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Brief note

STRESS AND DISPLACEMENT INTENSITY FACTORS OF CRACKS IN ANISOTROPIC MEDIA

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A relation connecting stress intensity factors (SIF) with displacement intensity factors (DIF) at the crack front is derived by solving a pseudodifferential equation connecting stress and displacement discontinuity fields for a plane crack in an elastic anisotropic medium with arbitrary anisotropy. It is found that at a particular point on the crack front, the vector valued SIF is uniquely determined by the corresponding DIF evaluated at the same point.

Key words: stress intensity factor, displacement intensity factor, anisotropy, asymptote, crack, pseudodifferential equation.

1. Introduction

The stress intensity factors (SIF) along with the displacement intensity factors (DIF) for a plane crack of arbitrary shape placed in homogeneous anisotropic media with arbitrary elastic anisotropy, are analyzed by constructing a pseudodifferential operator that connects asymptotes of the outer stress field (out of crack) evaluated at a particular point on the crack front with the inner asymptote of the displacement discontinuity field. A closed form relation between two intensity factors, SIF and DIF at the crack front, is derived by solving the corresponding pseudodifferential equation. The method is developed for a plane crack with a smooth crack front of Lyapunov type.

The crack occupying plane region Ω with smooth Lyapunov type boundary $\partial\Omega \in C^{1,\alpha}$, $\alpha > 1$, is shown in Fig.1, where $\boldsymbol{n}$ denotes the unit outward normal to Ω, and dashed arrows indicate directions for evaluating SIF and DIF. The plane with unit normal $\boldsymbol{\nu}$ supporting Ω is denoted by Π_ν.

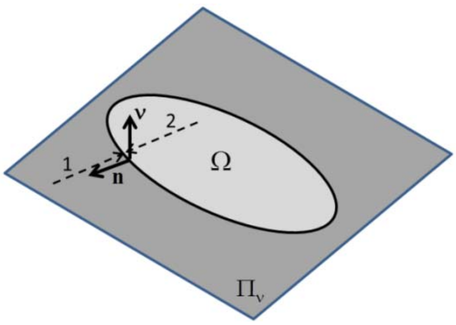

Fig.1. Plane crack Ω with smooth Lyapunov type boundary ∂Ω.

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Definition 1.1. The following definitions will be used for SIF and DIF

$$\begin{aligned}
&\boldsymbol{S}_{x_0} = (2\pi)^{+1/2} \lim_{\boldsymbol{x}'\cdot\boldsymbol{n}\to x_0+0} \left(\left| \boldsymbol{x}'\cdot\boldsymbol{n}_{x_0} - x_0 \right|^{+1/2} \boldsymbol{t}_\nu(\boldsymbol{x}') \right), \\
&\boldsymbol{D}_{x_0} = (2\pi)^{-1/2} \lim_{\boldsymbol{x}'\cdot\boldsymbol{n}\to x_0-0} \left(\left| \boldsymbol{x}'\cdot\boldsymbol{n}_{x_0} - x_0 \right|^{-1/2} \boldsymbol{b}(\boldsymbol{x}') \right), \quad \boldsymbol{x}'\in\Pi_\nu
\end{aligned} \tag{1.1}$$

where $x_0 \in \partial\Omega$ is a particular point belonging to the crack front; $\boldsymbol{t}_\nu$ is the surface traction field acting on the Π_ν-plane

$$\boldsymbol{t}_\nu = \nu \cdot \boldsymbol{t}\,, \tag{1.2}$$

herein, $\boldsymbol{t}$ is the stress tensor; vector $\boldsymbol{b}$ in $(1.1)_2$ denotes the displacement discontinuity field, defined by the limits taken along normal direction to the Π_ν-plane

$$\boldsymbol{b}(\boldsymbol{x}') = \lim_{\boldsymbol{x}\cdot\nu\to\boldsymbol{x}'\cdot\nu+0} \boldsymbol{u}(\boldsymbol{x}) - \lim_{\boldsymbol{x}\cdot\nu\to\boldsymbol{x}'\cdot\nu-0} \boldsymbol{u}(\boldsymbol{x})\,. \tag{1.3}$$

In Eq.(1.3) $\boldsymbol{x}'$ denotes projection of a point $\boldsymbol{x}$ onto the Π_ν-plane. Figure 2 illustrates SIF and DIF.

