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EXACT SOLUTIONS FOR FRACTIONALIZED SECOND GRADE FLUID FLOWS WITH BOUNDARY SLIP EFFECTS

S.DEHRAJ*, R.A.MALOOKANI and S.K.AASOORI
Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology, 67480, Nawabshah, PAKISTAN
E-mail: sanaullahdehraj@quest.edu.pk

G.M.BHUTTO
Department of Electrical Engineering, Quaid-e-Awam University of Engineering, Science and Technology, 67480, Nawabshah, PAKISTAN

L.ARAIN
Department of Business Administration, Shaheed Benazir Bhutto University
67480, Nawabshah, PAKISTAN

In this paper, an exact analytical solution for the motion of fractionalized second grade fluid flows moving over accelerating plate under the influence of slip has been obtained. A coupled system of partial differential equations representing the equation of motion has been re-written in terms of fractional derivatives form by using the Caputo fractional operator. The Discrete Laplace transform method has been employed for computing the expressions for the velocity field $u(y,t)$ and the corresponding shear stress $\tau(y,t)$. The obtained solutions for the velocity field and the shear stress have been written in terms of Wright generalized hypergeometric function ${}_p\Psi_q$ and are expressed as a sum of the slip contribution and the corresponding no-slip contribution. In addition, the solutions for a fractionalized, ordinary second grade fluid and Newtonian fluid in the absence of slip effect have also been obtained as special case. Finally, the effect of different physical parameters has been demonstrated through graphical illustrations.

Key words: second grade fluid, Caputo fractional operator, slip effects, discrete Laplace transform

1. Introduction

Polymer solutions, blood and certain oils are non-Newtonian fluids. They play a very important role in modern technological applications and industries. The wide applications of these fluids in many areas of life attract researcher and scientist to study them and describe their behaviors. Generally, non-Newtonian fluids differ in rate, differential and integral types. The differential type non-Newtonian fluids are the second-grade fluids [1]. The exact solutions for the second grade fluid which is the sub-class of non-Newtonian fluids are also obtainable. For many reasons, the computations of exact solutions are very important. For example, in order to examine the accuracies of many approximate solutions for complex flow problems, exact solutions are essential. Thus, for describing the behavior of non-Newtonian fluids, close-form solutions are mandatory. The exact solutions of such systems are obtained by various researches; see for instance [2-7]. Nowadays, fractional calculus, the branch of mathematics deals with an arbitrary order of differentiation and integration, has become important owing to its vast range of use in engineering and science. Fractional differential equations are extensively employed to model the problems in fluid flow, relaxation, diffusion, reaction–diffusion relaxation, oscillation and retardation processes in complex systems, dynamical processes and many

* To whom correspondence should be addressed

more physical and engineering processes [8, 9]. The merits of fractional differential equations in these applications are their non-local property. We are familiar that the integer order differential operator falls in the category of local operator, whereas the fractional order differential operator falls in the category of non-local operator; which indicates that the next state of a system not only depends on its current state but also on its future states [8,10]. For last few decades, the fractional calculus approach has extensively been employed to formulate and compute solution of fluid flows. The time derivative of integer order in the constitutive equation is replaced by the Reimann/Caputo operator. Specifically, it is proved to be a valuable tool for treating viscoelastic properties. Bagley [11], Friedrich [12], Junqi *et al.* [13], Guangyu *et al.* [14], Xu and Tan [15,16] and Tan *et al.* [17-22] have extensively developed the fractional calculus approach. The viscoelastic type studies are discussed in detail; see for instance [4,17,18,23-25]. In literature, most of the studies are focused on flow problems with no-slip condition, since no-slip condition is not valid for thin films problems. The problems including the multiple interfaces and the flow of rarefied fluids are brought under consideration. Experimentalists usually associate "spurt" with slip at the wall [26]. Slip conditions play a role in shear skin, hysteresis effects and spurt, whereas the insufficiency of the no-slip condition is quite visible in polymer melts which most often show microscopic wall slip. Much research has been done to study flow problems related to Newtonian and non-Newtonian fluids subject to no-slip condition [27-32]. Few investigations for the existence of slip at the solid boundary are discussed in [33,34]. In 1823, the possibility of fluid slip at the solid boundary was first time indicated by Navier as a general boundary condition. This boundary condition states that the tangential velocity of the fluid relative to the solid at a point on its surface is proportional to the tangential stress acting at that point [35].

This paper is structured as follows. In Section 2, the mathematical model with appropriate initial and boundary conditions is established. The exact analytic expressions for the velocity field and the corresponding shear stress have been computed in Section 3 and Section 4 respectively. In Section 5, some special cases have been discussed and expressions derived from obtained solutions by using different physical parameters. In Section 6, results and discussion have been presented and graphs have been drawn to analyse the effect of different parameters on the flow. Finally, in Section 7, some important conclusions have been presented.

