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SLOWLY VIBRATING AXIALLY SYMMETRIC BODIES-TRANSVERSE FLOW

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Stokes drag on axially symmetric bodies vibrating slowly along the axis of symmetry placed under a uniform transverse flow of the Newtonian fluid is calculated. The axially symmetric bodies of revolution are considered with the condition of continuously turning tangent. The results of drag on sphere, spheroid, deformed sphere, egg-shaped body, cycloidal body, Cassini oval, and hypocycloidal body are found to be new. The numerical values of frictional drag on a slowly vibrating needle shaped body and flat circular disk are calculated as particular cases of deformed sphere.

Key words: Stokes drag, axially symmetric bodies, Newtonian fluid, axial flow.

AMS Subject Classification. 76D05, 76D07

1. Introduction

In continuum mechanics, the researchers and workers are always interested in drag experienced by minute objects moving or swimming microorganisms through a viscous fluid. The study of flow features of slowly moving particles comes under the class of low Reynolds number hydrodynamics. A Newtonian fluid is the fluid which follows Newton's law of motion. This type of motion is often observed and studied in bio-engineering, chemical engineering and naval engineering. In such studies, it is the Stokes drag which matters for the mechanical engineers to study the motion of minute and small particles or microorganisms.

It was George Gabriel Stokes [1], the British physicist and engineer, who gave the idea of drag on objects by solving the Navier-Stokes equation along with continuity equation under no-slip boundary conditions for zero Reynolds number. But at that time he did not distinguish the axial and transverse drag on particles or objects. Kanwal [2] tackled the rotatory and longitudinal oscillations of axi-symmetric bodies in a viscous fluid. Payne and Pell [3] gave the general expression of drag on an axially symmetric body in terms of a stream function. Kanwal [4] obtained drag on axially symmetric bodies vibrating slowly along the axis of symmetry in terms of classical Stokes drag by using the inner and outer expansion method provided by Proudman and Pearson [5].

Chwang and Wu [6] also gave the closed form Stokes drag expressions for prolate and oblate spheroids by using singularity method approach in axial and transverse flow situations. Datta and Srivastava [7] presented a new approach for calculating Stokes drag on axially symmetric bodies based on geometric variables related to the body of revolution with the condition of continuously turning tangent in both axial and transverse flow. In this paper, the closed form solution of the classical Stokes drag for sphere, spheroid, deformed sphere, egg-shaped body and cycloidal body was calculated. Further, Srivastava [8] continued the work and evaluated the classical axial and transverse Stokes drag on Cassini oval, hypocycloidal body and cylindrical capsule by utilizing DS-conjecture. Srivastava *et al.* [9-10] tackled the problem of a steady Stokes flow around a deformed sphere for both oblate and prolate class of axi-symmetric bodies.

The drag values on sphere, spheroid, needle shaped body, circular disk etc. for various aspect ratios are presented by Lamb [11], Landau and Lifshitz [12], Happel and Brenner [13], Kohr and Pop [14] and Kim and Karilla [15].

In this paper, we targeted to find the transverse frictional drag on axially symmetric bodies of revolution of curve with the condition of continuously turning tangent vibrating slowly along the axis of symmetry by using the expression of drag given by Kanwal [4] in terms of the classical Stokes drag provided by Datta and Srivastava [7] and Srivastava [8] followed by a new idea provided in section 4.

2. Description of an axially symmetric body

Datta and Srivastava [7] gave the expressions of Stokes drag on axially symmetric particle of revolution of curve in meridional plane with the condition of continuously turning tangent (Fig.1) in an axial flow(along the x-axis) as

$$D_x = \frac{1}{2}\frac{\lambda b^2}{h_x} = \frac{4}{3}\frac{\lambda b^2}{\int_0^\pi R\ \sin^3\alpha d\alpha}, \qquad where \qquad \lambda = 6\pi\mu U_x, \tag{2.1}$$

$$D_x = \frac{8\pi\mu b^2 U_x}{\int_0^\pi R\ \sin^3\alpha d\alpha}.$$

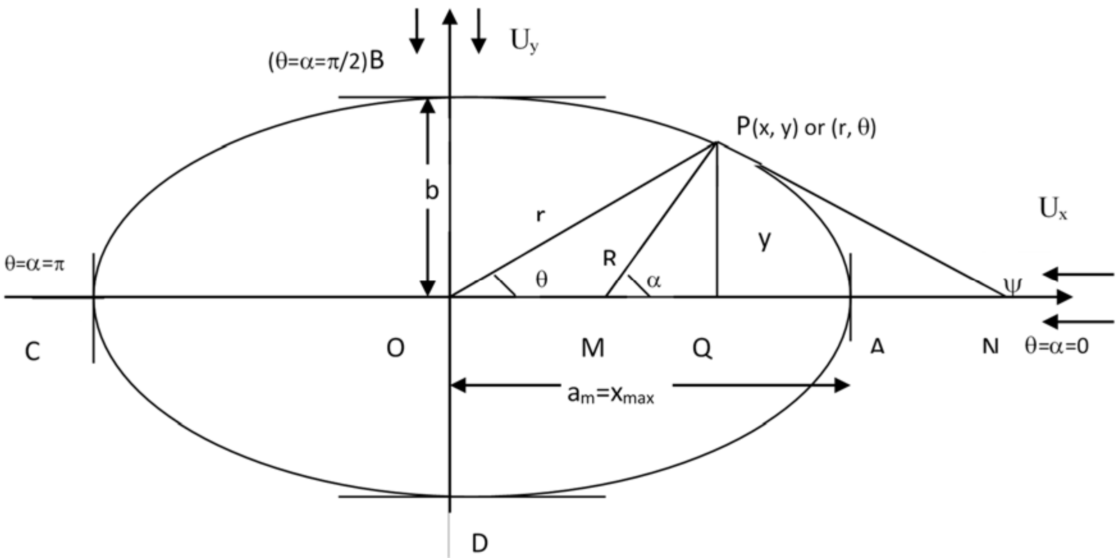

Fig.1. Geometry of axially symmetric body of revolution of curve in meridional plane.

The expression of Stokes drag on axially symmetric bodies placed in a transverse flow(uniform flow perpendicular to the axis of symmetry) is given by Datta and Srivastava [7]

$$D_y = \left(\frac{1}{2}\right)\frac{\lambda b^2}{h_y}, \qquad \text{where} \qquad \lambda = 6\pi\mu U, \tag{2.2}$$

and

$$h_y = \left(\frac{3}{16}\right)\int_0^{\pi} R\left(2\sin\alpha - \sin^3\alpha\right)d\alpha$$

where the suffix 'y' has been introduced to assert that the force is in the transverse direction.
Now, on dividing (2.2) and (2.3), we get

$$\frac{F_x}{F_y} = \frac{h_y}{h_x} = \frac{1}{2}\frac{\int_0^{\pi} R\left(2\sin\alpha - \sin^3\alpha\right)d\alpha}{\int_0^{\pi} R\sin^3\alpha d\alpha} = K(say), \tag{2.3}$$

a real factor.

In the above formulae, variable R is the intercepting length of normal a point on the body curve in the meridional plane and α is the angle made by the normal with the axis of symmetry. Also, h_x represents the height of the centre of gravity of force system in both the flow configurations and 'b' is the semi-transverse length of body curve in the merdional plane xy and 'μ' is the viscosity coefficient [7]. In every fluid-body interaction problems in fluid dynamics, the important quantity to evaluate is the drag experienced by a body which in a traditional approach is always found as a solution of a linear steady Stokes equation

$$\boldsymbol{0} = -\left(\frac{1}{\rho_1}\right) grad\ p + \nu\,\nabla^2\boldsymbol{u}, \qquad \mathrm{div}\,\boldsymbol{u} = 0\,, \tag{2.4}$$

under no-slip boundary condition [13]. In equation 2.4, ρ_1 is the fluid density, p is the pressure, ν is the kinematic viscosity and $\boldsymbol{u}$ is the fluid velocity.

3. Formulation of problem

We consider the unsteady flow of an incompressible, viscous fluid when an axially symmetric body of finite size is vibrating longitudinally along its axis of symmetry. It vibrates with velocity $U = U_0 e^{i\sigma t}$ about its center of inertia. Such a motion is governed by Navier-Stokes equations:

$$\rho_1\left(\left(\mathbf{v}.\nabla\right)\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t}\right) = -\mathrm{grad}\,p + \mu\nabla^2\boldsymbol{u}\,, \tag{3.1a}$$

$$\nabla.\boldsymbol{u} = 0\,. \tag{3.1b}$$

Following the inner and outer series expansion method developed by Proudman and Pearson [5], Kanwal [4] provided the general expression of drag on axially symmetric bodies vibrating slowly longitudinally along the axis of symmetry(x-axis) with no-slip boundary conditions as

$$\frac{D}{D_x} = \left[1 + \frac{D_x}{6\sqrt{2}\,\pi\mu U_0 a}(1+i)M + O(M^2)\right]e^{i\sigma t}\,, \tag{3.2}$$

where D_x is Stokes drag on an axially symmetric body moving parallel to its axis with velocity U_0 through an unbounded fluid, 'a' is any characteristic particle dimension and $M^2 = \dfrac{a^2 \sigma \rho_1}{\mu}$ is a non-dimensional number with σ as circular frequency. The part

$$\frac{D_f}{D_x} = \left[1 + \frac{D_x}{6\sqrt{2}\ \pi\mu U_0 a} M + O\left(M^2\right) \right] e^{i\sigma t}. \tag{3.3}$$

Gives the frictional force while the other part

$$\frac{D}{D_x} = i\left[1 + \frac{D_x}{6\sqrt{2}\ \pi\mu U_0 a} M + O\left(M^2\right) \right] e^{i\sigma t}, \tag{3.4}$$

is $\pi / 2$ out of phase with the velocity of the body and describes the virtual mass of the surrounding fluid associated with the motion [4].

