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# MODELLING OF EQUIVALENT MASS AND RIGIDITY OF CONTINUAL SEGMENT OF THE INTER-RESONANCE VIBRATION MACHINE

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The article deals with a continual segment of an inter-resonance vibration machine. In the form of a solid with distributed parameters this segment combines two defining parameters, namely: the inertial parameter of reactive: masses and appropriate rigidity of elastic coupling. These operation factors are revealed only in dynamic processes and are clearly not included in the parameters of the continual segment. Analytical dependences are developed for modeling of defining parameters of an inter-resonance system, namely: reactive mass and appropriate rigidity of elastic: coupling. Parameters of the reference point of the continual segment passing through its center of velocity are studied. The inertial parameter of the reactive mass and the rigidity of elastic coupling were modeled by the Rayleigh-Ritz method. The reliability of the results of theoretical research was confirmed experimentally and the parameters of the partial frequency of the continual segment were determined.

**Key words:** three-mass vibration system, Rayleigh-Ritz method, continual segment, equivalent mass, equivalent rigidity.

## 1. Introduction

One of the promising directions of development of vibrating technological equipment is the creation of inter-resonant vibration machines. Inter-resonant modes of operation provide a significant reduction in power consumption in the drives. Lanets *et al.* [1] supposes that for the efficient use of energy-saving inter-resonant modes of operation it is necessary to use small inertial values of reactive mass and rigidity of the corresponding elastic assembly. Such properties are characteristic of a flexible body - a continuous system that

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optimally combines inertial and rigid parameters.

It is known that a straight rod or beam has many natural forms of vibration, where each form of vibration corresponds to a certain value of frequency. Gursky *et al.* [2] studied the natural frequency of beam vibrations by the finite element method. However, this method can create a number of errors, which forces researchers to vary the complexity (density) of the structural grid to obtain acceptable results. In the work of Sharma *et al.* [3] the vibration of functionally graduated plates was analyzed using the finite element method. Jaworski *et al.* [4] conducted the experiment using the finite element method in ANSYS software. Raju *et al.* [5] analyzed the large amplitudes of free vibrations of conical beams using the continuum and the finite element method. Gharaibeh *et al.* [6] investigated the natural vibrations of rectangular plates with partially clamped edges (at the corners) by the finite element method.

Srinivasa *et al.* [7] experimentally studied the free vibration frequencies of isotropic and laminated composite plates. Jaroszewicz [8] analyzed the vibration of homogeneous and isotropic circular thin plates with nonlinear variable thickness, which are clamped at the edges. Zur [9] considered the natural vibration of homogeneous and isotropic circular thin plates with variable distributed parameters using Green's functions, which depend on the Poisson coefficient and the coefficient of distribution of the rigidity of the plate on the bend. In the works of Amabili *et al.* [10], analytical dependences were obtained using the nonlinear Von Karman plate theory and global sampling, as well as large amplitudes of forced vibrations of a thin rectangular plate with different concentrated masses were experimentally studied.

In Buchacz articles [11, 12], the beam is considered as a homogeneous beam with constant length parameters and the graph theory is used to establish the natural frequency of vibrations. Clementi *et al.* [13] studied the frequency response curves of a non-uniform beam by the MTS asymptotic expansion method. This non-uniform beam oscillates nonlinearly. Using the multiple time scale method, in which axial inertia is neglected, the equations of motion are statically compressed only at the transverse displacement. Firouz-Abadi *et al.* [14] presented an analytical solution based on the approximation of Wentzel, Kramers, Brillouin for free transverse vibrations of beams of various cross sections. Ece *et al.* [15] solved the problem of vibration of beams with exponentially variable cross-sectional width for three different types of boundary conditions: free, jointed and clamped ends. Mahmoud *et al.* [16] analyzed nonlinear free vibrations (of large amplitudes) of conical rods using the Max-Min Approach and Homotopy Perturbation Method. After comparing the obtained results, the authors confirmed the convergence of vibration. Kisa *et al.* [17] proposed a method for determining the frequency of natural vibrations of a uniform and stepped cracked beam with a circular cross section. Free vibrations of tapered beams were considered by Lee *et al.* [18]. The authors solved differential equations by numerical methods and established natural frequencies by combining the Runge Kutta method and the determinant search method. Shin *et al.* [19] applied the generalized method of differential quadrature and the method of differential transformation to the vibrational analysis of circular arches, declaring rapid convergence and accuracy. Lee [20] developed a method for finding the natural vibration of a Bernoulli-Euler tapered beam, in which the roots of the differential equation were determined using the Frobenius method, which allows obtaining solutions of the power series for bending vibrations.

The analysis confirms the relevance of further research in the direction of studying the natural frequencies of continual systems.

Analytical study of natural frequency by the Ritz method was considered in Vescovini *et al.* [21], where free vibrations and stability losses in highly anisotropic plates were studied. Dozio [22] expanded the potential of the Ritz method for predicting the natural bending frequencies of plates with various complicating factors. Study results confirmed the efficiency and accuracy of the method.