2. Mathematical model

The mathematical formulations of the continuity and momentum equation describing the motion of incompressible fluid flow are presented as under:

$$\nabla \cdot \boldsymbol{V} = 0, \ \nabla \cdot \boldsymbol{T} = \rho \frac{\partial \boldsymbol{V}}{\partial t} + \rho (\boldsymbol{V}.\nabla)\boldsymbol{V} \tag{2.1}$$

where the term ρ represents the density of the fluid, the velocity of the fluid flow is denoted by $\boldsymbol{V}$, t represents the time variable and ∇ represents the Nabla operator and the Cauchy tensor $\boldsymbol{T}$ for a second grade homogenous incompressible fluid flow is represented by the following equation [36-41]:

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}, \quad \boldsymbol{S} = \mu \boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2. \tag{2.2}$$

where α_1 and α_2 represent normal stress moduli or material moduli, p represents the hydrostatic pressure, unit tensor identity is represented by $\boldsymbol{I}$, spherical stress is represented by $-p\boldsymbol{I}$, and the extra-stress tensor and dynamic viscosity are represented by $\boldsymbol{S}$ and μ, respectively. The functions $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the kinematic tensors and are presented as under:

$$\boldsymbol{A}_1 = (\nabla \boldsymbol{V}) + (\nabla \boldsymbol{V})^T, \quad \boldsymbol{A}_2 = \frac{d\boldsymbol{A}_1}{dt} + \boldsymbol{A}_1(\nabla \boldsymbol{V}) + (\nabla \boldsymbol{V})^T \boldsymbol{A}_1. \tag{2.3}$$

In the present study the following form of velocity field is considered:

$$\boldsymbol{V}=\boldsymbol{V}(y,t)=u(y,t)\boldsymbol{i},\quad \boldsymbol{S}=\boldsymbol{S}(y,t) \tag{2.4}$$

where $\boldsymbol{i}$ represents the unit vector in the x direction. The constraint of incompressibility is satisfied automatically for considered flows. Initially, there is no fluid flow, thus we have the following equations:

$$\boldsymbol{V}=\boldsymbol{V}(y,0)=0,\quad \boldsymbol{S}=\boldsymbol{S}(y,0)=0, \tag{2.5}$$

By solving Eq.(2.1) to Eq.(2.5) and after long but elementary calculations, we get the following coupled system of partial differential equations:

$$\begin{cases} \dfrac{\partial u(y,t)}{\partial t}=\left(v+\alpha\dfrac{\partial}{\partial t}\right)\dfrac{\partial^2 u(y,t)}{\partial y^2}, \\ \\ \tau(y,t)=\left(\mu+\alpha_1\dfrac{\partial}{\partial t}\right)\dfrac{\partial u(y,t)}{\partial y} \end{cases} \tag{2.6}$$

where the velocity field and the shear stress are represented by $u(y,t)$ and $\tau(y,t)$, respectively. Furthermore, the kinematic viscosity and the viscoelastic parameter for the second grade fluid flow are represented by v and α respectively. The governing equations of motion given in Eqs (2.6) can be re-written in terms of fractional derivative by using the Caputo fractional operator:

$$\begin{cases} \dfrac{\partial u(y,t)}{\partial t}=\left(v+\alpha D_t^{\beta}\right)\dfrac{\partial^2 u(y,t)}{\partial y^2}, \\ \tau(y,t)=\left(\mu+\alpha_1 D_t^{\beta}\right)\dfrac{\partial u(y,t)}{\partial y} \end{cases} \tag{2.7}$$

where D_t^{β} represents the Caputo fractional operator and is represented as:

$$D_t^{\beta} f(t)=\begin{cases} \dfrac{1}{\Gamma(1-\beta)} & \displaystyle\int_0^t \dfrac{f'(\tau)}{(t-\tau)^{\beta}}d\tau,\ 0\le\beta<1, \\ \dfrac{df(t)}{dt}, & \beta=1 \end{cases} \tag{2.8}$$

where $\Gamma(\bullet)$ represents the gamma function. The appropriate initial and the boundary conditions for the developed equation of motion are given below:

$$u(y,0)=0 \quad \text{for} \quad y>0, \tag{2.9}$$

$$u(0,t)=UH(t)t^n+\theta H(t)\left.\dfrac{\partial u(y,t)}{\partial y}\right|_{y=0} \tag{2.10}$$

where $H(t)$ represents the Heaviside function and the slip effect parameter is denoted by θ. Furthermore, it has been assumed that there is no motion of fluid at infinity and no shear is taken in free stream, thus we have the following natural condition for the given problem:

$$u(y,t)=0, \quad \text{for } t\to\infty, \; t>0. \tag{2.11}$$

In the following section, we shall solve the coupled system of fractionalized partial differential equations given in Eq.(2.7) under appropriate initial and boundary conditions given in Eq.(2.9) to Eq.(2.11) by using the discrete Laplace transform [6,19-23,25,45-46].

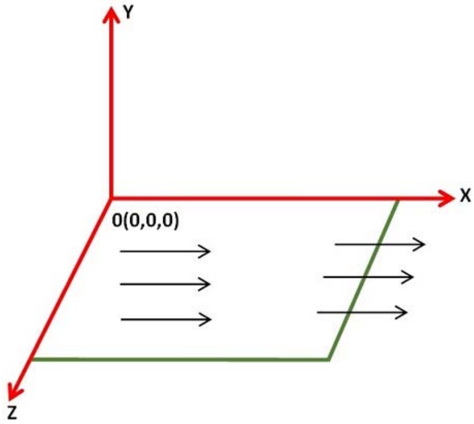

Fig.1. Fractionalized second grade fluid over an accelerating plate with slip effect.