4. Method.

We provide the method to find frictional drag on same body vibrating with velocity $U = U_0 e^{i\sigma t}$ about its center of inertia as the mean position with U_0 as the transverse uniform flow (perpendicular to the axis of symmetry) velocity and σ is circular frequency. Such a motion is governed by Navier-Stokes equations (3.1a, b). Let D_x and D_y be the axial and transverse Stokes drag on the axially symmetric particle described in Fig.1, obtained as the solution of linear Stokes equations (2.3) under no-slip boundary conditions. Datta and Srivastava [7] gave the expressions of drag D_x and D_y (Eqs (2.1)-(2.2)). The frictional drag coefficient for a slowly vibrating body placed in an axial flow stream, as in Fig.1, D_f / D_x is given by Eq.(3.3) [4].
Then, we can have the transverse frictional drag coefficient D_f / D_x as

$$\frac{D_f}{D_y} = K \frac{D_f}{D_x} \tag{4.1}$$

where 'K' is the real factor based on geometric variables 'R' and 'α' (see Fig.1) is defined in Eq.(2.3). Equation (4.1) indicates that the product of this real factor 'K' and axial frictional Stokes drag (3.3) on a slowly vibrating body with velocity $U = U_0 \exp(i\sigma t)$ gives the transverse frictional Stokes drag which was not covered earlier by Kanwal [4].

In the following section, we calculate transverse frictional drag on slowly oscillating bodies utilizing Eq.(3.3) first followed by Eq.(4.1) up to the order of M. Sphere, spheroids, deformed sphere, egg-shaped body, cycloidal bodies, Cassini oval and hypocycloidal bodies are considered as examples. These results are not available in the literature and are supposed to be new. In the following sections, we consider $U_0 = U_x = U_y$.

5. Sphere

Let us consider the parametric equation of sphere of revolution of circle about the x-axis having radius 'a' as

$$x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le \pi. \tag{5.1}$$

The axial and transverse Stokes drag on this sphere placed in a Newtonian fluid by using Eq.(2.1) and Eq.(2.2) with fact $b = a$, $U_x = U_0$, comes out to be

$$D_x = 6\pi\mu U_0 a, \tag{5.2a}$$

and

$$D_y = 6\pi\mu U_0 a. \tag{5.2b}$$

Then, from Eq.(2.3), on dividing the above two equations, we get the value of the real factor 'K' as

$$K = \frac{D_x}{D_y} = 1\,. \tag{5.3}$$

Now, the frictional drag on a sphere vibrating slowly longitudinally along the axis of symmetry (x-axis) with velocity $U = U_0 e^{i\sigma t}$ up to the order of M is given by using Eq.(5.2a)

$$D_f = D_x\left[1 + \frac{1}{\sqrt{2}}M\right]e^{i\sigma t} = 6\pi\mu U_0 a\left[1 + \frac{M}{\sqrt{2}}\right]e^{i\sigma t}\,,$$

or

$$\frac{D_f}{D_x} = \left[1 + \frac{M}{\sqrt{2}}\right], \qquad U = U_0 e^{i\sigma t}\,. \tag{5.4}$$

It is clear from the above expression of drag that frictional drag on a vibrating sphere is greater by $\frac{M}{\sqrt{2}}D_x$ than the classical Stokes drag on a sphere having a radius 'a' and moving slowly with uniform velocity U through the infinite fluid. This frictional drag D_f immediately reduces to the classical Stokes drag D_x as $M = 0$ and $t = 0$ (i.e. on considering the steady case and removing the oscillating effect). Further, upon using of Eq.(4.1), we can get the transverse frictional Stokes drag on a sphere as

$$\frac{D_f}{D_y} = K\frac{D_f}{D_x} = \left[1 + \frac{M}{\sqrt{2}}\right], \qquad U = U_0 e^{i\sigma t}\,. \tag{5.5}$$

This expression of frictional drag in the transverse flow case matches the frictional drag in the axial flow for slowly vibrating sphere along the axis of symmetry. This may be the reason why the sphere has fore and aft symmetry about mid of the body in both flows.

6. Prolate spheroid

Let us consider the prolate spheroid of revolution of ellipse about the x-axis. The parametric equation of ellipse is

$$x = a\cos t, \qquad y = b\sin t, \qquad 0 \le t \le \pi. \tag{6.1}$$

After careful calculation, the expressions of axial Stokes drag D_x on this prolate spheroid placed in a Newtonian fluid by using Eq.(2.1), assuming that where $U_x = U_0$, comes out to be

$$D_x = 16\pi\mu U_0 ae^3\left[\left(1+e^2\right)L-2e\right]^{-1} = 8\pi\mu U_0 a\delta_1 = \frac{4}{3}\left(6\pi\mu U_0 a\delta_1\right) \tag{6.2}$$

where

$$\delta_1 = 2e^3\left[\left(1+e^2\right)L-2e\right]^{-1} = \frac{3}{4}\left(1-\frac{2}{5}e^2-\frac{17}{175}e^4....\right). \tag{6.3}$$

In the above formulas, $L=\ln\left[\left(1+e\right)/\left(1-e\right)\right]$ with 'e' as eccentricity of ellipse. Further, the expressions of transverse Stokes drag on this prolate spheroid placed in a Newtonian fluid by using Eq.(2.2), when $U_y = U_0$, are as follows:

$$D_y = 32\pi\mu U_0 ae^3\left[2e+\left(3e^2-1\right)\ln\frac{1+e}{1-e}\right]^{-1} = 8\pi\mu U_0 a\delta_2 = \frac{4}{3}\left(6\pi\mu U_0 a\delta_2\right) \tag{6.4}$$

where

$$\delta_2 = 4e^3\left[\left(3e^2-1\right)L+2e\right]^{-1} = \frac{3}{4}\left(1-\frac{3}{10}e^2-\frac{285}{3500}e^4.....\right). \tag{6.5}$$

By using Eqs (2.3), (6.2) and (6.4), the value of the real factor 'K' is obtained as

$$K = \frac{1}{2}\left[2e+\left(3e^2-1\right)\ \ln\frac{1+e}{1-e}\right]\left[-2e+\left(1+e^2\right)\ \ln\frac{1+e}{1-e}\right]^{-1} = 1-\frac{1}{10}e^2-\frac{8}{175}e^4..... \tag{6.6}$$

Now, the axial frictional drag on prolate spheroid vibrating slowly longitudinally along this axis of symmetry (x-axis) with velocity $U=U_0e^{i\sigma t}$ up to the order of M is given by using Eq.(3.3) [4]

$$D_f = 8\pi\mu Ua\delta_1\left[1+\frac{2\sqrt{2}}{3}\delta_1 M\right], \qquad U=U_0e^{i\sigma t},$$

or

$$\frac{D_f}{8\pi\mu U_0 a\delta_1} = \left[1+\frac{1}{\sqrt{2}}\left(\frac{4}{3}\delta_1\right)M\right]e^{i\sigma t}. \tag{6.7}$$

It is clear from the above expression of drag that frictional drag on a vibrating prolate spheroid is greater by quantity $\frac{2\sqrt{2}\ \delta_1 M}{3}D_x$ than the classical Stokes drag on the prolate spheroid moving slowly with uniform velocity U through the infinite fluid. This frictional drag D_f immediately reduces to the classical Stokes drag $D_x = 8\pi\mu U_0 a\delta_1$ as $M=0$ and $t=0$ (i.e. on considering the steady case and removing the oscillating effect).

Now, by using Eq.(4.1) and (6.7), the transverse frictional drag on prolate spheroid can be written as

$$\begin{aligned}\frac{D_f}{D_y} &= \left(1-\frac{1}{10}e^2-\frac{8}{175}e^4.....\right)\left[1+\frac{1}{\sqrt{2}}\left(\frac{4}{3}\delta_1\right)M\right]e^{i\sigma t} = \\ &= \left(1-\frac{1}{10}e^2-\frac{8}{175}e^4.....\right)\left[1+\frac{1}{\sqrt{2}}\left(1-\frac{2}{5}e^2-\frac{17}{175}e^4.....\right)M\right]e^{i\sigma t} = \\ &= \left[\left(1-\frac{1}{10}e^2-\frac{8}{175}e^4.....\right)+\frac{1}{\sqrt{2}}\left(1-\frac{1}{2}e^2+\frac{9}{350}e^4.....\right)M\right]e^{i\sigma t},\end{aligned} \tag{6.8}$$

this tends to $\left[1+\frac{M}{\sqrt{2}}\right]e^{i\sigma t}$ as eccentricity e→0, the case of frictional drag of a vibrating sphere of radius '*a*' [4], and further approaches the classical Stokes drag $D = D_x = D_y$ [7] as $M = 0$ and $t = 0$ i.e. the case of motion of sphere without oscillation.