The Rayleigh-Ritz method ensures sufficient accuracy of the results (Yuan *et al.* [23], Kumar [24], Rahbar-Ranji *et al.* [25], Mazanoglu [26], Babakov [27]). This method can be applied to a wide range of plates, bars, rods, beams, etc., with any aspect ratio. Due to its versatility and accuracy, the Rayleigh-Ritz method is used to study the natural vibration frequency in this article.

## 2. Description of the problem

According to known methods for calculating inter-resonance vibration machines, Lanets *et al.* set five defining parameters of the vibration system (Fig.1a), namely: the inertial values of the three masses of $m_1$, $m_2$

and $m_3$ and the values of $c_{12}$ and $c_{23}$ rigidity of the elastic units, connecting the three vibrating masses, respectively. The oscillating system is driven with $\omega$ circular frequency of forced vibrations by a crank mechanism with the $\varepsilon$ eccentricity, where the $F_0$ perturbation force is $\varepsilon c_{23}$. The vibration system bases on the foundation through vibration insulators with $c_{is}$ rigidity.

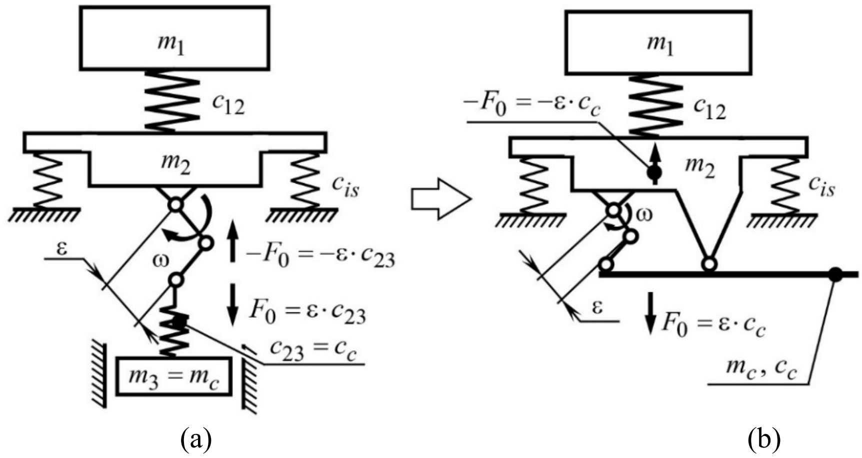



Fig.1. Schematic structures of three-mass mechanical vibration systems with eccentric drive: a - discrete; b - discrete-continual

The problem is that in the discrete-continual system (Fig.1b) the continual segment (beam) simultaneously connects two parameters – mass of $m_3$ and rigidity of $c_{23}$, which are revealed only in dynamic processes and are clearly not included in the parameters of the beam.

The task of this article is to develop analytical dependences that will make it possible to study the influence of two defining parameters of the inter-resonance system, namely: the mass of $m_3$ and rigidity of $c_{23}$, which are summary values of the inertial parameter of $m_c$ and rigidity of $c_c$ of the beam taking into consideration that $m_3 \equiv m_c$, and $c_{23} \equiv c_c$.

## 3. Results and discussions

### 3.1. The reference point of the continual segment

To determine the consolidated rigidity of $c_c$ and the consolidated mass $m_c$ of the beam in vibrating motion, the point of reference at the certain coordinate of $X_c$ is analytically determined. This point is formed as the interaction of the $X_{c1}$, $X_{c2}$ and $X_{c3}$ points of reference respectively, the left, middle and right segments of the beam, the lengths of which are $L_1$, $L_2$ and $L_3$ ($L$ is the length of the beam). It is at this point that the inertial value and rigidity of the entire beam are concentrated by convention (Fig.2).

There may be many reference points of the beam. For each case, we can find a specific value of mass and rigidity and enter them into a discrete model.

A question arises as to how adequate these values are and whether they are really perceived by the vibration system.

We assume that the discretized inter-resonance vibration system perceives the continual segment with distributed parameters discretely just concerning the consolidated or reference point at the $X_c$ certain coordinate relative to the center of velocities of the beam (Fig.2). Mathematically, the velocity center is defined as the ratio of the sum of the static moments of the velocities of each segment to the sum of the velocities along the entire length of the beam. For amplitude values, this can be written in integral form as follows:

$$X_c = \frac{\omega X_{c1} \int\limits_0^{L_1} w_l(x)\mathrm{d}x + \omega X_{c2} \int\limits_{L_1}^{L_1+L_2} w_m(x)\mathrm{d}x + \omega X_{c3} \int\limits_{L_1+L_2}^{L} w_r(x)\mathrm{d}x}{\omega \int\limits_0^{L_1} w_l(x)\mathrm{d}x + \omega \int\limits_{L_1}^{L_1+L_2} w_m(x)\mathrm{d}x + \omega \int\limits_{L_1+L_2}^{L} w_r(x)\mathrm{d}x}. \tag{3.1}$$