3. Computation of the velocity field

To compute the exact analytic expression for the velocity field $u(y,t)$ we use the discreate Laplace transform to Eq.(2.7)$_1$ on both sides and by considering the initial condition (2.9) it yields:

$$\left(\frac{\partial^2}{\partial y^2}-\frac{q}{\nu+\alpha q^{\beta}}\right)\bar{u}(y,q)=0 \tag{3.1}$$

where $\bar{u}(y,q)$ is the Laplace transorm of the funciton of $u(y,t)$ and q represents the transform parameter. Furthermore, by taking the discrete Laplace of the boundary condition given in Eq.(2.10) and natural condition given in Eq.(2.11) we get

$$\bar{u}(0,q)=\frac{Un!}{q^{n+1}}+\theta\frac{\partial\bar{u}(y,q)}{\partial y}\Big|_{y=0}, \tag{3.2}$$

and

$$\bar{u}(y,q)\to 0 \text{ as } y\to\infty, \tag{3.3}$$

By setting Eq.(3.2) and Eq.(3.3) into Eq.(3.1) , we obtain:

$$\bar{u}(y,q)=\frac{Un!}{q^{n+1}\left[1+\theta\sqrt{\frac{q}{v+\alpha q^{\beta}}}\right]}\exp\left\{-\left(\frac{q}{v+\alpha q^{\beta}}\right)^{\frac{1}{2}}y\right\}. \tag{3.4}$$

For the sake of convience, we can re-wrtite the each term of Eq.(3.4) in terms of infinite series and by using the fact given in Eq.(3.5)

$$(-1)^k\frac{\Gamma(m+1)}{\Gamma(m-k+1)}=\frac{\Gamma(k-m)}{\Gamma(-m)}, \tag{3.5}$$

after long calculations, we get Eq.(3.6):

$$\begin{aligned}\bar{u}(y,q)=&\frac{Un!}{q^{n+1}}+Un!\sum_{k=1}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^k\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{k}{2}\right)}{n!\Gamma\left(\frac{k}{2}\right)}\left(\frac{-v}{\alpha}\right)^n\frac{1}{q^{(\beta-1)\left(\frac{k}{2}\right)+n(\beta+1)+1}}+\\&+Un!\sum_{k=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^k\sum_{m=1}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^m\frac{1}{m!}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{m+k}{2}\right)}{n!\Gamma\left(\frac{m+k}{2}\right)}\left(\frac{-v}{\alpha}\right)^n\frac{1}{q^{(\beta-1)\left(\frac{m+k}{2}\right)+(\beta+1)n+1}}.\end{aligned} \tag{3.6}$$

In order to compute the expression for the velocity $u(y,t)$ from Eq.(3.6), we apply the discrete inverse Laplace transform on both sides of Eq.(3.6), thus we obtain

$$\begin{aligned}u(y,t)=&UH(t)t^n+UH(t)n!\sum_{k=1}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^k\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{k}{2}\right)\left(\frac{-vt^{\beta+1}}{\alpha}\right)^n t^{(\beta-1)\frac{k}{2}}}{n!\Gamma\left(\frac{k}{2}\right)\Gamma\left((\beta-1)\frac{k}{2}+n(\beta+1)+1\right)}+\\&+Un!H(t)\sum_{k=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^k\sum_{m=1}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^m\frac{1}{m!}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{m+k}{2}\right)\left(\frac{-vt^{\beta+1}}{\alpha}\right)^n t^{(\beta-1)\left(\frac{m+k}{2}\right)}}{n!\Gamma\left(\frac{m+k}{2}\right)\Gamma\left((\beta-1)\frac{m+k}{2}+n(\beta+1)+1\right)}.\end{aligned} \tag{3.7}$$

The expression in Eq.(3.7) represnts the excact analytic solution for the velocity field and can further be re-written in terms of Wright generalized hypergeometric funciton which is denoted by ${}_p\Psi_q$ and is defined as:

$${}_p\Psi_q\left[z\left|\begin{matrix}(a_1,A_1),\ldots(a_p,A_p)\\ \\(b_1,B_1),\ldots(b_q,\mathrm{B}_q)\end{matrix}\right.\right]=\sum_{n=0}^{\infty}\frac{z^n\Pi_{j=1}^{p}\Gamma(a_j+A_jn)}{n!\Pi_{j=1}^{q}\Gamma(b_j+B_jn)}. \tag{3.8}$$

Equation (3.7) can further be re-written as below by using Eq.(3.8)

$$
\begin{aligned}
&u(y,t)=UH(t)t^{n}+n!UH(t)\times \\
&\times\sum_{k=1}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^{k}{}_{1}\Psi_{2}\left[\frac{-vt^{\beta+1}}{\alpha}\left|\begin{matrix}\left(\frac{k}{2},1\right)\\ \\ \left(\frac{k}{2},0\right),\left((\beta-1)\frac{k}{2}+1,\beta+1\right)\end{matrix}\right.\right]t^{(\beta-1)\frac{k}{2}}+ \\
&+n!UH(t)\sum_{k=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^{k}\sum_{m=1}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^{m}\frac{1}{m!}{}_{1}\Psi_{2}\times \\
&\times\left[\frac{-vt^{\beta+1}}{\alpha}\left|\begin{matrix}\left(\frac{m+k}{2},1\right)\\ \\ \left(\frac{m+k}{2},0\right),\left((\beta-1)\frac{m+k}{2}+1,\beta+1\right)\end{matrix}\right.\right]t^{(\beta-1)\frac{m+k}{2}}.
\end{aligned}
\tag{3.9}
$$

Thus Eq.(3.9) represents the expression for an exact analytical solution of fractionalized second grade fluid flows and is written in terms of Wright generalized hypergeometric function with sum of slip and corresponding no-slip contribution.