7. Oblate spheroid

Let us consider the oblate spheroid of revolution of ellipse about the x-axis. The parametric equation of ellipse is

$$x = b\cos t, \qquad y = a\sin t, \qquad 0 \le t \le \pi. \tag{7.1}$$

After careful calculation, the axial Stokes drag on this oblate spheroid placed in a Newtonian fluid, by using Eq.(2.2) when $U_x = U_0$, is obtained as

$$D_x = 8\pi\mu U_0 ae^3\left[e\sqrt{1-e^2} - \left(1-2e^2\right)\sin^{-1}e\right]^{-1} = 8\pi\mu U_0 a\beta_1 = \frac{4}{3}\left(6\pi\mu U_0 a\beta_1\right), \tag{7.2}$$

where

$$\beta_1 = e^3\left[e\sqrt{1-e^2}-\left(1-2e^2\right)\sin^{-1}e\right]^{-1} = \frac{3}{4}\left(1-\frac{1}{10}e^2-\frac{31}{1400}e^4.....\right). \tag{7.3}$$

Further, the expressions of transverse Stokes drag on this oblate spheroid placed in a Newtonian fluid by using Eq.(2.2), when $U_y = U_0$, are:

$$\begin{aligned}D_y &= 16\pi\mu U_0 ae^3\left[-e\sqrt{1-e^2}+\left(1+2e^2\right)\sin^{-1}e\right]^{-1} = \\ &= 16\pi\mu U_0 a\beta_2 = \frac{8}{3}\left(6\pi\mu U_0 a\beta_2\right),\end{aligned} \tag{7.4}$$

where

$$\beta_2 = e^3\left[-e\sqrt{1-e^2}+\left(1+2e^2\right)\sin^{-1}e\right]^{-1} = \frac{3}{8}\left(1-\frac{1}{5}e^2-\frac{79}{1400}e^4+.....\right). \tag{7.5}$$

By using Eq.(2.3), (6.2) and (6.4), the value of the real factor 'K' is obtained as

$$K = \frac{D_x}{D_y} = \frac{1}{2}\left[-e\sqrt{1-e^2} + \left(1+2e^2\right)\sin^{-1} e\right]\left[e\sqrt{1-e^2} - \left(1-2e^2\right)\sin^{-1} e\right]^{-1} = \\ = 1 - \frac{7}{30}e^2 - \frac{199}{33600}e^4 ..... \tag{7.6}$$

Now, the frictional drag on the oblate spheroid vibrating slowly longitudinally along the axis of symmetry (x-axis) with velocity $U = U_0 e^{i\sigma t}$ up to the order of M is given by using Eq.(3.3)

$$D_f = 8\pi\mu U a \beta_1 \left[1 + \frac{2\sqrt{2}}{3}\beta_1 M\right], \qquad U = U_0 e^{i\sigma t}. \tag{7.7}$$

It is clear from the above expression of drag that frictional drag on the vibrating prolate spheroid is greater by quantity $\frac{2\sqrt{2}\,\beta_1 M}{3} D_x$ than the classical Stokes drag on the prolate spheroid moving slowly with uniform velocity U through the infinite fluid. This frictional drag D_f immediately reduces to the classical Stokes drag $D_x = 8\pi\mu U_0 a\beta_1$ as $M = 0$ and $t = 0$ (i.e. on considering the steady case and on removing the oscillating effect).
Now, by using Eq.(4.1) and (6.7), the transverse frictional drag on the oblate spheroid can be written as

$$\begin{aligned} \frac{D_f}{D_y} &= \left(1 - \frac{7}{30}e^2 - \frac{199}{33600}e^4 .....\right)\left[1 + \frac{1}{\sqrt{2}}\left(\frac{4}{3}\beta_1\right)M\right]e^{i\sigma t} = \\ &= \left(1 - \frac{7}{30}e^2 - \frac{199}{33600}e^4 .....\right)\left[1 + \frac{1}{\sqrt{2}}\left(1 - \frac{1}{10}e^2 - \frac{31}{1404}e^4 .....\right)M\right]e^{i\sigma t} = \\ &= \left[\left(1 - \frac{7}{30}e^2 - \frac{199}{33600}e^4 .....\right) + \frac{1}{\sqrt{2}}\left(1 - \frac{1}{3}e^2 + ...\right)M\right]e^{i\sigma t}, \end{aligned} \tag{7.8}$$

this tends to $\left[1 + \frac{M}{\sqrt{2}}\right]e^{i\sigma t}$ as eccentricity $e \to 0$, the frictional drag case of the vibrating sphere of radius 'a' [4], and further approaches the classical Stokes drag $D = D_x = D_y$ [7] as $M = 0$ and $t = 0$ i.e. the case of motion of the sphere without oscillation.

8. Prolate deformed axi-symmetric body

Let us consider the polar equation of the deformed sphere

$$r = a\left[1 + \varepsilon\, g_1(\theta) + \varepsilon^2\, g_2(\theta)\right], \qquad g_1(\theta) = -\sin^2\theta, \qquad g_2(\theta) = -\frac{3}{2}\cos^2\theta\,\sin^2\theta. \tag{8.1}$$

In this polar equation, 'e' is the deformation parameter. We consider the z-ρ plane with $z = r\cos\theta,\ r = r\sin\theta$, where the z-axis is the polar axis and the ρ-axis is the equatorial axis. The axial Stokes drag on this deformed sphere can be obtained by careful algebraic calculation from Eq.(2.1) and is [10]

$$D_z = 6\pi\mu U_0 a\left[1-\frac{4}{5}\varepsilon-\frac{2}{175}\varepsilon^2......\right] = 6\pi\mu U_0 a\gamma_1(\varepsilon) \tag{8.2}$$

where

$$\gamma_1(\varepsilon) = \left[1-\frac{4}{5}\varepsilon+\frac{2}{175}\varepsilon^2......\right]. \tag{8.3}$$

Also, the transverse Stokes drag on this deformed sphere can be obtained by careful algebraic calculation from Eq.(2.2) and is [10]

$$D_\rho = 6\pi\mu U_0 a\left[1-\frac{3}{5}\varepsilon-\frac{9}{350}\varepsilon^2......\right] = 6\pi\mu U_0 a\alpha_1(\varepsilon) \tag{8.4}$$

where

$$\alpha_1(\varepsilon) = \left[1-\frac{3}{5}\varepsilon-\frac{9}{350}\varepsilon^2......\right]. \tag{8.5}$$

Let 'a' be the semi-polar axis length and $b = a(1-e)$ be the semi-equatorial axis length of the prolate deformed spheroid, then the eccentricity $e = \sqrt{1-\left(\frac{b}{a}\right)^2}$, with $b = a(1-\varepsilon)$, which further gives us the relation between the eccentricity 'e' and deformation parameter 'e' as

$$e = \sqrt{1-(1-\varepsilon)^2} = \sqrt{2\varepsilon-\varepsilon^2} = \sqrt{2\varepsilon} \quad \text{(leaving the square term).} \tag{8.6}$$

By using relationship (8.6), the expression of axial Stokes drag (8.2) and transverse Stokes drag (8.4) may be written in terms of eccentricity as [10]

$$D_z = 6\pi\mu U_0 a\left[1-\frac{2}{5}e^2-\frac{17}{175}e^4......\right] = 6\pi\mu U_0 a\gamma_2(e) \tag{8.7}$$

where

$$\gamma_2(e) = \left[1-\frac{2}{5}e^2-\frac{17}{175}e^4......\right], \tag{8.8}$$

and

$$D_{\rho} = 6\pi\mu U_0 a\left[1 - \frac{3}{10}e^2 - \frac{285}{3500}e^4 ......\right] = 6\pi\mu U_0 a\alpha_2(e) \quad (8.9)$$

where

$$\alpha_2(e) = \left[1 - \frac{3}{10}e^2 - \frac{285}{3500}e^4 ......\right]. \quad (8.10)$$

By using Eqs (2.3), (8.2) and (8.4), the value of the real factor '$K(e)$' is obtained as

$$K(\varepsilon) = \frac{D_z}{D_{\rho}} = \frac{\gamma_1(\varepsilon)}{\alpha_1(\varepsilon)} = \frac{1}{2}\left[1 - \frac{4}{5}\varepsilon + \frac{2}{175}\varepsilon^2 ......\right]\left[1 - \frac{3}{5}\varepsilon - \frac{9}{350}\varepsilon^2 ......\right]^{-1} = = 1 - \frac{1}{5}\varepsilon - \frac{29}{350}\varepsilon^2 ..... \quad (8.11)$$