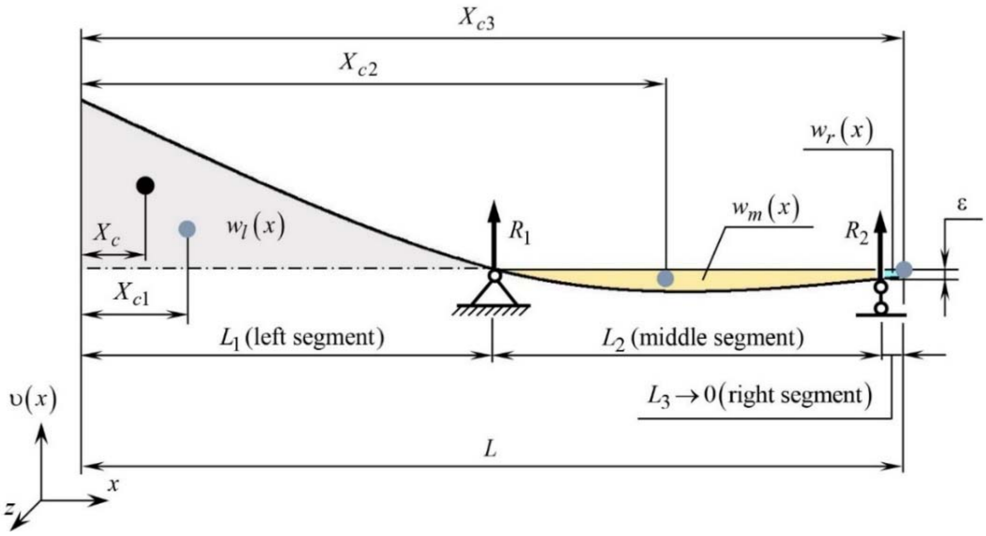



Fig.2. The scheme of velocities of the $\upsilon(x) = \omega w(x)$ rod segments deflection in the *x* coordinate, where: $R_1$, $R_2$ – reaction of supports; $w_l(x)$, $w_m(x)$ and $w_r(x)$ – deflection of the beam in the left, middle and right segments respectively.

Velocity centers of $X_{c1}$, $X_{c2}$ and $X_{c3}$ individual sections are searched according to:

$$X_{c1} = \frac{\omega \int\limits_0^{L_1} x \cdot w_l(x)\mathrm{d}x}{\omega \int\limits_0^{L_1} w_l(x)\mathrm{d}x}; \quad X_{c2} = \frac{\omega \int\limits_{L_1}^{L_1+L_2} x \cdot w_m(x)\mathrm{d}x}{\omega \int\limits_{L_1}^{L_1+L_2} w_m(x)\mathrm{d}x}; \quad X_{c3} = \frac{\omega \int\limits_{L_1+L_2}^{L} x \cdot w_r(x)\mathrm{d}x}{\omega \int\limits_{L_1+L_2}^{L} w_r(x)\mathrm{d}x}. \tag{3.2}$$

After substituting Eqs (3.2) into Eq.(3.1) and simplifying the obtained expression, the equation for determining the center of velocity of the elastic beam, which is analogous to finding the center of mass of the solid, will take the form of:

$$X_c = \frac{\int_0^{L_1} x \cdot w_l(x)\,\mathrm{d}x + \int_{L_1}^{L_1+L_2} x \cdot w_m(x)\,\mathrm{d}x + \int_{L_1+L_2}^{L} x \cdot w_r(x)\,\mathrm{d}x}{\int_0^{L_1} w_l(x)\,\mathrm{d}x + \int_{L_1}^{L_1+L_2} w_m(x)\,\mathrm{d}x + \int_{L_1+L_2}^{L} w_r(x)\,\mathrm{d}x}. \tag{3.3}$$

Knowing the $X_c$ coordinate of the reference point, the equivalent mass and equivalent rigidity of the elastic beam can be determined.

### 3.2. Modeling of equivalent mass and equivalent rigidity by Rayleigh–Ritz method

The method is based on the fact that the amplitude values of the $\mathrm{K}_r$ kinetic and potential $\Pi_r$ energies of the elastic beam for the period of vibrations are the same. That is:

$$\mathrm{K}_r = \Pi_r . \tag{3.4}$$

The $\mathrm{K}_r$ kinetic energy of the beam consists of the kinetic energy from the $w(x)$ displacement and from the rotation at the $\theta(x)$ angle:

$$\mathrm{K}_r = \frac{1}{2}\int_0^L m_{rr}\omega^2 w(x)^2\,\mathrm{d}x + \frac{1}{2}\int_0^L J_{rr}(x)\omega^2\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x, \tag{3.5}$$

where $m_{rr}$ is the linear mass of beam; $w(x)$ is the displacement of the cross section of the rod in the vertical direction depending on the $x$ coordinate (along the length of the elastic beam); $J_{rr}$ is the moment of inertia of beam length unit (linear unit) as the function of $x$ coordinate.