4. Computation of the shear stress

To compute the exact analytic expressions for the corresponding shear stress $\tau(y,t)$, for the fractionalized second grade fluid flow, we apply the discrete Laplace transform to Eq.(2.7)$_2$, on both sides, and we get:

$$\overline{\tau}(y,q)=\left(\mu+\alpha_{1}q^{\beta}\right)\frac{\partial\overline{u}(y,q)}{\partial y}, \tag{4.1}$$

where $\overline{\tau}(y,q)=L\{\tau(y,t)\}$ and q is the transform parameter. Using Eq.(3.4) into Eq.(4.1), we get

$$\overline{\tau}(y,q)=\frac{-\rho\sqrt{q}\,n!U\sqrt{v+\alpha q^{\beta}}}{q^{n+1}\left[1+\theta\sqrt{\frac{q}{v+\alpha q^{\beta}}}\right]}\exp\left\{-\left(\frac{q}{v+\alpha q^{\beta}}\right)^{\frac{1}{2}}y\right\}. \tag{4.2}$$

In order to compute the inverse Laplace transform of Eq.(4.2), we first expand each term of Eq.(4.2) in terms of infinite series as:

$$\overline{\tau}(y,q)=-\sqrt{\alpha}\rho n!U\sum_{k=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^{k}\sum_{m=0}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^{m}\frac{1}{m!}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{m+k-1}{2}\right)\left(\frac{-v}{\alpha}\right)^{n}}{n!\Gamma\left(\frac{m+k-1}{2}\right)q^{(\beta-1)\left(\frac{m+k-1}{2}\right)+n(\beta+1)+1}}. \tag{4.3}$$

By applying the discrete inverse Laplace transform to Eq.(4.3) on both sides, we get

$$\tau(y,t)=-\sqrt{\alpha}\rho n!UH(t)\sum_{k=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^{k}\sum_{m=0}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^{m}\frac{1}{m!}\times$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{m+k-1}{2}\right)\left(\frac{-vt^{\beta+1}}{\alpha}\right)^{n}t^{(\beta-1)\left(\frac{m+k-1}{2}\right)}}{n!\Gamma\left(\frac{m+k-1}{2}\right)\Gamma\left((\beta-1)\left(\frac{m+k-1}{2}\right)+n(\beta+1)+1\right)}. \tag{4.4}$$

By using Eq.(3.8), we re-write the expression for the shear stress in terms of Wright generalized hypergeometric function as

$$\tau(y,t)=-\sqrt{\alpha}\rho n!UH(t)\times\sum_{k=0}^{\infty}\left(\frac{-\theta}{\sqrt{\alpha}}\right)^{k}\sum_{m=0}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^{m}\frac{1}{m!}{}_{1}\Psi_{2}\times$$
$$\times\left[\frac{-vt^{\beta+1}}{\alpha}\left|\begin{matrix}\left(\frac{m+k-1}{2},1\right)\\ \\ \left(\frac{m+k-1}{2},0\right)\left((\beta-1),\left(\frac{m+k-1}{2}\right)+1,\beta+1\right)\end{matrix}\right.\right]t^{(\beta-1)\left(\frac{m+k-1}{2}\right)}. \tag{4.5}$$

Equation (4.5) represents the exact analytic expression for the shear stress for the fractionalized second grade fluid flow and is written in terms of Wright generalized hypergeometric function.

5. Special cases

5.1. Ordinary second grade fluid with slip effects $\beta=1$

By setting the fractional parameter $\beta=1$ into Eq.(3.9) and Eq.(4.5), we obtain the expressions for the velocity field and the corresponding shear stress respectively, for an ordinary second grade fluid with slip effect.

5.2. Fractionalized second grade fluid without slip effect for $\theta=0$

The exact analytic expressions for the velocity field and the corresponding shear stress can be obtained for the fractionalized second grade fluid flow without slip effect by setting the slip parameter $\theta=0$ in Eq.(3.9) and Eq.(4.5), respectively. The expressions are given below:

$$u(y,t)=UH(t)t^{n}+n!UH(t)\sum_{m=1}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^{m}\frac{1}{m!}{}_{1}\times$$
$$\times\Psi_{2}\left[\frac{-vt^{\beta+1}}{\alpha}\left|\begin{matrix}\left(\frac{m}{2},1\right)\\ \\ \left(\frac{m}{2},0\right),\left((\beta-1)\frac{m}{2}+1,\beta+1\right)\end{matrix}\right.\right]t^{(\beta-1)\frac{m}{2}} \tag{5.1}$$

and corresponding shear stress

$$\tau(y,t) = -\sqrt{\alpha}\rho n! UH(t)\times$$
$$\times\sum_{m=0}^{\infty}\left(\frac{-y}{\sqrt{\alpha}}\right)^m \frac{1}{m!}\,{}_1\Psi_2\left[\frac{-\nu t^{\beta+1}}{\alpha}\left|\begin{matrix}\left(\frac{m-1}{2},1\right)\\ \\ \left(\frac{m-1}{2},0\right),\left((\beta-1)\left(\frac{m-1}{2}\right)+1,\beta+1\right)\end{matrix}\right.\right]t^{(\beta-1)\left(\frac{m-1}{2}\right)}. \tag{5.2}$$

5.3. Ordinary second grade fluid without slip effects $\beta = 1$

The expressions for the velocity field and the corresponding shear stress for an ordinary second grade fluid flow without slip effect can be obtained by setting the fractional parameter $\beta = 1$ into Eq.(5.1) and Eq.(5.2), respectively.