Similarly, by using Eqs (2.3), (8.7) and (8.9), the value of the real factor '$K(e)$' is

$$K(e) = \frac{D_z}{D_{\rho}} = \frac{\gamma_2(e)}{\alpha_2(e)} = \frac{1}{2}\left[1 - \frac{2}{5}e^2 - \frac{17}{175}e^4 ......\right]\left[1 - \frac{3}{10}e^2 - \frac{285}{3500}e^4 ......\right]^{-1} = = 1 - \frac{1}{10}e^2 - \frac{8}{175}e^4 ..... \quad (8.12)$$

This results of drag (8.2 and 8.4) may directly be obtained from closed form solution of drag on the prolate spheroid from Eq.(6.2) and (6.4) with the use of relationship (8.6). Now, the frictional drag on the deformed prolate spheroid vibrating slowly longitudinally along the axis of symmetry (polar axis) with velocity $U = U_0 e^{i\sigma t}$ up to the order of M is given by (using Eqs (3.3), (8.2) and (8.7))

$$\frac{D_f}{D_z} = \frac{D_f}{6\pi\mu U_0 a\gamma_1(\varepsilon)} = \left[1 + \frac{\gamma_1(\varepsilon)}{\sqrt{2}}M\right]e^{i\sigma t} \quad (8.13)$$

and

$$\frac{D_f}{D_z} = \frac{D_f}{6\pi\mu U_0 a\gamma_2(e)} = \left[1 + \frac{\gamma_2(e)}{\sqrt{2}}M\right]e^{i\sigma t} \quad (8.14)$$

where $\gamma_1(\varepsilon)$ and $\gamma_2(e)$ are defined in (8.3) and (8.8). It is clear from the above expressions of drag (8.13) and (8.14) that frictional drag on the vibrating prolate spheroid is greater $\frac{\gamma_1(\varepsilon)M}{\sqrt{2}}D_z$ and $\frac{\gamma_2(e)M}{\sqrt{2}}D_z$ respectively than the classical Stokes drag on the prolate spheroid moving slowly with uniform velocity U through the infinite fluid. This frictional drag D_f immediately reduces to the classical Stokes drag $D_z = 6\pi\mu U_0 a\gamma_1(\varepsilon)$ and $6\pi\mu U_0 a\gamma_2(e)$ as $M = 0$ and $t = 0$ (i.e. on considering the steady case and removing the oscillating effect), which further reduces to the classical Stokes drag on sphere viz., $D_z = 6\,pmUa$ in both form as eccentricity $e \to 0$ or deformation parameter $e \to 0$.

Now, by using Eqs (4.1), (8.11) and (8.13), the transverse frictional drag on the oblate spheroid can be written in terms of the deformation parameter '*e*' as

$$\frac{D_f}{D_\rho}=\left(1-\frac{1}{5}\varepsilon-\frac{29}{350}\varepsilon^2.....\right)\left[1+\frac{\gamma_1(\varepsilon)}{\sqrt{2}}M\right]e^{i\sigma t}=$$
$$=\left(1-\frac{1}{5}\varepsilon-\frac{29}{350}\varepsilon^2.....\right)\left[1+\frac{1}{\sqrt{2}}\left(1-\frac{4}{5}\varepsilon+\frac{2}{175}\varepsilon^2......\right)M\right]e^{i\sigma t}= \tag{8.15}$$
$$=\left[\left(1-\frac{1}{5}\varepsilon-\frac{29}{350}\varepsilon^2.....\right)+\frac{1}{\sqrt{2}}\left(1-\varepsilon+\frac{31}{350}\varepsilon^2.....\right)M\right]e^{i\sigma t},$$

which tends to $\left[1+\frac{M}{\sqrt{2}}\right]e^{i\sigma t}$ as the deformation parameter $\varepsilon'\to 0$, the frictional drag case of vibrating sphere of radius '*a*' [4], and further approaches the classical Stokes drag $D=D_z=D_\rho$ [7] as $M=0$ and $t=0,$ i.e. the case of motion of the sphere without oscillation.

Similarly, by using Eqs (4.1), (8.12) and (8.14), the transverse frictional drag on the oblate spheroid can be written in terms of eccentricity '*e*' as

$$\frac{D_f}{D_\rho}=\left(1-\frac{1}{10}e^2-\frac{8}{175}e^4.....\right)\left[1+\frac{\gamma_2(e)}{\sqrt{2}}M\right]e^{i\sigma t}=$$
$$=\left(1-\frac{1}{10}e^2-\frac{8}{175}e^4.....\right)\left[1+\frac{1}{\sqrt{2}}\left(1-\frac{2}{5}e^2-\frac{17}{175}e^4......\right)M\right]e^{i\sigma t}= \tag{8.16}$$
$$=\left[\left(1-\frac{1}{10}e^2-\frac{8}{175}e^4.....\right)+\frac{1}{\sqrt{2}}\left(1-\frac{1}{2}e^2-\frac{18}{175}e^4.....\right)M\right]e^{i\sigma t},$$

which tends to $\left[1+\frac{M}{\sqrt{2}}\right]e^{i\sigma t}$ as the eccentricity $e\to 0$, the frictional drag case of vibrating sphere of radius '*a*' [4], and further approaches the classical Stokes drag $D=D_z=D_\rho$ [7] as $M=0$ and $t=0,$ i.e. the case of motion of sphere without oscillation.

Particular case. **Slender elongated prolate spheroid(needle shaped body)**

The eccentricity '*e*' of a prolate spheroid is

$$e=\sqrt{1-\frac{b^2}{a^2}}\ ,\quad b=a(1-\varepsilon),$$

ε is smal and positive,

$$e=\sqrt{1-(1-\varepsilon)^2}\ ,\qquad \varepsilon=1-\sqrt{1-e^2}\ . \tag{8.17}$$

The eccentricity 'e' of a slender elongated prolate spheroid of radius 'a' is unity, whence by (8.17), the value of the deformation parameter 'e' is also unity. For this case, the expressions for the axial drag(8.2) and (8.7) due to Stokes is

$$\frac{D_z}{6\pi\mu U_0\ a} = \gamma_1(\varepsilon) = 0.2114, \quad \text{up to } O\left(e^2\right), \tag{8.18}$$

$$\frac{D_z}{6\pi\mu U_0 a} = \gamma_2(e) = 0.5028, \quad \text{up to } O\left(e^4\right). \tag{8.19}$$

The numerical value of the real factor $K(\varepsilon)$ can be written as (on taking $\varepsilon = 1$), first by using (8.3) and (8.5)

$$K(\varepsilon) = \frac{\gamma_1(\varepsilon)}{\alpha_1(\varepsilon)} = 1 - \frac{1}{5}\varepsilon - \frac{29}{350}\varepsilon^2 ..... = 0.71714286\ , \tag{8.20}$$

and then by using (8.8) and (8.10)

$$K(e) = \frac{\gamma_2(e)}{\alpha_2(e)} = 1 - \frac{1}{10}e^2 - \frac{8}{175}e^4 ..... = 0.854285\,. \tag{8.21}$$

Now, the frictional drag on the slender deformed prolate spheroid vibrating slowly longitudinally along the axis of symmetry (polar axis) with velocity $U = U_0 e^{i\sigma t}$ up to the order of M is given by using Eqs (3.3), (8.2) and Eq.(8.7) with the use of numerical values of $\gamma_1 = 0.2114$ and $\gamma_2 = 0.5028$

$$\begin{gathered}\frac{D_f}{6\pi\mu U a} = \gamma_1\left(1 + \frac{\gamma_1}{\sqrt{2}}M\right) = (0.2114)[1 + 0.1495M] = 0.2114 + 0.03160M, \\ U = U_0 e^{i\sigma t},\end{gathered} \tag{8.22}$$

and

$$\begin{gathered}\frac{D_f}{6\pi\mu U a} = \gamma_2\left(1 + \frac{\gamma_2}{\sqrt{2}}M\right) = (0.5028)[1 + 0.3555M] = 0.5028 + 0.17878M, \\ U = U_0 e^{i\sigma t}.\end{gathered} \tag{8.23}$$

These two forms of frictional drag immediately reduce to the classical frictional drag on the prolate spheroid $6\pi\mu U_0 a\gamma_1$ and $6\pi\mu U_0 a\gamma_2$ as $M \to 0,\ t = 0$ which further reduce to the classical Stokes drag on the sphere of radius a, i.e. $6\pi\mu U_0 a$ or $\varepsilon \to 0$ or $e \to 0$.

Now, by using Eqs (4.1), (8.20) and (8.22), the transverse frictional drag on the oblate spheroid can be written in terms of the deformation parameter 'ε' as

$$\frac{D_f}{D_\rho} = K(\varepsilon)\left[1 + \frac{\gamma_1(\varepsilon)}{\sqrt{2}} M\right] e^{i\sigma t} = 0.71714286[1 + 0.1495M]e^{i\sigma t} = \\ = [0.71714286 + 0.10721285757M]e^{i\sigma t}, \tag{8.24}$$

which tends to 0.71714286 as $M = 0$ and $t = 0$ i.e. the case of motion of the deformed prolate spheroid(needle shaped body) without oscillation.