Considering the solid as a beam, the moment of inertia of the unit length of the beam is written as:

$$J_{rr}(x) = \frac{\rho h b_r}{3}x^2 = \frac{m_{rr}}{3}x^2 \tag{3.6}$$

where $\rho$ is the unit weight for beam material; $h$ is the width of the elastic beam; $b_r$ is the thickness of the elastic beam.

The displacement of the cross section of the rod of $w(x)$ in general can be represented as [27]:

$$w(x) = A\cos(\xi x) + B\sin(\xi x) + C\,\mathrm{ch}(\xi x) + D\,\mathrm{sh}(\xi x) \tag{3.7}$$

where $A$, $B$, $C$ and $D$ – four arbitrary constants are chosen so that for Eq.(3.7) the boundary conditions are satisfied, i.e. the conditions of fixing the ends of the beam (segments of the beam);

$$\xi = \sqrt[4]{\frac{m_{rr}\,\omega^2}{EJ_z}} \tag{3.8}$$

where $E$ is the (tensile) elastic modulus; $J_z$ is moment of inertia of beam rectangular cross-section area relative to the neutral line of the section; $EJ_z$ is the rigidity of the beam cross-section area.

The $\Pi_r$ potential energy of the bent axis of the rod was determined as the sum of the potential energy of its bending and the potential energy of shear in the layers of the rod and the result can be written as:

$$\Pi_r = \frac{1}{2} EJ_z \int_0^L \left( \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} \right)^2 \mathrm{d}x + \frac{1}{2} \frac{k_y (EJ_z)^2}{GF_r} \int_0^L \left( \frac{\mathrm{d}^3 w(x)}{\mathrm{d}x^3} \right)^2 \mathrm{d}x \tag{3.9}$$

where $G$ is the modulus of rigidity; $F_r$ is the cross-section area of the beam (constant along the length of the beam); $k_y$ is the coefficient that depends on the shape of the cross section of the beam [27].

According to formula Eq.(3.4), after a comparison Eq.(3.5) and Eq.(3.9) and selection from each term of the equation of the amplitude of vibration at the $w(X_c)$ reference point which is a constant value, the result can be written as:

$$\begin{gathered} \frac{\omega^2 w(X_c)^2}{2} \left[ \frac{m_{rr}}{w(X_c)^2} \int_0^L w(x)^2 \, \mathrm{d}x + \frac{1}{w(X_c)^2} \int_0^L J_{rr}(x) \left( \frac{\mathrm{d}w(x)}{\mathrm{d}x} \right)^2 \mathrm{d}x \right] = \\ = \frac{w(X_c)^2}{2} \left[ \frac{EJ_z}{w(X_c)^2} \int_0^L \left( \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} \right)^2 \mathrm{d}x + \frac{k_y (EJ_z)^2}{w(X_c)^2 \, GF_r} \int_0^L \left( \frac{\mathrm{d}^3 w(x)}{\mathrm{d}x^3} \right)^2 \mathrm{d}x \right]. \end{gathered} \tag{3.10}$$

Equation (3.10) is the equality of kinetic and potential energy. Given that the $\omega \cdot w(X_c)$ is a linear velocity of $\upsilon(X_c)$, Eq.(3.10) will take the form:

$$\frac{\upsilon(X_c)^2 \cdot m_c}{2} = \frac{w(X_c)^2 \cdot c_c}{2} \tag{3.11}$$

where

$$m_c = \frac{m_{rr}}{w(X_c)^2} \int_0^L w(x)^2 \, \mathrm{d}x + \frac{1}{w(X_c)^2} \int_0^L J_{rr}(x) \left( \frac{\mathrm{d}w(x)}{\mathrm{d}x} \right)^2 \mathrm{d}x, \tag{3.12}$$

$$c_c = \frac{EJ_z}{w(X_c)^2} \int_0^L \left( \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} \right)^2 \mathrm{d}x + \frac{k_y (EJ_z)^2}{w(X_c)^2 \, GF_r} \int_0^L \left( \frac{\mathrm{d}^3 w(x)}{\mathrm{d}x^3} \right)^2 \mathrm{d}x \tag{3.13}$$

are equivalent mass and equivalent rigidity, respectively, of the elastic beam with distributed parameters at the $X_c$ point of reduction.

### 3.3. Model of the continual segment

The continual segment is considered as the beam with several sections. Three independent sections will be connected by the $R_1$ and $R_2$ reactions in the supports. The left end of the rod, which is in the free state, is taken as the starting point (Fig.3).