5.4. Newtonian fluid under influence of slip effect $\alpha = 0$

For computing the exact analytic solutions for a Newtonian fluid with slip effect, we set $\alpha = 0$, Eq.(3.4) and we get:

$$\bar{u}(y,q) = \frac{Un!}{q^{n+1}\left[1+\theta\sqrt{\frac{q}{\nu}}\right]}\exp\left\{-\left(\sqrt{\frac{q}{\nu}}\right)y\right\}. \tag{5.3}$$

In order to find the inverse Laplace transform we can re-write each term of Eq.(5.3) in series form as under:

$$\bar{u}(y,q) = Un!\sum_{m=0}^{\infty}\left(\frac{-y}{\sqrt{\nu}}\right)^m\frac{1}{m!}\frac{1}{q^{1+n-\frac{m}{2}}} + Un!\sum_{k=1}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^k\sum_{m=0}^{\infty}\left(\frac{-y}{\sqrt{\nu}}\right)^m\frac{1}{m!}\frac{1}{q^{1+n-\left(\frac{m+k}{2}\right)}} \tag{5.4}$$

By applying the inverse Laplace transform on both sides of Eq.(5.4) we get:

$$\begin{aligned} u(y,t) &= Un!t^n H(t)\sum_{m=0}^{\infty}\left(\frac{-y}{\sqrt{t\nu}}\right)^m\frac{1}{m!\Gamma\left(1-\frac{m}{2}+n\right)}+\\ &+t^n Un! H(t)\sum_{k=1}^{\infty}\left(\frac{-\theta}{\sqrt{t\nu}}\right)^k\sum_{m=0}^{\infty}\left(\frac{-y}{\sqrt{t\nu}}\right)^m\frac{1}{m!}\frac{1}{\Gamma\left(1-\left(\frac{m+k}{2}\right)+n\right)}.\end{aligned} \tag{5.5}$$

The expression for a Newtonian fluid with slip effect as given in Eq.(5.5) can be re-written in terms of Wright function which is defined as under:

$$W_{a,b}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(an+b)}, \qquad a>-1. \tag{5.6}$$

Thus we have the following expression of the exact analytic solution for a Newtonian fluid with slip effect:

$$u(y,t)=n!t^n UH(t)\left[W_{-\frac{1}{2},1+n}\left(\frac{-y}{\sqrt{t\nu}}\right)+\sum_{k=1}^{\infty}\left(\frac{-\theta}{\sqrt{t\nu}}\right)^k W_{-1/2,n+1-\frac{k}{2}}\left(\frac{-y}{\sqrt{t\nu}}\right)\right]. \tag{5.7}$$

Similarly, for computing the corresponding shear stress $\tau(y,t)$ for a Newtonian fluid under the influence of slip effect, we set $\alpha=0$ into Eq.(4.2). It yields following expression:

$$\overline{\tau}(y,q)=\frac{-\rho\sqrt{q}n!U\sqrt{\nu}}{q^{n+1}\left[1+\theta\sqrt{\frac{q}{\nu}}\right]}\exp\left\{-\left(\frac{q}{\nu}\right)^{\frac{1}{2}}y\right\}. \tag{5.8}$$

To obtain $L^{-1}\left\{\overline{\tau}(y,q)\right\}=\tau(y,t)$ easily, we expand each term of Eq.(5.8) in series form:

$$\overline{\tau}(y,q)=-n!U\sqrt{\rho\mu}\sum_{k=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^k\sum_{m=0}^{\infty}\left(\frac{-y}{\sqrt{\nu}}\right)^m\frac{1}{m!}\frac{1}{q^{n-\left(\frac{m+k-1}{2}\right)}}. \tag{5.9}$$

By applying the inverse Laplace transform to both sides of Eq.(5.9), we get the following expression for the shear stress for a Newtonian fluid:

$$\tau(y,t)=-n!UH(t)t^{n-\frac{1}{2}}\sqrt{\rho\mu}\sum_{k=0}^{\infty}\left(\frac{-\theta}{\sqrt{t\nu}}\right)^k\sum_{m=0}^{\infty}\left(\frac{-y}{\sqrt{t\nu}}\right)^m\frac{1}{m!}\frac{1}{\Gamma\left(n-\left(\frac{m+k-1}{2}\right)\right)}. \tag{5.10}$$

The expression for a Newtonian fluid under the influence of slip effect as given in Eq.(5.10) can further be re-written in terms of Wright function as follows:

$$\tau(y,t)=-n!UH(t)t^{n-\frac{1}{2}}\sqrt{\rho\mu}\sum_{k=0}^{\infty}\left(\frac{-\theta}{\sqrt{t\nu}}\right)^k W_{-\frac{1}{2},\frac{n-k+1}{2}}\left(\frac{-y}{\sqrt{t\nu}}\right). \tag{5.11}$$

5.5. Newtonian fluid without slip effect $\theta=0$

By setting the slip parameter $\theta=0$ in Eq.(5.7) and Eq.(5.11), we obtained the exact analytic expressions for the velocity filed and shear stress, respectively, for Newtonian fluids without slips.

6. Results and discussion

The close-form solutions for a fractionalized second grade fluid moving under the influence of slip effect over the accelerating plate have been obtained by the application of the discrete Laplace transform method. These solutions are re-written in terms of Wright generalized hypergeometric function as a sum of slip and no-slip contribution. Furthermore, many special cases have also been discussed. Many graphs have been drawn to analyze the effect of different physical parameters on the motion of the fractionalized second grade fluid due to an accelerating plate. The graphical illustration for the velocity field $u(y,t)$ and the corresponding shear stresses $\tau(y,t)$ are produced for different parameters by using Mathcad software. For simplicity, all diagrams are plotted by taking the values $U = 1$, $\nu = 0.295$, $\mu = 26$, $\alpha = 0.5$ and $\beta = 0.2$.