Similarly, by using Eqs (4.1), (8.21) and (8.23), the transverse frictional drag on the oblate spheroid can be written in terms of the eccentricity 'e' as

$$\frac{D_f}{D_\rho} = K(e)\left[1 + \frac{\gamma_2(e)}{\sqrt{2}} M\right] e^{i\sigma t} = 0.854285[1 + 0.3555M]e^{i\sigma t} = \\ = [0.854285 + 0.303698317M]e^{i\sigma t}, \tag{8.25}$$

which tends to 0.854285 as $M = 0$ and $t = 0$, i.e. the case of motion of the deformed prolate spheroid (needle shaped body) without oscillation.

9. Oblate deformed axi-symmetric body

Let us consider the polar equation of the deformed sphere

$$r = a\left[1 + \varepsilon' f_1(\theta) + \varepsilon'^2 f_2(\theta)\right], \quad f_1(\theta) = -\cos^2\theta, \quad f_2(\theta) = -\frac{3}{2}\cos^2\theta \sin^2\theta. \tag{9.1}$$

In this polar equation, 'ε'' is the deformation parameter. We consider the z-ρ plane with $z = r\cos\theta$, $r = r\sin\theta$, where the z-axis is the polar axis and the ρ-axis is the equatorial axis. The axial Stokes drag on this deformed sphere can be obtained by careful algebraic calculation from Eq.(2.1) and is [9]

$$D_z = 6\pi\mu U_0 a\left[1 - \frac{1}{5}\varepsilon' + \frac{2}{175}\varepsilon'^2 ......\right] = 6\pi\mu U_0 a \xi_1(\varepsilon) \tag{9.2}$$

where

$$\xi_1(\varepsilon') = \left[1 - \frac{1}{5}\varepsilon' + \frac{2}{175}\varepsilon'^2 ......\right]. \tag{9.3}$$

Also, the transverse Stokes drag on this deformed sphere can be obtained by careful algebraic calculation from Eq.(2.2) and is a follows [9]

$$D_\rho = 6\pi\mu U_0 a\left[1 - \frac{2}{5}\varepsilon' - \frac{9}{350}\varepsilon'^2 ......\right] = 6\pi\mu U_0 a \zeta_1(\varepsilon') \tag{8.4}$$

where

$$\zeta_1(\varepsilon')=\left[1-\frac{2}{5}\varepsilon'-\frac{9}{350}\varepsilon'^2......\right]. \tag{9.5}$$

Let ‘a’ be the semi-polar axis length and $b=a(1-\varepsilon')$ be the semi-equatorial axis length of the prolate deformed spheroid, then the eccentricity $e=\sqrt{1-\left(\frac{b}{a}\right)^2}$, with $b=a(1-\varepsilon')$, which further gives us the relation between the eccentricity ‘e’ and deformation parameter ‘ε'’ as

$$e=\sqrt{1-(1-\varepsilon')^2}=\sqrt{2\varepsilon'-\varepsilon'^2}=\sqrt{2\varepsilon'}\ \text{(leaving the square term).} \tag{9.6}$$

By using relationship (8.6), the expression of the axial Stokes drag (9.2) and transverse Stokes drag (9.4) may be written in terms of eccentricity as [9]

$$D_z=6\pi\mu U_0 a\left[1-\frac{1}{10}e^2-\frac{31}{1400}e^4......\right]=6\pi\mu U_0 a\xi_2(e) \tag{9.7}$$

where

$$\xi_2(e)=\left[1-\frac{1}{10}e^2-\frac{31}{1400}e^4......\right], \tag{9.8}$$

and [9]

$$D_\rho=6\pi\mu U_0 a\left[1-\frac{1}{5}e^2-\frac{79}{1400}e^4......\right]=6\pi\mu U_0 a\zeta_2(e) \tag{9.9}$$

where

$$\zeta_2(e)=\left[1-\frac{1}{5}e^2-\frac{79}{1400}e^4......\right]. \tag{9.10}$$

By using Eqs (2.3), (9.2) and (9.4), the value of the real factor ‘$K(\varepsilon')$’ is obtained as

$$\begin{aligned}K(\varepsilon')&=\frac{D_z}{D_\rho}=\frac{\xi_1(\varepsilon')}{\zeta_1(\varepsilon')}=\frac{1}{2}\left[1-\frac{1}{5}\varepsilon'+\frac{2}{175}\varepsilon'^2......\right]\left[1-\frac{2}{5}\varepsilon'-\frac{9}{350}\varepsilon'^2......\right]^{-1}=\\&=1+\frac{1}{5}\varepsilon'+\frac{41}{350}\varepsilon'^2.....\end{aligned} \tag{9.11}$$

Similarly, by using Eqs (2.3), (9.7) and (9.9), the value of the real factor ‘$K(e)$’ is obtained as

$$K(e)=\frac{D_z}{D_\rho}=\frac{\xi_2(e)}{\zeta_2(e)}=\frac{1}{2}\left[1-\frac{1}{10}e^2-\frac{31}{1400}e^4......\right]\left[1-\frac{1}{5}e^2-\frac{79}{1400}e^4......\right]^{-1}= \\ =1+\frac{1}{10}e^2+\frac{19}{350}e^4..... \tag{9.12}$$

Those results of drag (9.2) and (9.4) may directly be obtained from closed form solution of drag on the oblate spheroid from Eqs (7.2) and (7.4) with the use of relationship (9.6). Now, the frictional drag on the deformed prolate spheroid vibrating slowly longitudinally along the axis of symmetry (polar axis) with velocity $U=U_0e^{i\sigma t}$ up to the order of M is given by(using Eqs (3.3), (9.2) and (9.7)

$$\frac{D_f}{D_z}=\frac{D_f}{6\pi\mu U_0 a\xi_1(\varepsilon)}=\left[1+\frac{\xi_1(\varepsilon)}{\sqrt{2}}M\right]e^{i\sigma t} \tag{9.13}$$

and

$$\frac{D_f}{D_z}=\frac{D_f}{6\pi\mu U_0 a\xi_2(e)}=\left[1+\frac{\xi_2(e)}{\sqrt{2}}M\right]e^{i\sigma t} \tag{9.14}$$

where $\xi_1(\varepsilon)$ and $\xi_2(e)$ are defined in (9.3) and (9.8). It is clear from the above expressions of drag (9.13 and 8.14) that the frictional drag on the vibrating oblate spheroid is greater by quantity $\frac{\xi_1(\varepsilon)M}{\sqrt{2}}D_z$ and $\frac{\xi_2(e)M}{\sqrt{2}}D_z$ respectively the classical Stokes drag on the oblate spheroid moving slowly with uniform velocity U through the infinite fluid. This frictional drag D_f immediately reduces to the classical Stokes drag $D_z=6\pi\mu U_0 a\xi_2(\varepsilon)$ and $6\pi\mu U_0 a\xi_2(e)$ as $M=0$ and $t=0$ (i.e. on considering the steady case and removing the oscillating effect), which further reduces to the classical Stokes drag on the sphere, viz., $D_z=6\pi\mu Ua$ in both forms as the eccentricity $e\to 0$ or deformation parameter $e\to 0$.
Now, by using Eqs (4.1), (9.11) and (9.13), the transverse frictional drag on the oblate spheroid can be written in terms of the deformation parameter 'ε'' as

$$\frac{D_f}{D_\rho}=\left(1+\frac{1}{5}\varepsilon'+\frac{41}{350}\varepsilon'^2.....\right)\left[1+\frac{\xi_1(\varepsilon')}{\sqrt{2}}M\right]e^{i\sigma t}= \\ =1+\frac{1}{5}\varepsilon'+\frac{41}{350}\varepsilon'^2.....\left[1+\frac{1}{\sqrt{2}}\left(1-\frac{1}{5}\varepsilon'+\frac{2}{175}\varepsilon'^2......\right)M\right]e^{i\sigma t}= \\ =\left[\left(1+\frac{1}{5}\varepsilon'+\frac{41}{350}\varepsilon'^2.....\right)+\frac{1}{\sqrt{2}}\left(1+\frac{31}{350}\varepsilon'.....\right)M\right]e^{i\sigma t}, \tag{9.15}$$

which tends to $\left[1+\frac{M}{\sqrt{2}}\right]e^{i\sigma t}$ as the deformation parameter $\varepsilon'\to 0$, the frictional drag case of vibrating sphere of radius 'a' [4], and further approaches the classical Stokes drag $D=D_z=D_\rho$ [7] as $M=0$ and $t=0$ i.e. the case of motion of sphere without oscillation.

