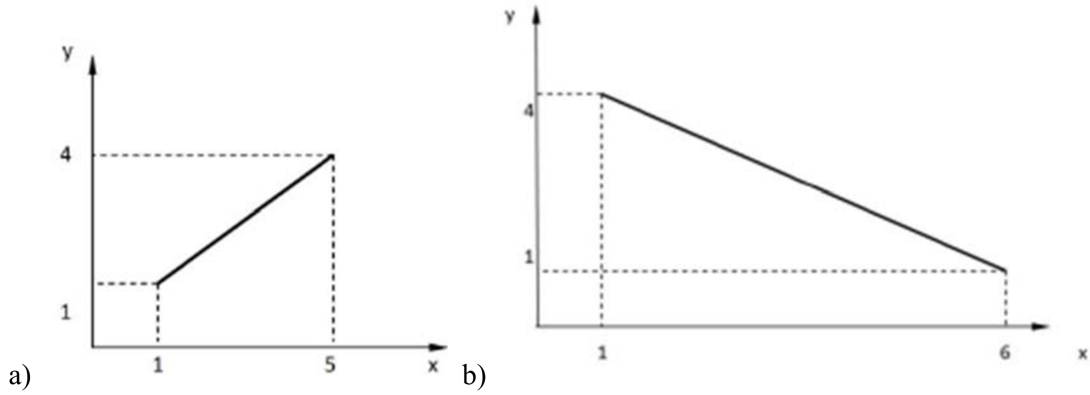

Fig.3. Coordinates of: a) member 1; b) member 2.

Table 1. Value of rod members.

Member No.	Lengths of the member $L_c\,[cm]$	Direction cosine angle c_x	Direction cosine angle c_y
1	*50*	*0.8*	*0.6*
2	*58.3*	*0.857*	*0.514*

The stiffness matrix for member No.1 is

$$[\boldsymbol{K}]_1 = \frac{EA}{L_{c1}}\begin{bmatrix} c_x^2 & c_x c_y & -c_x^2 & -c_x c_y \\ c_x c_y & c_y^2 & -c_x c_y & -c_y^2 \\ -c_x^2 & -c_x c_y & c_x^2 & c_x c_y \\ -c_x c_y & -c_y^2 & c_x c_y & c_y^2 \end{bmatrix} = \begin{bmatrix} 3215 & 2411 & -3215 & -2411 \\ 2411 & 1808 & -2411 & -1808 \\ -3215 & -2411 & 3215 & 2411 \\ -2411 & -1808 & 2411 & 1808 \end{bmatrix}. \tag{4.4}$$

The stiffness matrix for member No.2 is

$$[\boldsymbol{K}]_2 = \frac{EA}{L_{c2}}\begin{bmatrix} c_x^2 & c_x c_y & -c_x^2 & -c_x c_y \\ c_x c_y & c_y^2 & -c_x c_y & -c_y^2 \\ -c_x^2 & -c_x c_y & c_x^2 & c_x c_y \\ -c_x c_y & -c_y^2 & c_x c_y & c_y^2 \end{bmatrix} = \begin{bmatrix} 3084 & 1851 & -3084 & -1851 \\ 1851 & 2160 & -1851 & -2160 \\ -3084 & -1851 & 3084 & 1851 \\ -1851 & -2160 & 1851 & 2160 \end{bmatrix}. \tag{4.5}$$

The stiffness matrix of the whole system is

$$[\boldsymbol{K}]_c = [\boldsymbol{K}]_1 + [\boldsymbol{K}]_2 . \tag{4.6}$$

Since the members are connected at points 2 and 4 as shown in Fig.2, we therefore sum the two stiffness matrices in the corresponding rows and columns corresponding to these points. In our case, this will be the 1st and 2nd rows and columns from the $[\boldsymbol{K}]_1$ matrix, and the 3rd and 4th rows and columns from the $[\boldsymbol{K}]_2$ matrix, thus

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 & 0 & 0 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 \\ k_{41}^1 & k_{42}^1 & k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{23}^2 & k_{24}^2 \\ 0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 \\ 0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_{1,2} \\ x_{3,4} \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F\cos\gamma \\ F\sin\gamma \\ 0 \\ 0 \end{bmatrix}. \tag{4.7}$$