Since the left end of the beam is free, the $w(x)$ cross motion and the $\theta(x) = w'(x)$ angle of rotation are in it, and the $M(x)=EJ_z w''(x)$ bending moment and $Q(x)=EJ_z w'''(x)$ transverse force are absent. The equations of $w_l(x)$, $w_m(x)$ in $w_r(x)$ deflections in the left, middle and right beam sections, respectively, will take the form:

$$w_l(x)=\frac{A}{2}\left(\mathrm{ch}(\xi x)+\cos(\xi x)\right)+\frac{B}{2}\left(\mathrm{sh}(\xi x)+\sin(\xi x)\right), \tag{3.14}$$

$$w_m(x)=w_l(x)+\frac{R_1}{2\xi^3 EJ_z}\cdot\left(\mathrm{sh}\left(\xi(x-L_1)\right)-\sin\left(\xi(x-L_1)\right)\right), \tag{3.15}$$

$$w_r(x)=w_l(x)+w_m(x)+\frac{R_2}{2\xi^3 EJ_z}\cdot\left(\mathrm{sh}\left(\xi\left(x-(L-L_2)\right)\right)-\sin\left(\xi\left(x-(L-L_2)\right)\right)\right). \tag{3.16}$$

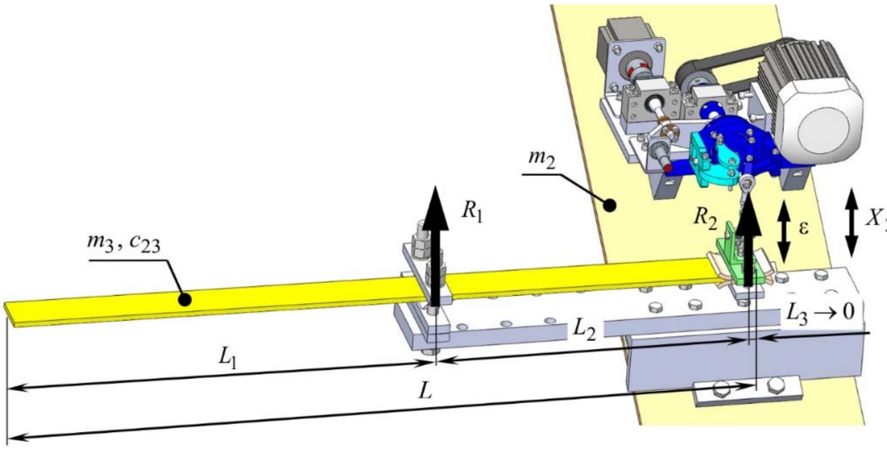



Fig.3. Calculation scheme of the beam continual segment.

Given that for the left section in the $R_1$ support the displacement is $\delta_1 = X_2$ ($X_2$ – is the amplitude of vibration of the $m_2$ intermediate mass) at $x = L_1$ (Fig.3), and for the middle section ($R_2$ support) the displacement is $\delta_2 = X_2 + \varepsilon$ at $x = L_1 + L_2$, the system of four equations of forced vibration of the beam will be:

$$\begin{cases} w_r''(L)=0, \\ w_r'''(L)=0, \\ w_l(L_1)=\delta_1, \\ w_m(L_1+L_2)=\delta_2. \end{cases} \tag{3.17}$$

### 3.4. Results of modeling of equivalent mass and equivalent rigidity

It is established that to ensure $\zeta = 5g$ overload ($g$ - acceleration of gravity) at the frequency of $\omega_f = 99.484 rad/s$ ($n_f = 950 rpm$) forced vibrations the amplitude of vibration of the working body should be $X_1 = 4.955 \cdot 10^{-3} m$, and the intermediate mass of $X_2 = -5.868 \cdot 10^{-3} m$. As above determined $X_1$ and $X_2$ parameters were substituted into Eqs (3.17), the values of $A$ and $B$ constants, the reactions of the $R_1$ and $R_2$ supports were determined, which are: $A = 0.083$, $B = -0.066$, $R_1 = -592.066 N$ and $R_1 = 341.729 N$. After determined values were substituted in Eq.(3.3), the coordinate of the reference point along the $x$-axis was defined which is $X_c = 0.01 m$.

To establish the equivalent mass and the equivalent rigidity of the beam according to expressions (3.12) and (3.13), these parameters will be formed as an algebraic sum of the corresponding indices on each section of the beam, i.e.:

$$c_c = c_{c1} + c_{c2} + c_{c3} + c_{c1i} + c_{c2i} + c_{c3i}, \tag{3.18}$$

$$m_c = m_{c1} - m_{c2} - m_{c3} + m_{c1i} - m_{c2i} - m_{c3i} \tag{3.19}$$

where $c_{c1}$, $c_{c2}$, $c_{c3}$ – fractions of equivalent rigidities of the beam bending on left, middle and right sections, respectively; $c_{c1i}$, $c_{c2i}$, $c_{c3i}$ – fractions of equivalent rigidities of the beam bending on left, middle and right sections, respectively; $m_{c1}$, $m_{c2}$, $m_{c3}$ – fractions of equivalent mass of linear movement of the beam on left, middle and right segments, respectively; $m_{c1i}$, $m_{c2i}$, $m_{c3i}$ – fractions of equivalent mass of the beam twist on left, middle and right sections, respectively. The values of the fractions of the equivalent masses and rigidities in the third section can be neglected due to the scantiness of its length.