Similarly, by using Eqs (4.1), (9.12) and (9.14), the transverse frictional drag on the oblate deformed spheroid can be written in terms of the eccentricity ‘e’ as

$$\frac{D_f}{D_\rho}=\left(1+\frac{1}{10}e^2+\frac{19}{350}e^4.....\right)\left[1+\frac{\xi_2(e)}{\sqrt{2}}M\right]e^{i\sigma t}=$$
$$=\left(1+\frac{1}{10}e^2+\frac{19}{350}e^4.....\right)\left[1+\frac{1}{\sqrt{2}}\left(1-\frac{1}{10}e^2-\frac{31}{1400}e^4......\right)M\right]e^{i\sigma t}= \tag{9.15}$$
$$=\left[\left(1+\frac{1}{10}e^2+\frac{19}{350}e^4.....\right)+\frac{1}{\sqrt{2}}\left(1+\frac{31}{1400}e^4.....\right)M\right]e^{i\sigma t},$$

which tends to $\left[1+\frac{M}{\sqrt{2}}\right]e^{i\sigma t}$ as the eccentricity e→0, the frictional drag case of vibrating sphere of radius ‘*a*’ [4], and further approaches the classical Stokes drag $D=D_z=D_\rho$ [7] as $M=0$ and $t=0$, i.e. the case of motion of the sphere without oscillation.

Particular case. **Flat circular disk (broadside on case)**

The eccentricity ‘*e*’ of a prolate spheroid is

$$e=\sqrt{1-\frac{b^2}{a^2}}\ ,\quad b=a(1-\varepsilon'),$$

ε' is small and positive

$$e=\sqrt{1-(1-\varepsilon')^2},\qquad \varepsilon'=1-\sqrt{1-e^2}\ . \tag{9.16}$$

The eccentricity ‘*e*’ of a flat circular disk of radius ‘a’ is unity, whence by (9.16), the value of the deformation parameter ‘ε'’ is also unity. For this case, the expressions for the axial drag (9.2), (9.7) due to Stokes give

$$\frac{D_z}{6\pi\mu U_0 a}=\xi_1(\varepsilon)=0.8114,\ ,\qquad (\text{up to } O(e^2)), \tag{9.17}$$

$$\frac{D_z}{6\pi\mu U_0 a}=\xi_2(e)=0.8842\ ,\qquad (\text{up to } O(e^4)). \tag{9.18}$$

The numerical value of the real factor $K(\varepsilon')$ can be written as (on taking $\varepsilon'=1$), first by using (9.3) and (9.5)

$$K(\varepsilon')=\frac{\xi_1(\varepsilon')}{\zeta_1(\varepsilon')}=1+\frac{1}{5}\varepsilon+\frac{41}{350}\varepsilon'^2.....\cong 1.317142857142\ , \tag{9.19}$$

and then by using (9.8) and (9.10)

$$K(e)=\frac{\xi_2(e)}{\zeta_2(e)}=1+\frac{1}{10}e^2+\frac{19}{350}e^4.....\cong 1.154285714285\,. \tag{9.20}$$

Now, the frictional drag on the flat circular disk vibrating slowly longitudinally along the axis of symmetry (polar axis) with velocity $U=U_0e^{i\sigma t}$ up to the order of M is given by using Eqs (3.3), (9.2) and (9.7) with the use of numerical values of $\xi_1=0.8114$ and $\xi_2=0.8842$

$$\frac{D_f}{6\pi\mu U_0 a\xi_1(\varepsilon')}=\left(1+\frac{\xi_1(\varepsilon')}{\sqrt{2}}M\right)e^{i\sigma t} = \left[1+0.5738M\right]e^{i\sigma t}, \tag{9.21}$$

and

$$\frac{D_f}{6\pi\mu U_0 a\xi_2(e)}=\left(1+\frac{\xi_2(e)}{\sqrt{2}}M\right)e^{i\sigma t}=\left[1+0.6253M\right]e^{i\sigma t}. \tag{9.22}$$

These two forms of the frictional drag immediately reduce to the classical frictional drag on the oblate deformed spheroid $6\pi\mu U_0 a\xi_1(\varepsilon')$ and $6\pi\mu U_0 a\xi_2(e)$ as $M\to 0,\ t=0.$ The drag further reduces to the classical Stokes drag on the sphere of radius a, i.e. $6\pi\mu U_0 a$ either $\varepsilon'\to 0$ or $e\leftarrow 0.$

Now, by using Eqs (4.1), (9.19) and (9.21), the transverse frictional drag on the oblate deformed spheroid(flat circular disk) can be written in terms of the deformation parameter 'ε'' as

$$\begin{aligned}\frac{D_f}{D_\rho}&=K(\varepsilon')\left[1+\frac{\xi_1(\varepsilon')}{\sqrt{2}}M\right]e^{i\sigma t}=1.31714285\left[1+0.5738M\right]e^{i\sigma t}=\\&=\left[1.31714285+0.755776571428M\right]e^{i\sigma t},\end{aligned} \tag{9.23}$$

which tends to 1.31714285 as $M=0$ and $t=0$ i.e. the case of motion of the deformed oblate spheroid (flat circular disk) without oscillation.

Similarly, by using Eqs (4.1), (9.20) and (9.22), the transverse frictional drag on the oblate deformed spheroid (flat circular disk) can be written in terms of the eccentricity 'e' as

$$\begin{aligned}\frac{D_f}{D_\rho}&=K(e)\left[1+\frac{\xi_2(e)}{\sqrt{2}}M\right]e^{i\sigma t}=1.154285714285\left[1+0.6253M\right]e^{i\sigma t}=\\&=\left[1.154285714285+0.7217748571427M\right]e^{i\sigma t},\end{aligned} \tag{9.24}$$

which tends to 1.154285714285 as $M=0$ and $t=0,$ i.e. the case of motion of the deformed oblate spheroid (flat circular disk) without oscillation.

10. Egg-shaped body

Let us consider the parametric equation of the left half of the closed curve as a semi-circle with radius b as

$$x = b\cos t, \qquad y = b\sin t, \qquad 0 \le t \le \frac{\pi}{2} \tag{10.1a}$$

and the parametric equation of the right half of the closed curve as a semi-ellipse with the semi-major axis length a and semi-minor axis length b as

$$x = b\cos t, \qquad y = a\sin t, \qquad \frac{\pi}{2} \le t \le \pi. \tag{10.1b}$$

The axial Stokes drag on the egg-shaped body of revolution of closed curve with continuously turning tangent about the x-axis is given by utilizing Eq.(2.1)

$$D_x = 8\pi\mu U_0 a\sqrt{1-e^2}\left[\frac{2}{3} + \frac{\sqrt{1-e^2}}{4e^3}\left\{-2 + \left(1+e^2\right)\log\left(\frac{1+e}{1-e}\right)\right\}\right]^{-1} = 6\pi\mu U_0 a\tau_1(e), \tag{10.2}$$

where [7]

$$\tau_1(e) = \frac{4}{3}\sqrt{1-e^2}\left[\frac{2}{3} + \frac{\sqrt{1-e^2}}{4e^3}\left\{-2 + \left(1+e^2\right)\log\left(\frac{1+e}{1-e}\right)\right\}\right]^{-1}, \tag{10.3}$$

$$\tau_1(\mathrm{e}) = \left(1 + \frac{1}{20}\mathrm{e}^2 + \frac{51}{1400}\mathrm{e}^4 \ldots..\right). \tag{10.4}$$

The transverse Stokes drag on the egg-shaped body of revolution of closed curve with continuously turning tangent about the x-axis is given by using Eq.(2.2)

$$D_y = 16\pi\mu U_0 a\sqrt{1-e^2}\left[\frac{4}{3} + \frac{\sqrt{1-e^2}}{4e^3}\left\{2e + \left(3e^2-1\right)\log\left(\frac{1+e}{1-e}\right)\right\}\right]^{-1} = 6\pi\mu U_0 a\tau_2(e), \tag{10.5}$$

where [7]

$$\tau_2(e) = \frac{8}{3}\sqrt{1-e^2}\left[\frac{4}{3} + \frac{\sqrt{1-e^2}}{4e^3}\left\{2e + \left(3e^2-1\right)\log\left(\frac{1+e}{1-e}\right)\right\}\right]^{-1}, \tag{10.6}$$

$$\tau_2(e) = \left(1 - \frac{2}{5}e^2 - \frac{47}{2800}e^4 + \ldots..\right). \tag{10.7}$$

By using Eqs (2.3), (10.4) and (10.7), the value of the real factor ‘$K(e)$’ is obtained as

$$K(e)=\frac{D_x}{D_y}=\frac{\tau_1(e)}{\tau_2(e)}=\left[1+\frac{1}{20}e^2+\frac{51}{1400}e^4......\right]\left[1-\frac{2}{5}e^2-\frac{47}{2800}e^4......\right]^{-1}, \tag{10.8}$$

$$K(e)= 1+\frac{5}{12}e^2+\frac{733}{1680}e^4+..... \,. \tag{10.9}$$

Now, the frictional drag on the slowly vibrating egg-shaped body moving along the axis of symmetry (x-axis) is obtained from Eq.(3.3)

$$D_f=D_x\left[1+\frac{D_x}{6\sqrt{2}\pi\mu U_0 a}M\right]e^{i\sigma t}=6\pi\mu U_0 a\tau_1(e)[1+\frac{2\sqrt{2}}{3}\tau_1(e)M]e^{i\sigma t} \tag{10.10}$$