Since both members are connected at both ends to another member, the boundary conditions for both members are

$$x_1^1 = x_1^2 = 0,$$
$$x_2^3 = x_2^4 = 0. \tag{4.8}$$

After taking into account the boundary conditions in the global stiffness matrix, we obtain a reduced system of the form

$$\begin{bmatrix} \mathrm{k}_{33}^1 + \mathrm{k}_{11}^2 & \mathrm{k}_{34}^1 + \mathrm{k}_{12}^2 \\ \mathrm{k}_{43}^1 + \mathrm{k}_{21}^2 & \mathrm{k}_{44}^1 + \mathrm{k}_{22}^2 \end{bmatrix} \begin{bmatrix} \mathrm{x}_{1,2} \\ \mathrm{x}_{3,4} \end{bmatrix} = \begin{bmatrix} F\cos\gamma \\ F\sin\gamma \end{bmatrix} \tag{4.9}$$

where

$$[\boldsymbol{K}]_c = \begin{bmatrix} 6299 & 4262 \\ 4262 & 3968 \end{bmatrix}. \tag{4.10}$$

By solving the reduced algebraic equation we obtain the values of the displacements of the nodes

$$\begin{bmatrix} \mathrm{x}_{1,2} \\ \mathrm{x}_{3,4} \end{bmatrix} = \begin{bmatrix} 0.0573 \\ -0.0238 \end{bmatrix}. \tag{4.11}$$

Consider a modified system by changing the cross-section of the members $d = 60\,mm$, i.e., by changing the cross-sectional area.
The modified stiffness matrices of the individual members are

$$\boldsymbol{K}_1^* = \frac{EA}{L_{c1}} \begin{bmatrix} c_x^2 & c_x c_y & -c_x^2 & -c_x c_y \\ c_x c_y & c_y^2 & -c_x c_y & -c_y^2 \\ -c_x^2 & -c_x c_y & c_x^2 & c_x c_y \\ -c_x c_y & -c_y^2 & c_x c_y & c_y^2 \end{bmatrix} = \begin{bmatrix} 7232 & 5424 & -7232 & -5424 \\ 5424 & 4068 & -5424 & -4068 \\ -7232 & -5424 & 7232 & 5424 \\ -5424 & -4068 & 5424 & 4068 \end{bmatrix},$$

$$\boldsymbol{K}_2^* = \frac{EA}{L_{c1}} \begin{bmatrix} c_x^2 & c_x c_y & -c_x^2 & -c_x c_y \\ c_x c_y & c_y^2 & -c_x c_y & -c_y^2 \\ -c_x^2 & -c_x c_y & c_x^2 & c_x c_y \\ -c_x c_y & -c_y^2 & c_x c_y & c_y^2 \end{bmatrix} = \begin{bmatrix} 7119 & 4267 & -7119 & -4267 \\ 4267 & 2560 & -4267 & -2560 \\ -7119 & -4267 & 7119 & 4267 \\ -4267 & -2560 & 4267 & 2560 \end{bmatrix}. \tag{4.12}$$

Adding the modified stiffness matrices gives the total stiffness matrix *e* of the modified system

$$\boldsymbol{K}_c^* = \boldsymbol{K}_1^* + \boldsymbol{K}_2^* \,. \tag{4.13}$$

The modified algebraic equation is

$$\boldsymbol{K}_c^*\{\boldsymbol{x}_0\}=\{\boldsymbol{p}_0\}. \tag{4.14}$$

Solving the equation gives the displacements of the nodes

$$\begin{bmatrix} x_{1,2} \\ x_{3,4} \end{bmatrix}=\begin{bmatrix} -0.1298 \\ 0.2190 \end{bmatrix}. \tag{4.15}$$

The stiffness matrix difference $[\Delta\boldsymbol{K}]$ between the original and the modified member system is

$$[\Delta\boldsymbol{K}]=\boldsymbol{K}_c^*-\boldsymbol{K}_c\ ,$$

$$[\Delta\boldsymbol{K}]=\begin{bmatrix} k_{33}^{1*}+k_{11}^{2*} & k_{34}^{1*}+k_{12}^{2*} \\ k_{43}^{1*}+k_{21}^{2*} & k_{44}^{1*}+k_{22}^{2*} \end{bmatrix}-\begin{bmatrix} k_{33}^{1}+k_{11}^{2} & k_{34}^{1}+k_{12}^{2} \\ k_{43}^{1}+k_{21}^{2} & k_{44}^{1}+k_{22}^{2} \end{bmatrix}=\begin{bmatrix} 8052 & 5429 \\ 5429 & 2660 \end{bmatrix}. \tag{4.16}$$