To establish the fraction of the $m_{c1}$ equivalent mass of movement at the left end of the beam (from $0$ to $L_1$), the left part of expression (3.12) and expression (3.14) were used:

$$m_{c1} = m_{rr} \int_0^{L_1} \left( \frac{w_l(x)}{w(X_c)} \right)^2 \mathrm{d}x = 0.243 \ kg \tag{3.20}$$

where

$$w(X_c) = \frac{A}{2}\left(\mathrm{ch}(\xi X_c) + \cos(\xi X_c)\right) + \frac{B}{2}\left(\mathrm{sh}(\xi X_c) + \sin(\xi X_c)\right)$$

is the deflection of the beam at the point of reference.

On the middle section of the beam (from $L_1$ to $L_1 + L_2$) we can determine the fraction of the equivalent mass of movement using Eq.(3.15):

$$m_{c2} = m_{rr} \int_{L_1}^{L_1+L_2} \left( \frac{w_m(x)}{w(X_c)} \right)^2 \mathrm{d}x = 9.27 \cdot 10^{-3} \ kg. \tag{3.21}$$

To establish the fraction of the $m_{c1i}$ equivalent mass of twist at the left end of the beam (from $0$ to $L_1$) the right part of expression (3.12) and expression (3.14) were used:

$$m_{c1i} = \int_0^{L_1} \left( \frac{m_{rr}}{3} x^2 \right) \left( \frac{\mathrm{d}}{\mathrm{d}x} \left( \frac{w_l(x)}{w(X_c)} \right) \right)^2 \mathrm{d}x = 0.082 \ kg. \tag{3.22}$$

On the middle section of the beam (from $L_1$ to $L_1 + L_2$) we can determine the fraction of the equivalent mass of twist using Eq.(3.15):

$$m_{c2i} = \int_{L_1}^{L_1+L_2} \left( \frac{m_{rr}}{3} (x - L_1)^2 \right) \left( \frac{\mathrm{d}}{\mathrm{d}x} \left( \frac{w_m(x)}{w(X_c)} \right) \right)^2 \mathrm{d}x = 2.22 \cdot 10^{-3} \ kg. \tag{3.23}$$

To establish the fraction of the $c_{c1}$ equivalent rigidity of bending at the left end of the beam (from $0$ to $L_1$), the left part of expression (3.13) and expression (3.14) were used:

$$c_{c1} = EJ_z \int_0^{L_1} \left( \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left( \frac{w_l(x)}{w(X_c)} \right) \right)^2 \mathrm{d}x = 1.33 \cdot 10^3 \ N/m. \tag{3.24}$$

On the middle section of the beam (from $L_1$ to $L_1 + L_2$) we can determine the fraction of the equivalent rigidity of bending using Eq.(3.15):

$$c_{c2} = EJ_z \int_{L_1}^{L_1+L_2} \left( \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left( \frac{w_m(x)}{w(X_c)} \right) \right)^2 \mathrm{d}x = 1.416 \cdot 10^3 \ N/m. \tag{3.25}$$

To establish the fraction of the $c_{c1i}$ equivalent rigidity of beam layers shearing at the left end of the beam (from $0$ to $L_1$), the right part of expression (3.13) and expression (3.14) were used:

$$c_{c1i} = \frac{k_y (EJ_z)^2}{GF_r} \int_0^{L_1} \left( \frac{\mathrm{d}^3}{\mathrm{d}x^3} \left( \frac{w_l(x)}{w(X_c)} \right) \right)^2 \mathrm{d}x = 0.237 \ N/m. \tag{3.26}$$

On the middle section of the beam (from $L_1$ to $L_1 + L_2$) we can determine the fraction of the equivalent rigidity of beam layers shearing using Eq.(3.15):

$$c_{c2i} = \frac{k_y (EJ_z)^2}{GF_r} \int_{L_1}^{L_1+L_2} \left( \frac{\mathrm{d}^3}{\mathrm{d}x^3} \left( \frac{w_m(x)}{w(X_c)} \right) \right)^2 \mathrm{d}x = 0.273 \ N/m. \tag{3.27}$$

Therefore, the equivalent mass and the equivalent rigidity of the beam according to expressions of (3.20) and (3.27) are:

$$m_c = 0.313 \, kg \, , \tag{3.29}$$

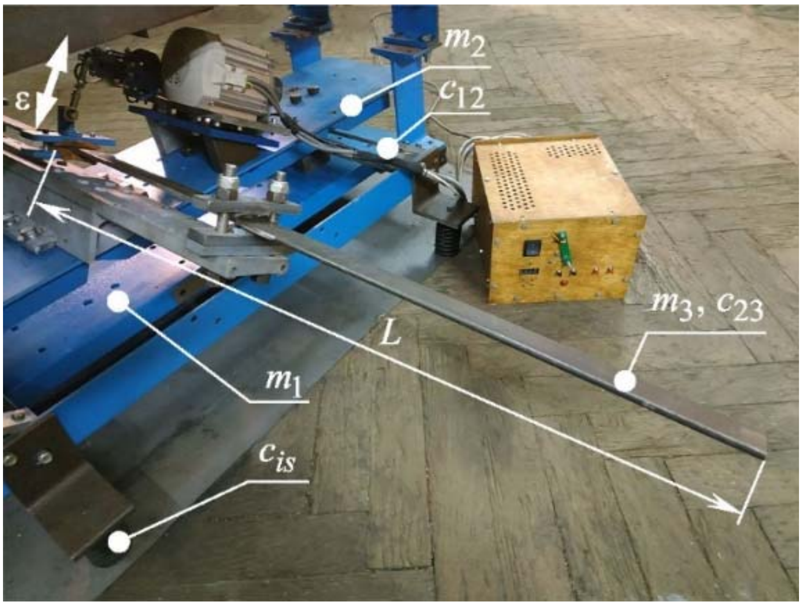
$$c_c = 2.747 \, N / m \, . \tag{3.28}$$

## 4. The results of the experimental study

The reliability of the calculated values of the $m_c$ equivalent mass and the $c_c$ equivalent rigidity can be determined by upholding of the value of the partial frequency of the $\omega_p$ continual segment. Due to the established technical requirements, the partial frequency should be of $\omega_p = 95.672 rad/s$. According to calculations using Eq.(3.29) and Eq.(3.28), the partial frequency is:

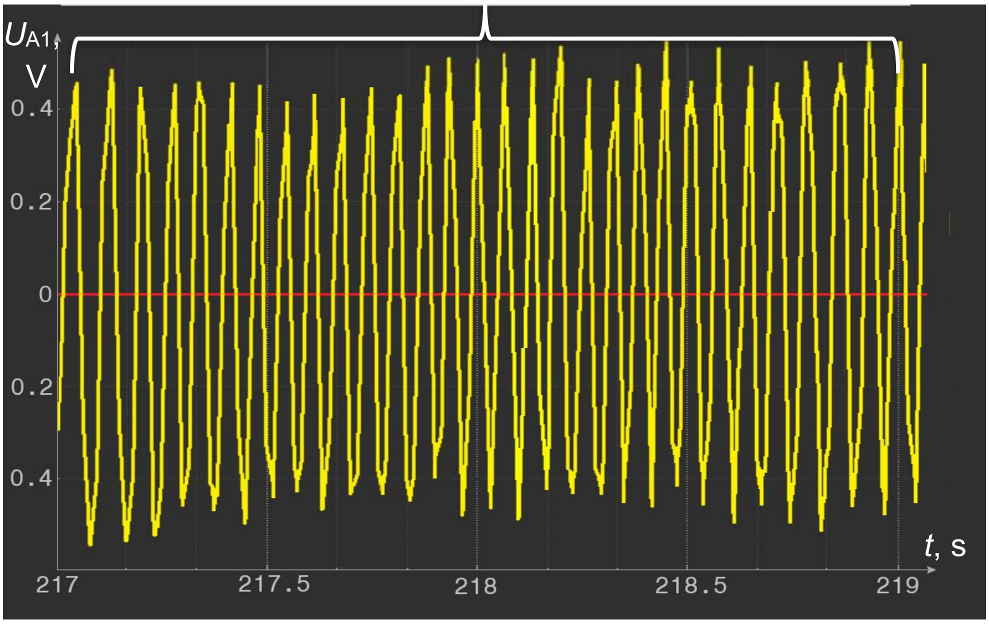
$$\omega_p = \sqrt{\frac{c_c}{m_c}} = 93.707\, rad/s, \tag{4.1}$$

the result coincides quite precisely with the set parameter.




(a)

Vibration cycle quantity




(b)

Fig.4. Experimental sample of the (a) conveyor-separator and amplitude-time characteristic of the (b) intermediate mass (the amplitude value of vibrations is read from the sensor in volts).

The continual segment in the form of the beam with the geometric parameters given in the article is implemented in an inter-resonance vibrating machine. The nature of vibrations of the intermediate mass, which was set in motion from the continual segment with vibration at its own (partial) frequency, was experimentally indicated. An experimental sample of the continual segment of the inter-resonance vibrating machine is shown in Fig.4a. Inertial parameters of sensors connected to the $m_2$ mass practically do not affect the vibrating system because of $m_2 = 62.1kg$ .

The time characteristics of the $m_2$ mass movement during vibrations of the beam at its own frequency are shown in Fig.4b. The value of the first natural cyclic frequency of vibrations of the beam was determined from the ratio of the number of peaks to the time interval, which is defined as the difference of *219-217*, Fig.4b:

$$\nu_{pe} = \frac{\text{vibration cycle quantity}}{\text{time interval}} \approx \frac{29.4}{219-217} \approx 14.7\,Hz. \tag{4.2}$$

The circular frequency of the beam, according to Eq.(4.2), will be $\omega_{pe} \approx 92.36 rad/s$ . The obtained value is consistent with the results of simulation ( $\omega_p = 93.707 rad/s$ ).