(up to order of M only).
This frictional drag immediately reduces to the classical axial Stokes drag on the egg-shaped body [7], viz., $8\pi\mu U_0 a\tau_1(e)$ as $M\to 0,\ t=0.$ It is interesting to note that the Stokes flow does not distinguish the fore and aft symmetry. That is why the drag will remain same on interchanging the left and right halves of the egg-shaped body.
Now, by using Eqs (4.1) and (10.10), the expression of the transverse frictional drag for the egg shaped body may be written as

$$\begin{aligned}&\frac{D_f}{D_y}=K\frac{D_f}{D_x}=\\&=\left(1+\frac{5}{12}e^2+\frac{733}{1680}e^4+.....\right)\left[1+\frac{1}{\sqrt{2}}\left(1-\frac{9}{20}e^2-\frac{159}{1400}e^4......\right)M\right]e^{i\sigma t}=\\&=\left[\left(1+\frac{5}{12}e^2+\frac{733}{1680}e^4+.....\right)+\frac{1}{\sqrt{2}}\left(1-\frac{1}{30}e^2+\frac{71}{525}e^4+....\right)M\right]e^{i\sigma t},\end{aligned} \tag{10.11}$$

which tends to $\left(1+\frac{M}{\sqrt{2}}\right)e^{i\sigma t}$ if the eccentricity $e\to 0$, the case of frictional drag on the oscillating sphere of radius ‘a’ and immediately further reduces to the classical Stokes drag on the sphere $D=D_x=D_y=6\pi\mu U_0 a$ as $M=0,\ t=0,$ i.e. on removing the oscillating effect.

11. Cycloidal body of revolution

Case-1

Let us take the inverted cycloid

$$x=a(t+\sin t),\qquad y=a(1+\cos t),\ -\pi<t<\pi, \tag{11.1}$$

with vertex at $(0,\ 2a)$, and revolve it about the x-axis, the base, to generate the cycloidal body of revolution. The axial Stokes drag on such cycloidal body of revolution can be obtained from Eq.(2.1) [7]

$$D_x = \frac{448}{33}\pi\mu U_0 a = \left(\frac{56}{33}\right)\left(8\pi\mu U_0 a\right) = \frac{224}{99}\left(6\pi\mu U_0 a\right) \cong 13.5757\pi\mu U_0 a\,. \tag{11.2}$$

The transverse Stokes drag on such cycloidal body of revolution can be obtained from Eq.(2.2) [7]

$$D_y = \frac{896}{55}\pi\mu U_0 a = \left(\frac{112}{55}\right)\left(8\pi\mu U_0 a\right) = \frac{448}{165}\left(6\pi\mu U_0 a\right) \cong 16.29090\pi\mu U_0 a\,. \tag{11.3}$$

Now, by using Eq.(2.3), the numerical value of the real factor K can be evaluated as

$$K = \frac{D_x}{D_y} = \frac{\frac{128}{3}\mu U_0 a}{\frac{256}{5}\mu U_0 a} = \frac{5}{6} \cong 0.8333\,. \tag{11.4}$$

Now, the frictional drag on the cycloidal body vibrating slowly along the axis of symmetry(x-axis) is obtained from Eq.(3.3)

$$\begin{gathered} D_f = \left(\frac{56}{33}\right)\left(8\pi\mu U_0 a\right)\left[1 + \frac{112\sqrt{2}}{99}M\right] = \left(\frac{224}{99}\right)\left(6\pi\mu U_0 a\right)\left[1 + \frac{112\sqrt{2}}{99}M\right] \cong \\ \cong 13.5757\pi\mu U_0 a\left[1 + 1.6M\right]e^{i\sigma t} \cong 13.5757\pi\mu U a\left[1 + 1.6M\right], \qquad U = U_0 e^{i\sigma t}, \end{gathered} \tag{11.5}$$

(up to order of M only).

This frictional drag immediately reduces to the classical Stokes drag $D_x = \frac{448}{33}\pi\mu U_0 a$, as $M \to 0,\ t = 0.$

The transverse frictional drag is given by using Eqs (4.1) and (11.5) as

$$\frac{D_f}{D_y} = K\frac{D_f}{D_x} = 0.8333\left[1 + 1.6M\right]e^{i\sigma t} = \left[0.8333 + 1.3333M\right]e^{i\sigma t}\,, \tag{11.6}$$

which tends to 0.8333, the classical Stokes drag on the cycloidal body as $M = 0,\ t = 0,$ i.e. without oscillating effect.

Case-II

Let us consider the body generated by rotation about the x-axis of the curve composed of arcs of two cycloidal parts represented parametrically by

$$x = a\left(1 + \cos t\right), \qquad y = a\left(t + \sin t\right), \qquad 0 \le t \le \pi, \tag{11.7a}$$

and

$$x = -a\left(1 + \cos t\right), \qquad y = a\left(t + \sin t\right), \qquad 0 \le t \le \pi. \tag{11.7b}$$

The axial Stokes drag on this axi-symmetric body may be obtained by Eq.(2.1) [7]

$$D_x = \left(\frac{96\pi^2}{3\pi^2 + 16} \right) \pi\mu U_0 a = \left(\frac{16\pi^2}{3\pi^2 + 16} \right) 6\pi\mu U_0 a \cong 20.779964 \pi\mu U_0 a \,. \tag{11.8}$$

The transverse Stokes drag on such cycloidal body of revolution can be obtained from Eq.(2.2) [7]

$$D_y = \left(\frac{192\pi^2}{9\pi^2 + 32} \right) \pi\mu U_0 a = \frac{32\pi^2}{\left(9\pi^2 + 32\right)} \left(6\pi\mu U_0 a\right) \,. \tag{11.9}$$

Now, by using Eq.(2.3), the numerical value of the real factor K can be evaluated as

$$K = \frac{D_x}{D_y} = \frac{1}{2} \left(\frac{9\pi^2 + 32}{3\pi^2 + 16} \right) \cong 1.3244 \,. \tag{11.10}$$

Now, the frictional drag on the cycloidal body (11.7a,b) vibrating slowly along the axis of symmetry(x-axis) is obtained from Eq.(3.3)

$$\begin{aligned} D_f &= \left(\frac{16\pi^2}{3\pi^2 + 16} \right) 6\pi\mu U_0 a \left[1 + \left(\frac{\left(8\sqrt{2}\right)\pi^2}{3\pi^2 + 16} \right) M \right] e^{i\sigma t} \cong \\ &\cong 20.779964 \pi\mu U_0 a \left[1 + \frac{20.779964}{6\sqrt{2}} M \right] e^{i\sigma t} \cong \\ &\cong 20.779964 \pi\mu U a \left[1 + 2.448572 M \right], \qquad U = U_0 e^{i\sigma t} \end{aligned} \tag{11.11}$$

(up to order of M only).

This frictional drag immediately reduces to the classical Stokes drag $D_x = \left(\frac{96\pi^2}{3\pi^2 + 16} \right) \pi\mu U_0 a$, as $M \to 0,\ t = 0.$

The transverse frictional drag is given by using Eqs (4.1) and (11.11) as

$$\frac{D_f}{D_y} = K \frac{D_f}{D_x} = 1.3244 \left[1 + 2.448572 M \right] e^{i\sigma t} = \left[1.3244 + 3.242888 M \right] e^{i\sigma t} \,, \tag{11.6}$$

which tends to 1.3244, the classical Stokes drag on the cycloidal body as $M = 0,\ t = 0,$ i.e. without oscillating effect.

12. Cassini oval

Let us consider a Cassini body(oval) of revolution of curve

$$y^2 = \frac{2}{3} \sqrt{1 + 3x^2} - x^2 - \frac{1}{3}, \qquad 0 \le x \le 1, \tag{12.1}$$

about the axis of symmetry(x-axis). For this axi-symmetric body, we take $x_{max} = a = 1.0$ units and $y_{max} = b = 0.577$ units. Then, the axial Stokes drag on the Cassini oval can be calculated by eq. (2.1) [8]

$$D_x \cong 0.8\pi\mu U_0 \, . \tag{12.2}$$

The transverse Stokes drag on the Cassini oval can be calculated by Eq.(2.2) [8]

$$D_y \cong 0.82\pi\mu U_0 \, . \tag{12.3}$$

The value of the real factor K can be evaluated by Eqs (2.3), (12.2) and (12.3)

$$K = \frac{D_x}{D_y} \cong 0.9756078 \, . \tag{12.4}$$

Now, the frictional drag on the slowly vibrating Cassini oval along the axis of symmetry is obtained from Eq.(3.3)

$$\frac{D_f}{D_x} \cong \left[1 + \frac{\sqrt{2}}{15} M \right] e^{i\sigma t} \text{, (up to order of M only).} \tag{12.5}$$

This frictional drag immediately reduces to the classical Stokes drag, $D_x \cong 0.8\pi\mu U_0$ as $M \to 0,\ t = 0$ i.e. the case of drag without oscillation.
Now, the transverse frictional drag on the Cassini oval is obtained from Eq.(4.1) and (12.5) as

$$\frac{D_f}{D_y} = K \frac{D_f}{D_x} = 0.9756098 [1 + 0.094266M] e^{i\sigma t} = \\ = [0.9756098 + 0.091966833M] e^{i\sigma t} , \tag{12.6}$$

which tends to 0.9756098, the classical Stokes drag $D = 0.9756098$, $D_y = D_x = 0.8pU_0m$ on the cycloidal body as $M = 0,\ t = 0,$ i.e. without oscillating effect.