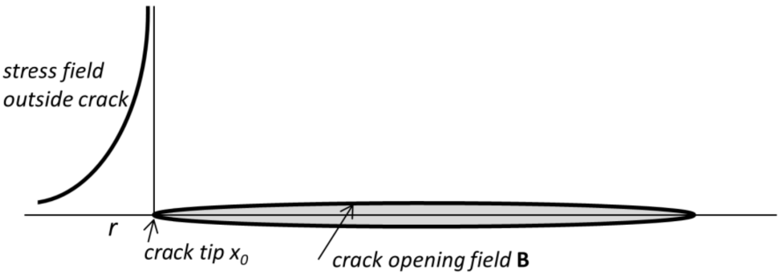

Fig.2. The crack opening field and stress field.

Note that definition $(1.1)_1$ is slightly more restrictive than sometimes used in the theory of cracks; see Zehnder [1] and Tada *et al* [2]; that is because of expression $(1.1)_1$ defining SIF only along directions normal to $\partial\Omega$. Thus, definition $(1.1)_1$ yields no dependence of SIF upon the direction of approaching, while a more general definition allows different approaching directions.

The discussed relation between $\boldsymbol{S}_{x_0}$ and $\boldsymbol{D}_{x_0}$ reads as follows

$$\boldsymbol{S}_{x_0} = \boldsymbol{M}(\boldsymbol{n}_{x_0}) \cdot \boldsymbol{D}_{x_0} \tag{1.4}$$

where $\boldsymbol{M}(\boldsymbol{n}_{x_0})$ is a specially constructed 3×3 matrix depending solely on elastic properties of the medium and direction of the normal $\boldsymbol{n}_{x_0}$ to the crack front $\partial\Omega$. The matrix $\boldsymbol{M}(\boldsymbol{n}_{x_0})$ structure is discussed later on.

The applied method utilizes asymptotic analysis. For the general definition of the wave front (WF) that is used for obtaining relation (1.4) and the corresponding applications, seeTrev [10], Shubin [11], Duduchava [12].

2. Stresses acting on the Π_ν - plane

The displacement field produced by the crack discontinuity can be represented by the following double-layer potential; see Kupradze *et al.* [13], Duduchava *et al.* [14]

$$\boldsymbol{u}(\boldsymbol{x}) = \int_\Omega \boldsymbol{b}(\boldsymbol{y}') \cdot \boldsymbol{T}(\nabla_y, \nu_y) \boldsymbol{E}(\boldsymbol{x} - \boldsymbol{y}') dy', \tag{2.1}$$

where $\boldsymbol{y}' \in \Omega$; $\boldsymbol{b}$ is the crack discontinuity field; $\boldsymbol{E}$ is the Green tensor in R^3; and $\boldsymbol{T}$ is the surface traction operator

$$\boldsymbol{T}(\nabla_y, \nu_y) \equiv \nu_y \cdot \boldsymbol{C} \cdot \nabla_y . \tag{2.2}$$

In Eq.(2.2) $\boldsymbol{C}$ is the fourth-order elasticity tensor, assumed to be strongly elliptic

$$\forall_{\boldsymbol{m},\boldsymbol{n} \neq 0} \boldsymbol{m},\boldsymbol{n} \quad \boldsymbol{m} \otimes \boldsymbol{n} \cdot\cdot \boldsymbol{C} \cdot\cdot \boldsymbol{n} \otimes \boldsymbol{m} > 0 . \tag{2.3}$$

In Eq.(2.3) $\boldsymbol{m}, \boldsymbol{n}$ are arbitrary non-zero vectors in R^3. Strong ellipticity condition (2.3) ensures ellipticity of the fundamental solution $\boldsymbol{E}$ [16]. Properties and methods of construction of the elasticity tensor at the case of general elastic anisotropy are discussed in [15, 16] and for wave dynamics in [18].