To analyze quantitatively of the consistency between the simulation results and the experimental data, the frequency error is calculated:

$$\varepsilon_p = \left|\frac{\omega_{pe}-\omega_p}{\omega_{pe}}\right| \cdot 100\% = \left|\frac{92.36-93.707}{92.36}\right| \cdot 100 = 1.46\ \% . \tag{4.3}$$

The frequency error does not exceed *2*%, which confirms the reliability of theoretical studies.

## 5. Conclusions

For inter-resonance vibration machines, the developed analytical model renders it possible to synthesize continual segments, the properties of which are consistent with the relevant sections of discrete systems. Using the developed model, the two defining parameters of the oscillating system of the vibrating machine are studied, namely: the inertial parameter of reactive mass and appropriate rigidity of the elastic coupling. These operation factors are revealed only in dynamic processes and are clearly not included in the parameters of the continual segment. In order to take into account these parameters the reference point of the continual segment passing through its centre of velocity is studied. The inertial parameter of the reactive mass and the rigidity of the elastic coupling were studied by the Rayleigh-Ritz method. The reliability of the results of theoretical research was confirmed experimentally and the parameters of the partial frequency of the continual segment were determined.

## Nomenclature

$A,\ B,\ C,\ D$ – arbitrary constants, [–]

$b_r$ – beam thickness, [*m*]

$c_c$ – equivalent rigidity of the beam, [*N/m*]

$c_{c1i},\ c_{c2i},\ c_{c3i}$ – fractions of equivalent rigidities of the beam shearing on left, middle and right sections, respectively, [*N/m*]

$c_{c1},\ c_{c2},\ c_{c3}$ – fractions of equivalent rigidities of the beam bending on left, middle and right sections, respectively, [*N/m*]

$c_{is}$ – rigidity of vibration isolators, [*N/m*]

$c_{12},\ c_{23}$ – rigidity of elastic couplings that connect the active mass to the intermediate and intermediate mass to the reactive mass, respectively, [*N/m*]

| | |
|---|---|
| $E$ | – (tensile) elastic modulus, [*Pa*] |
| $F_r$ | – cross-section area of the beam, [$m^2$] |
| $F_0$ | – amplitude value of excitation force, [*N*] |
| $G$ | – modulus of rigidity, [*Pa*] |
| $g$ | – acceleration of gravity, [$m/s^2$] |
| $h$ | – beam width, [*m*] |
| $J_{rr}$ | – moment of inertia of the beam length unit as the *x* function, [$m^4$] |
| $J_z$ | – moment of inertia of the beam cross-section area (rectangular cross-section) relative to the neutral line of the section, [$m^4$] |
| $\mathrm{K}_r$ | – kinetic energy, [*J*] |
| $k_y$ | – coefficient that depends on the shape of the rectangular cross section of the beam, |
| $L$ | – beam length, [*m*] |
| $L_1,\ L_2,\ L_3$ | – length of left, middle and right beam sections, respectively, [*m*] |
| $M(x)$ | – bending moment, [*N m*] |
| $m_c$ | – equivalent mass of the beam, [*kg*] |
| $m_{c1},\ m_{c2},\ m_{c3}$ | – fractions of equivalent mass of linear movement of the beam on left, middle and right sections, respectively, [*kg*] |
| $m_{c1i},\ m_{c2i},\ m_{c3i}$ | – fractions of equivalent mass of the beam twist on left, middle and right sections, respectively, [*kg*] |
| $m_{rr}$ | – linear mass of the beam, [*kg/m*] |
| $m_1,\ m_2,\ m_3$ | – inertial parameters of active, intermediate and reactive masses, respectively, [*kg*] |
| $n$ | – speed, [*rpm*] |
| $Q(x)$ | – shear distribution along the beam, [*N*] |
| $R_1,\ R_2$ | – reaction at supports, [*N*] |
| $X_c$ | – value along the *x*-axis of the reference point of the beam, [*m*] |
| $X_{c1},\ X_{c2},\ X_{c3}$ | – value along the *x*-axis of the reference points for left, middle and right sections respectively, [*m*] |
| $X_1,\ X_2$ | – vibration amplitude of the active and intermediate mass, respectively, [*m*] |
| $\delta_1,\ \delta_2$ | – displacement in the first and second supports, respectively, [*m*] |
| $\varepsilon$ | – eccentricity of crank mechanism, [*m*] |
| $\zeta$ | – overload, [–] |
| $\theta(x)$ | – angle of twist of beam segment, [*rad*] |
| $\Pi_r$ | – potential energy, [*J*] |
| $\rho$ | – unit weight for steel, [$kg/m^3$] |
| $\upsilon(X_c)$ | – linear velocity, [*m/s*] |
| $\omega$ | – circular frequency of beam oscillation, [*rad/s*] |
| $\omega_f$ | – forced circular frequency of system vibration, [*rad/s*] |
| $\omega_p$ | – partial circular frequency of vibrating system, [*rad/s*] |

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