By substituting $[\Delta\boldsymbol{K}]$ into the resulting reanalysis matrix, when the submatrix $[\boldsymbol{C}]$ takes the form $[\boldsymbol{C}]=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then based on Eq.(3.32), we get

$$\begin{bmatrix} 6299 & 4262 & -1 & 0 & 0 & 0 \\ 4262 & 3968 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 8052 & 5429 \\ 0 & 0 & 0 & 1 & 5429 & 2660 \end{bmatrix}\begin{bmatrix} \Delta\boldsymbol{x}_{1,2} \\ \Delta\boldsymbol{x}_{3,4} \\ \lambda_1 \\ \lambda_2 \\ \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0.0573 \\ -0.0238 \\ 259.8 \\ 150 \end{bmatrix},$$

$$\begin{bmatrix} \Delta\boldsymbol{x}_{1,2} \\ \Delta\boldsymbol{x}_{3,4} \\ \lambda_1 \\ \lambda_2 \\ \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}=\begin{bmatrix} 0.3889 \\ -0.5834 \\ -36.6574 \\ -657.6808 \\ 0.4462 \\ -0.6072 \end{bmatrix}. \tag{4.17}$$

Solution of this bar system are the increments of displacements

$$\begin{bmatrix} \Delta\boldsymbol{x}_3 \\ \Delta\boldsymbol{x}_4 \end{bmatrix}=\begin{bmatrix} 0.0573 \\ -0.0237 \end{bmatrix}+\begin{bmatrix} 0.3889 \\ -0.5834 \end{bmatrix}=\begin{bmatrix} 0.4462 \\ -0.6072 \end{bmatrix}. \tag{4.18}$$

5. Conclusion

When designing optimal structures, iterative analysis or reanalysis is often very necessary if solving a given design optimization requires a lot of computational and computer time. Reanalysis methods render it possible to reduce just this time (Wu *et al.* [15], Sága *et al.* [16]). The reduction in computational time itself is based on the fact that a substantial part of the information about the modified design can be obtained by using the results from the solution of the original design. Regarding the suitability of the above methods, it can be stated that direct methods are more suitable for small changes in the design, but even though iterative and approximate methods are more efficient, their accuracy of solution is often insufficient. In the above example, it has been shown that by using an efficient reanalysis method, the computational time can be reduced considerably when calculating the necessary quantities due to the reduced forms of the equations and the resulting relationship. The method we have presented for solving the constructions is relatively simple, but a small disadvantage that makes the calculation itself more difficult is the necessity to add values to the original system of equations after each change of the element and to repeat the whole calculation cycle again. Thus, the reanalysis does not refer to the previous calculation, but to the first analysis. When calculating complex member systems and we need to obtain accurate results, we cannot do without the use of computational techniques due to the very high number of equations.

Acknowledgements

The work was supported by the grant projects VEGA No. 1/0500/20, KEGA 027TUKE-4/2020.

Nomenclature

$\boldsymbol{A}_i$	– cross-sectional area of the element
$\boldsymbol{C}$	– diagonal matrix
$[\boldsymbol{C}]$	– Boolean matrix
$\boldsymbol{f}_{ji}$	– internal force in element j
$\boldsymbol{I}$	– unit matrix
$[\overline{\boldsymbol{K}}]$	– diagonal submatrix of matrix $[\boldsymbol{K}_0]^{-1}$
$[\boldsymbol{K}_0]$	– stiffness matrix of the structure
$\{\boldsymbol{p}\},\{\boldsymbol{p}_v\}$	– vector of external forces
$\{\boldsymbol{p}_0\}$	– load vector
$\boldsymbol{P}_j$	– internal forces in element j of the original structure
$\boldsymbol{P}_j^*$	– internal forces in element j of the modified structure
$\overline{\boldsymbol{p}}_1$	– internal forces of the modified structure
$\{\boldsymbol{q}\}$	– vector representing the forces introduced together with the added element and acting on the edge of this element
$\boldsymbol{r}_i$	– coefficient of variation
$\boldsymbol{S}_{21}$	– sensitivity matrix of the forces
$\{\boldsymbol{x}_0\}$, $\{\boldsymbol{x}\}$	– displacement vector of nodal points
$\{\boldsymbol{y}\}$	– vector that corresponds to the degrees of freedom
$[\Delta\boldsymbol{K}_0]$	– singular matrix, stiffness matrix difference