Now, the surface traction field on the Π_ν - plane can be defined by applying operator (2.2) to the potential (2.1) and transition to the non-tangential limits to Π_ν

$$\boldsymbol{t}(\boldsymbol{x}') = \lim_{\boldsymbol{x} \to \boldsymbol{x}'} \boldsymbol{T}(\nabla_x, -\nu_{x'}) \int_\Omega \boldsymbol{b}(\boldsymbol{y}') \cdot \boldsymbol{T}(\nabla_y, \nu_y) \boldsymbol{E}(\boldsymbol{x} - \boldsymbol{y}') dy', \quad \boldsymbol{x}' \in \Pi_\nu . \tag{2.4}$$

These limits are correctly defined due to the Lyapunov – Tauber theorem for elastic potentials; see Kupradze *et al.* [13], Duduchava *et al.* [14].

3. Operator of the theory of cracks

Application of the Fourier transformation to expression (2.4) gives an amplitude (Trev [10]) of the corresponding integro-differential operator as

$$\boldsymbol{G}^{\sim}(\xi) = (2\pi)^2 \nu_y \cdot \boldsymbol{C} \cdot\cdot \xi \otimes \boldsymbol{E}^{\sim}(\xi) \otimes \xi \cdot\cdot \boldsymbol{C} \cdot \nu_x \tag{3.1}$$

where symbol "~" stands for the integral Fourier transform**;** and ν_x, ν_y correspond to the same unit normal ν.

Restricting amplitude (3.1) on the Π_ν - plane yields the (principal) symbol of the desired operator that depends on $\xi' \in \Pi_\nu$ variable only

$$\boldsymbol{Z}^{\sim}(\xi') = (2\pi)^2 \, \text{F.P.} \int_{-\infty}^{\infty} \boldsymbol{G}^{\sim}(\xi) \, d\xi'' \tag{3.2}$$

where $\xi \in R^3$, $\xi' = \mathrm{Pr}_{\Pi_\nu}\,\xi$, $\xi'' = \mathrm{Pr}_\nu\,\xi'$, and thus $\xi = \xi' + \xi''\boldsymbol{\nu}$; in Eq.(3.2) F.P. stands for the Finite Part of the divergent improper integral, this can be evaluated by a regularization technique (Kuznetsov [3]). The following properties of symbol $\boldsymbol{Z}^{\sim}$ immediately follow from its definition:

PROPOSITION 3.1. a) Symbol $\boldsymbol{Z}^{\sim}$ is symmetric and strongly elliptic; b) Symbol $\boldsymbol{Z}^{\sim}$ is positively homogeneous of degree 1 with respect to $|\xi'|$; c) Symbol $\boldsymbol{G}^{\sim}$ is real-analytic in $R^2 \setminus 0$.

The symbol $\boldsymbol{Z}^{\sim}$ can be written in the form

$$\boldsymbol{Z}^{\sim}(\xi') = -|2\pi\xi'|^2\,\boldsymbol{K}^{\sim}(\xi') \tag{3.3}$$

where symbol $\boldsymbol{K}^{\sim}(\xi')$ is of the order -1, and hence corresponds to a smoothing integral operator in R^2.

Now, integral Fourier transform inversion of expression (3.3) yields

$$\boldsymbol{Z}(\boldsymbol{x}') = \Delta_{x'} \circ \boldsymbol{K}(\boldsymbol{x}'), \quad \boldsymbol{x}' \in \Pi_\nu \tag{3.4}$$

where $\Delta_{x'}$ is Laplacian in Π_ν and $\boldsymbol{K}(\boldsymbol{x}')$ is the homogeneous 3×3-matrix kernel of degree -1 (with respect to $|\boldsymbol{x}'|$) of the corresponding matrix integral operator in Π_ν. In Eq.(3.4) symbol "$\circ$" denotes composition of two matrix operators.

4. Relation between SIF and DIF: preliminary results

Decomposition (3.4) of the operator $\boldsymbol{G}$ yields

$$\boldsymbol{t} = \underbrace{\left(\Delta_{x'} \circ \boldsymbol{K}\right)}_{\boldsymbol{Z}(x')}\,\boldsymbol{b}\,. \tag{4.1}$$

Restriction of the operators in Eq.(4.1) onto the crack domain Ω yields

$$\boldsymbol{t}_\Omega = \left(\Delta_{x'} \circ \boldsymbol{K}_\Omega\right) \circ \boldsymbol{b}\,. \tag{4.2}$$

Now, inverting the latter formula yields

$$\boldsymbol{b} = \left(r_\Omega \circ \boldsymbol{K}_\Omega^{\,-1}\right)\boldsymbol{t}_\Omega \tag{4.3}$$

where r_Ω is Green's function for the Dirichlet problem in Ω; $\boldsymbol{K}_\Omega^{\,-1}$ denotes restriction on Ω of the inverse operator $\boldsymbol{K}^{-1}$, in view of Eq.(3.4) the latter is an integro-differential operator with the corresponding symbol of the order +1.