In Fig.2, the graphs for the velocity field and the corresponding shear stress have been drawn for different time positions. It can easily be observed from these graphs that the velocity profile and the shear stress profile grow with respect to time t and reduce with respect to the height y. In Fig.3, the effects of the material parameter α on the velocity profile and the corresponding shear stress profile are shown. It can be seen in these graphs that both the velocity as well as shear stress are increasing functions with the growth of the material parameter.

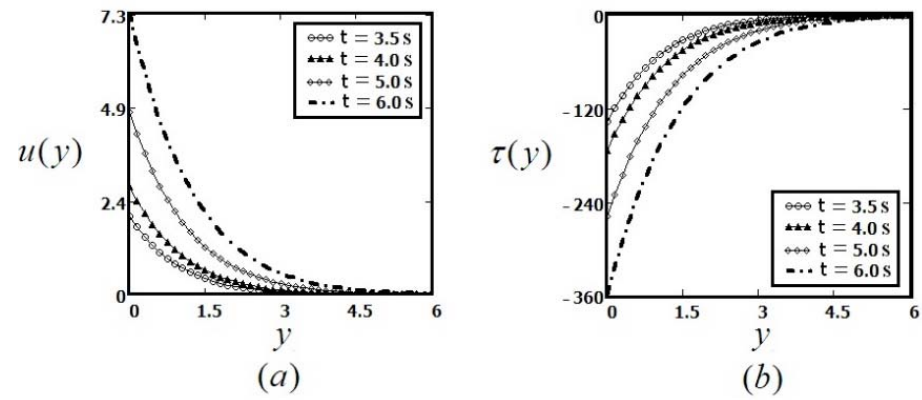

Fig.2. The velocity and shear stress profiles for $U = 1$, $\nu = 0.295$, $\mu = 26$, $\alpha = 0.5$, $\beta = 0.2$, $\theta = 5$, $n = 2$ and different values of *t*.

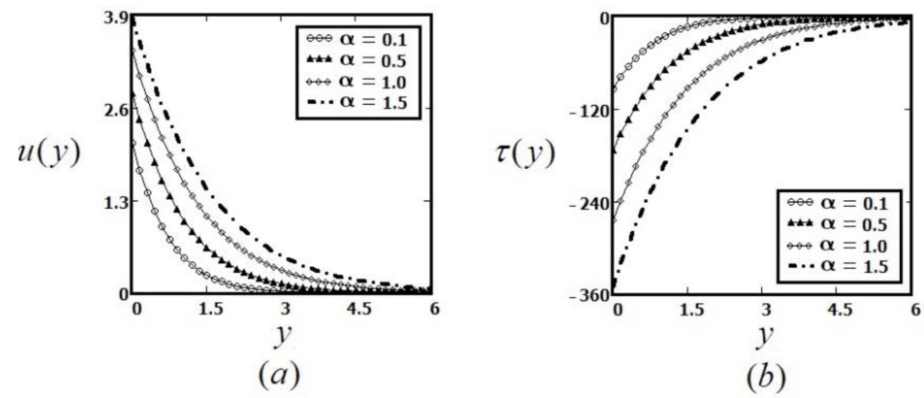

Fig.3. The velocity and shear stress profiles for $U = 1$, $\nu = 0.295$, $\mu = 26$, $\beta = 0.2$, $\theta = 5$, $n = 2$, $t = 4s$ and different values of α.

The effect of the kinematic viscosity ν over the motion of the fractionalized second grade fluid is depicted in Fig.4. It is shown that the velocity and the shear stress grow as the parametric values of the kinematic viscosity are increasing. The effect of the fractional parameter β on motion of the fluid flow is depicted in Fig.5. The velocity field $u(y,t)$ and the corresponding shear stress $\tau(y,t)$ have opposite behavior for different fractional parametric values. The velocity field increases and shear stress decreases with regard to the fractional parameter β.

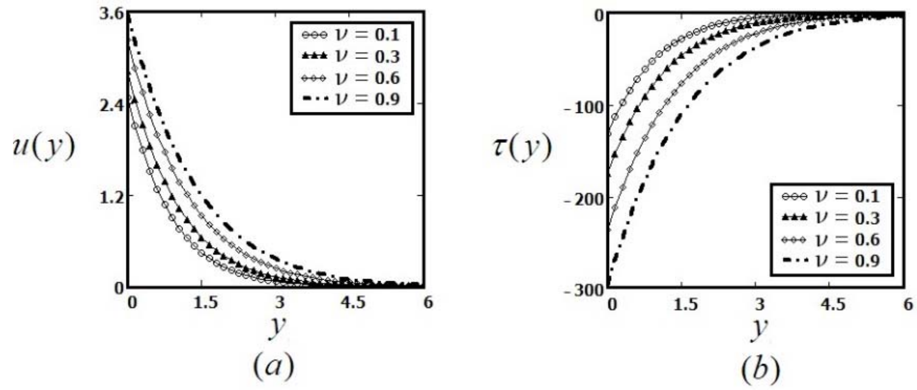

Fig.4. The velocity and shear stress profiles for $U=1$, $\rho=88$, $\alpha=0.5$, $\beta=0.2$, $\theta=5$, $n=2$, $t=4s$ and different values of ν.