$\{\Delta \boldsymbol{x}_0\}$ – increment of the displacement vector

δ – displacements of the nodes

$\{\lambda\}$ – vector of Lagrange coefficients

References

[1] Kołakowski P., Wikło M. and Holnicki-Szulc J. (2008): *The virtual distortion method-a versatile reanalysis tool for structures and systems*.– Structural and Multidisciplinary Optimization, vol.36, No.3, pp.217-234.

[2] Kirsch U. (2008): *Reanalysis of Structures* (*in Slovak*).– Springer Netherlands, pp.93-120.

[3] Bocko J. (1989): *Implementation of an effective reanalysis method*.– Journal of Mechanical Engineering, vol.40, No.1, pp.59-67

[4] Delyová I., Frankovský P., Bocko J., Trebuňa P., Živčák J., Schürger B. and Janigová S. (2021): *Sizing and topology optimization of trusses using genetic algorithm*.– Materials, vol.14, No.4, p.14, https://doi.org/10.3390/ma14040715.

[5] Rong F., Chen S.H. and Chen Y.D. (2003): *Structural modal reanalysis for topological modifications with extended Kirsch method*.– Computer Methods in Applied Mechanics and Engineering, vol.192, No.5-6, pp.697-707.

[6] Akgün M.A., Garcelon J.H. and Haftka R.T. (2001): *Fast exact linear and non-linear structural reanalysis and the Sherman–Morrison–Woodbury formulas*.– International Journal for Numerical Methods in Engineering, vol.50, No.7, pp.1587-1606.

[7] Wu B. and Li Z. (2006): *Static reanalysis of structures with added degrees of freedom*.– Communications in Numerical Methods in Engineering, vol.22, No.4, pp.269-281.

[8] Cao H., Li H., Wang M., Huang B. and Sun Y. (2022): *A structural reanalysis assisted harmony search for the optimal design of structures*.– Computers and Structures, vol.270, Article ID.106844, https://doi.org/10.1016/j.compstruc.2022.106844.

[9] Huang G., Wang H. and Li G. (2014): *A reanalysis method for local modification and the application in large-scale problems*.– Structural and Multidisciplinary Optimization, vol.49, No.6, pp.915-930.

[10] Mo K., Guo D. and Wang H. (2020): *Iterative reanalysis approximation-assisted moving morphable component-based topology optimization method*.– International Journal for Numerical Methods in Engineering, vol.121, No.22, pp.5101-5122.

[11] Materna D. and Kalpakides V.K. (2016): *Nonlinear reanalysis for structural modifications based on residual increment approximations*.– Computational Mechanics, vol.57, No.1, pp.1-18.

[12] Wu B., Li Z. and Li S. (2003): *The implementation of a vector-valued rational approximate method in structural reanalysis problems*.– Computer Methods in Applied Mechanics and Engineering, vol.192, No.13-14, pp.1773-1784.

[13] Tertel E., Kurylo P. and Papacz W. (2014): *The stress state in the three layer open conical shell during of stability loss*.– Acta Mechanica Slovaca, vol.18, No.2, pp.56-63.

[14] Bittnar Z. and Šejnoha J. (1996): *Numerical methods in structural mechanics*.– Thomas Telford, https://doi.org/10.1061/9780784401705.

[15] Wu Y., Wang H., Liu J., Zhang S. and Huang H. (2019): *A novel dynamic isogeometric reanalysis method and its application in closed-loop optimization problems*.– Computer Methods in Applied Mechanics and Engineering, vol.353, pp.1-23.

[16] Sága M., Vaško M., Handrik M. and Kopas P. (2019): *Contribution to random vibration numerical simulation and optimisation of nonlinear mechanical systems*.– Scientific Journal of Silesian University of Technology, Series Transport, vol.103, DOI: https://doi.org/10.20858/sjsutst.2019.103.11.

Received: June 15, 2022
Revised: August 15, 2022