12. Hypocycloidal body

Let us consider the hypocycloidal body obtained by revolving curve about the axis of symmetry (x-axis)

$$y^2 = -3x^2 + \sqrt{\left(1 + 8x^4\right)} \, , \qquad 0 \le x \le 1 \, . \tag{12.1}$$

For this axi-symmetric body, we take $x_{max} = a = 1.0$ units and $y_{max} = b = 1.0$ units. Then, the axial Stokes drag on the hypocycloidal body can be calculated by Eq.(2.1) [8]

$$D_x \cong 1.044\lambda \cong 6.264\pi\mu U_0 \times 1, \qquad \lambda = 6\pi\mu U_0 \, . \tag{12.2}$$

Similarly, the transverse Stokes drag on the hypocycloidal body can be calculated by Eq.(2.2) and is [8]:

$$D_y \cong 1.2\lambda \cong 7.92\pi\mu U_0 \times 1, \qquad \lambda = 6\pi\mu U_0 . \tag{12.3}$$

The value of the real factor K can be calculated by Eqs (2.3), (12.2) and (12.3)

$$K = \frac{D_x}{D_y} \cong 0.9756078 . \tag{12.4}$$

Now, the frictional drag on the slowly vibrating hypocycloidal body along the axis of symmetry is obtained from Eq.(3.3)

$$\begin{aligned} \mathrm{D_f} &\cong 6.264\pi\mu \mathrm{U_0}\left[1+\frac{6.264}{6\sqrt{2}}\mathrm{M}\right]\mathrm{e}^{\mathrm{i}\sigma \mathrm{t}} \cong 0.783\left(8\pi\mu \mathrm{U_0}\times 1\right)\left[1+0.73833\mathrm{M}\right]\mathrm{e}^{\mathrm{i}\sigma \mathrm{t}} \cong \\ &\cong 1.044\left(6\pi\mu \mathrm{U_0}\times 1\right)\left[1+0.73833\mathrm{M}\right]\mathrm{e}^{\mathrm{i}\sigma \mathrm{t}}, \end{aligned} \tag{12.5}$$

(up to order of M only).

This frictional drag immediately reduces to the classical Stokes drag, $D_x \cong 6.264\pi\mu U_0$ as $M \to 0,\ t = 0.$

Now, the transverse frictional drag on the Cassini oval is obtained from Eq.(4.1), (12.4) and (12.5) as

$$\begin{aligned} \frac{D_f}{D_y} &= K\frac{D_f}{D_x} = 0.9756098\left[1+0.73833M\right]e^{i\sigma t} = \\ &= \left[0.9756098+0.720321983634M\right]e^{i\sigma t}, \end{aligned} \tag{12.6}$$

which tends to 0.9756098, the classical Stokes drag $D = 0.9756098,\ D_y = D_x = 6.264\,pU_0 m$ on the cycloidal body as $M = 0,\ t = 0,$ i.e. without oscillating effect.

12. Conclusion

The problem of Stokes drag on axially symmetric bodies vibrating slowly along the axis of symmetry placed under a uniform transverse flow of a Newtonian fluid is studied analytically by a new idea explained in section-4. The definition of frictional drag $\left(D_f / D_x\right)$ on slowly vibrating axially symmetric particles placed under an axial flow provided by Kanwal [4] is utilized along with the real factor K, the ratio of classical Stokes drag on same axi-symmetric body in both axial and transverse flow configurations without vibrations. An axially symmetric body of revolution in the meridional plane with continuously turning tangent is considered for application purposes. The transverse frictional drag on slowly vibrating bodies like a sphere, spheroid, deformed sphere, egg-shaped body, cycloidal body, Cassini oval and hypocycloidal body is calculated. The numerical values of frictional drag on a slowly vibrating needle shaped body and flat circular disk are calculated as particular cases of a deformed sphere. The results of transverse frictional drag reported in this paper seem to be new and have never existed in the literature and in few cases like a sphere the result is reduces to the classical result available in the literature. The main advantage of the idea used in this paper is that we avoided solving the Stokes system of equations but we can get the optimum gain out of it mainly in transverse flow case. This idea may be exploited further to solve the problems of Oseen's flow, quadratic flow, magneto-hydrodynamic flow etc. The author is working on solving these types of problems which may be discussed in future papers. Also, the salient features of on axial flow and transverse flow may be exploited mainly in low Reynolds number hydrodynamics.

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Nomenclature

a – semi horizontal axis length
a_m – maximum horizontal axis length
b – maximum mid cross-section radius
D_f – frictional drag
D_x, D_z – axial Stokes drag
D_y, D_ρ – transverse Stokes drag
e – eccentricity of spheroid
h_x – height of centre of gravity of force system in axial flow
h_y – height of centre of gravity of force system in transverse flow
K – ratio of axial and transverse drag
M^2 – dimensionless number
p – fluid pressure
r,θ – polar coordinates
R – the intercepting length between the point at meridional curve and the axis of symmetry
$\boldsymbol{u}$ – fluid velocity vector
$U = U_0 e^{i\sigma t}$ – vibrational velocity of axially symmetric body
U_0 – uniform flow velocity
U_x – uniform fluid flow along the axis of symmetry
U_y – uniform fluid flow along the direction perpendicular to the axis of symmetry
x, y – Cartesian coordinates
α – angle made by intercepting length R with the axis of symmetry
ε – deformation parameter
μ – viscosity coefficient
ν – kinematic viscosity
ρ_1 – mass density of fluid
σ – circular frequency

References

[1] Stokes G.G. (1851): *On the effect of the internal friction of fluids on pendulums*.– Trans. Comb. Philos. Soc., vol.9, p.8.

[2] Kanwal R.P. (1955): *Rotatory and longitudinal oscillations of axi-symmetric bodies in a viscous fluid*.– J. Fluid Mech., vol.9, No.4, pp.631-636.

[3] Payne L.E. and Pell W.H. (1960): *The Stokes flow problem for a class of axially symmetric bodies*.– J. Fluid Mech., vol.7, No.4, pp.529-549.

[4] Kanwal R.P. (1964): *Drag on an axially symmetric body vibrating slowly along its axis in a viscous fluid*.– J. Fluid Mech., vol.19, No.4, pp.631-636.

[5] Proudman I. and Pearson J.R.A. (1957): *Expansions at small Reynolds numbers for flow past a sphere and a circular cylinder*.– J. Fluid Mech., vol.2, pp.237-262.

[6] Chwang A. T. and Wu T.Y. (1975): *Hydromechanics of low-Reynolds-number flow. Part 2. Singularity method for Stokes flows*.– J. Fluid Mech., vol.67, No.4, pp.787-815.

[7] Datta S. and Srivastava D.K. (1999): *Stokes drag on axially symmetric bodies: a new approach*.– Proceedings Math. Sci., Ind. Acad. Sci., vol.109, No.4, pp.441-452.

[8] Srivastava D.K. (2001): *Stokes drag on axially symmetric bodies: a note*.– Nepali Math. Sci. Report, vol.19, No.1 and No.2, pp.29-34.

[9] Srivastava D.K., Yadav R.R. and Yadav S. (2012): *Steady Stokes flow around deformed sphere: class of oblate axi-symmetric bodies*.– Int. J. of Appl. Math and Mech., vol.9, No.20, pp.16-44.

[10] Srivastava D.K., Yadav R.R. and Yadav S. (2013): *Steady Stokes flow around deformed sphere: class of prolate axi-symmetric bodies*.– Int. J. of Appl. Math and Mech., vol.9, No.20, pp.16-44.

[11] Lamb H. (1945): *Hydrodynamics*.– 6th ed., New York: Dover.

[12] Landau L.D. and Lifshitz E.M. (1959): *Fluid Mechanics*.– New York: Addison-Wesley.

[13] Happel J. and Brenner H. (1983): *Low Reynolds Number Hydrodynamics*.– Nijhoff, Dordrecht, The Nederlands.

[14] Kohr M. and Pop I. (2004): *Viscous Incompressible Flow for Low Reynolds Numbers*.– WIT Press, Southampton (UK), Boston.

[15] Sangtae Kim and Seppo J. Karrila (2005): *Microhydrodynamics: Principles and Selected Applications*.– Courier Corporation.

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