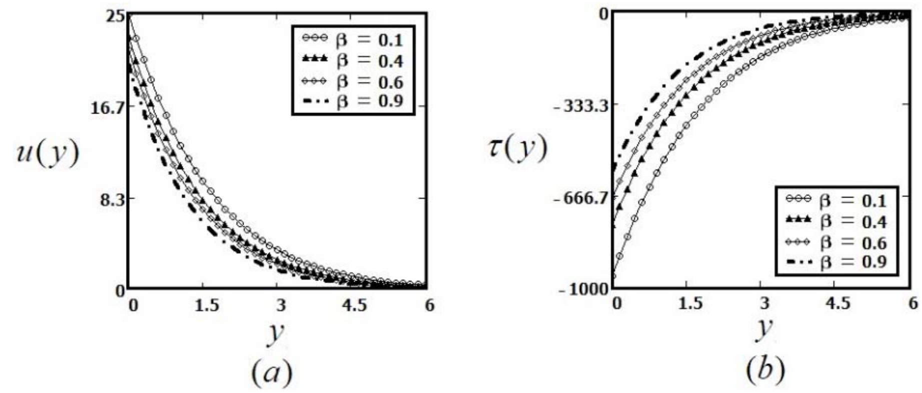

Fig.5. The velocity and shear stress profiles for $U=1$, $\nu=0.295$, $\mu=26$, $\alpha=0.5$, $\theta=5$, $n=2$, $t=4s$ and different values of β.

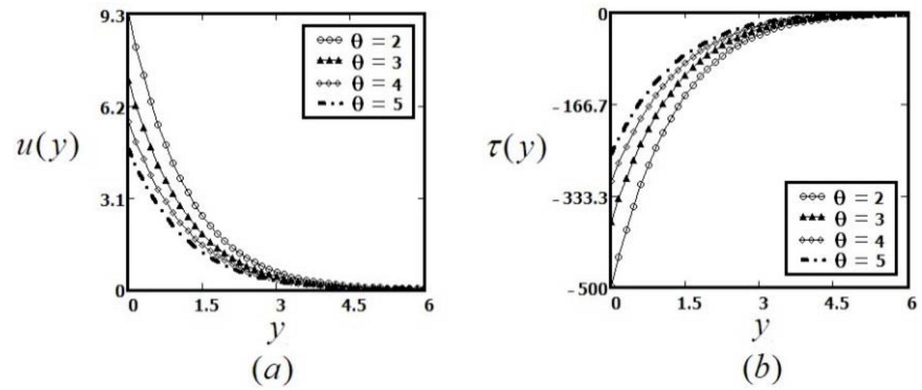

Fig.6. The velocity and shear stress profiles for $U=1$, $\nu=0.295$, $\mu=26$, $\alpha=0.5$, $\beta=0.2$, $n=2$, $t=5s$ and different values of θ.

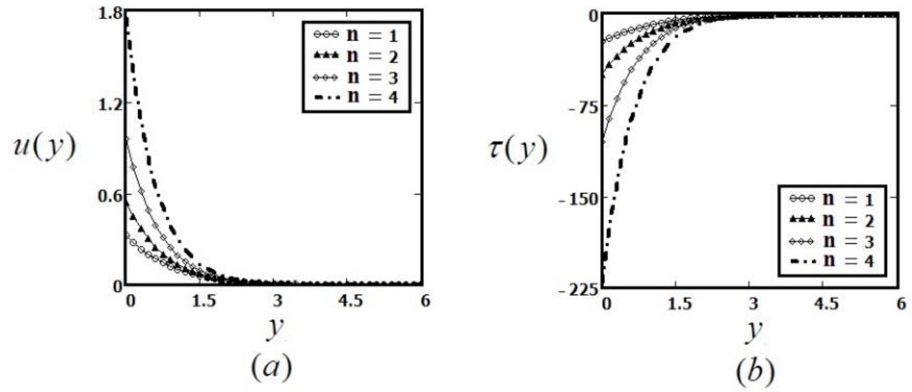

Fig.7. The velocity and shear stress profiles for $U = 1$, $\nu = 0.295$, $\mu = 26$, $\alpha = 0.5$, $\beta = 0.2$, $t = 2s$ and different values of n.

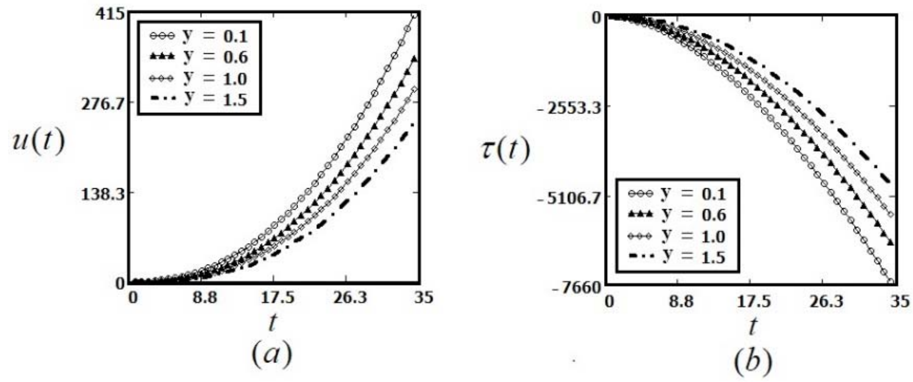

Fig.8. The velocity and shear stress profiles for $U = 1$, $\nu = 0.295$, $\mu = 26$, $\alpha = 0.5$, $\beta = 0.2$, $\theta = 5$, $n = 2$ and different values of y.

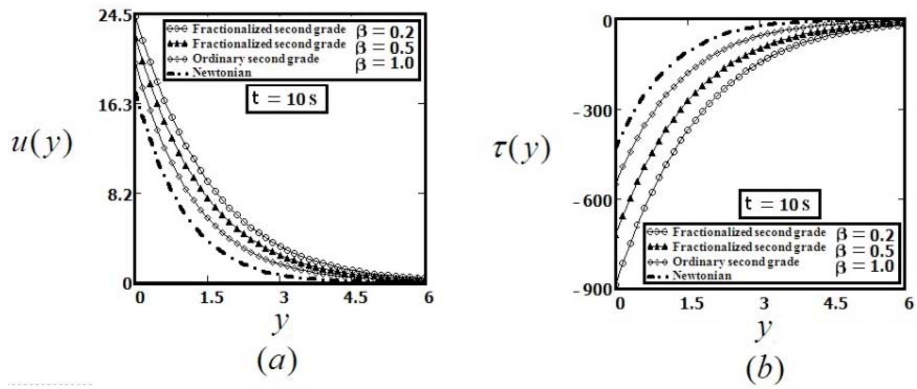

Fig.9. The velocity and shear stress profiles for $U = 1$, $\nu = 0.295$, $\mu = 26$, $\alpha = 0.5$, $\beta = 0.2,\ 0.5,\ 1$, $\theta = 5$, $n = 2$ and $t = 4s$.

The effect of the slip parameter θ over the motion of the fractionalized second grade fluid has been drawn in Fig.6. It can be seen in these graphs that the velocity and the corresponding shear stress are decreasing functions with regard to the slip parameter. Figure 7 depicts the effect of power n over the motion of the fluid. It can easily be seen that as the parametric values of power n are increased, then both entities are increasing.

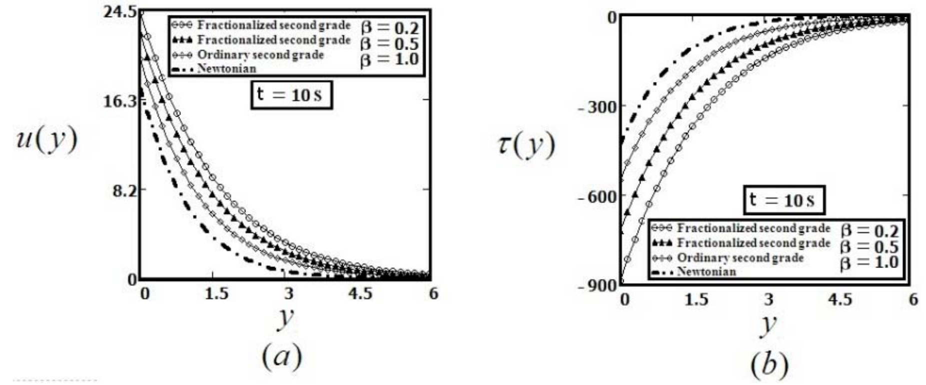

Fig.10. The velocity and shear stress profiles for $U = 1$, $\nu = 0.295$, $\mu = 26$, $\alpha = 0.5$, $\beta = 0.2,\ 0.5,\ 1$, $\theta = 5$ and $t = 10s$.

The velocity field and the corresponding shear stress decrease as we leave the plate. This phenomenon is clearly depicted in Fig.8. A comparison of the velocity profiles and the shear stress corresponding to the four models (fractionalized second grade fluid for $\beta = 0.2,\ 0.5$, ordinary second grade $\beta = 1$, Newtonian fluid) at two different values of $t = 4s,\ 10s$ is shown in Figs 9 and 10. It can be seen from these diagrams that the fractionalized second grade fluids have largest values and for Newtonian fluids the values of both entities of interest are smallest.

6. Conclusion

In this paper, a fractionalized second grade fluid flow over an accelerating plate under the influence of slip effect has been studied by means of discrete Laplace transforms. The mathematical model representing the motion of the fluid has been re-written in terms of fractional derivative by using the Caputo fractional operator and appropriate initial and boundary conditions has been considered. The exact analytic solutions for the velocity field $u(y,t)$ and the corresponding shear stress $\tau(y,t)$ have been obtained by applying the discrete Laplace transform and re-written in terms of Wright generalized hypergeometric function ${}_p\Psi_q$ with sum of slip and no-slip contribution. A similar type of solutions for ordinary second grade and Newtonian fluids can easily be obtained as limiting cases of general solutions. Moreover, the solutions for a fractionalized and ordinary second grade fluid in the absence of slip effect are also obtained as special cases. In addition, the effects of the material parameter and fractional parameters on the motion of fractionalized second grade fluids have been discussed via graphical illustrations by using Mathcad software. The difference among fractionalized second, ordinary second grade and Newtonian fluid models is also emphasized. With respect to time, material parameter α and kinematic viscosity ν the velocity field and the shear stress are increasing functions. The fluid motion is strongly influenced by the fractional parameter β, whereas the effects of the fractional parameter on velocity and shear stress profiles are shown to be opposite. The increasing values of the slip parameter θ slow down the fluid motion. Fluid motion decreases by increasing the values of n, whereas increases by increasing the height y. The fractionalized second grade fluid is moving fast in comparison to ordinary second grade and Newtonian fluids.

Nomenclature

$\boldsymbol{A_1}, \boldsymbol{A_2}$ – kinematic tensors
D_t^β – Caputo fractional operator
$H(t)$ – Heaviside function
p – hydrostatic pressure
${}_p\Psi_q$ – Wright generalized hypergeometric function
q – transform parmater under Laplace
$\boldsymbol{T}$ – Cauchy tensor
$u(y,t)$ – velocity field
$W_{a,b}(z)$ – Wright function
ρ – fluid density
α – viscoelastic parameter
β – fractional parameter
$\Gamma(\bullet)$ – gamma function
θ – slip effect parameter
$\tau(y,t)$ – shear stress
v – kinematic Viscosity
∇ – Del or Nabla operator

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