5. Relation between SIF and DIF: final expression

In view of Eq.(4.3), the Wave Front (WF) of the crack discontinuity field $\boldsymbol{b}$ at a particular point $\boldsymbol{x}'_0 \in \partial\Omega$ takes the form (Trev [10])

$$\mathrm{WF}(\boldsymbol{b}) = (\boldsymbol{x}_0', \xi' \cdot \boldsymbol{n}) \tag{5.1}$$

where, as before $\boldsymbol{n}$ denotes the (outward) unit normal to $\partial\Omega$ at $\boldsymbol{x}_0' \in \partial\Omega$. Now, Fourier integral transform of the (inner) asymptote of the field **b**at $\boldsymbol{x}_0'$, yields

$$\begin{aligned}&\left[((\boldsymbol{x}' - \boldsymbol{x}_0') \cdot \boldsymbol{n})_-^{1/2} \boldsymbol{D}_{\boldsymbol{x}_0}\right]^{\sim} = \\ &= \Gamma(\tfrac{3}{2}) \left| 2\pi\, \xi' \cdot \boldsymbol{n} \right|^{-3/2} \exp\left(\tfrac{3}{4}\pi i - 2\pi i \xi' \cdot \boldsymbol{x}_0'\right) \delta\left(\xi' - (\xi' \cdot \boldsymbol{n})\boldsymbol{n}\right) \boldsymbol{D}_{\boldsymbol{x}_0}.\end{aligned} \tag{5.2}$$

At the same time the wave front of the surface traction field at the same point $\boldsymbol{x}_0' \in \partial\Omega$ in view of Eq.(4.1) takes the form (Trev [10])

$$\mathrm{WF}(\boldsymbol{Zb}) = (\boldsymbol{x}_0', \xi' \cdot \boldsymbol{n}) . \tag{5.3}$$

Multiplying both sides of Eq.(5.2) by symbol $\boldsymbol{Z}^{\sim}$ yields the desired asymptote for the outer stresses

$$\boldsymbol{t}\left((\boldsymbol{x}' - \boldsymbol{x}_0') \cdot \boldsymbol{n}\right) \;\sim\; (4\pi)^{-1} ((\boldsymbol{x}' - \boldsymbol{x}_0') \cdot \boldsymbol{n})_+^{-1/2} \boldsymbol{Z}^{\sim}(\boldsymbol{n}) \cdot \boldsymbol{D}_{\boldsymbol{x}_0} . \tag{5.4}$$

Equation (5.4) yields the desired equation between SIF and DIF

$$\boldsymbol{S}_{\boldsymbol{x}_0} = 2^{-1} (2\pi)^{-1/2} \boldsymbol{Z}^{\sim}(\boldsymbol{n}) \cdot \boldsymbol{D}_{\boldsymbol{x}_0} . \tag{5.5}$$

The constructed relation between vector-valued SIF ($\boldsymbol{S}_{\boldsymbol{x}_0}$) and vector-valued DIF ($\boldsymbol{D}_{\boldsymbol{x}_0}$) reveals that up to the invariant with respect to elastic properties multiplier $2^{-1} (2\pi)^{-1/2}$ the SIF and DIF are connected by the matrix symbol $\boldsymbol{Z}^{\sim}(\boldsymbol{n})$, solely dependent upon elastic properties of a medium. Finally, in view of Eqs (1.4), (5.5), matrix $\boldsymbol{M}(\boldsymbol{n})$ takes the form

$$\boldsymbol{M}(\boldsymbol{n}) = 2^{-1} (2\pi)^{-1/2} \boldsymbol{Z}^{\sim}(\boldsymbol{n}) . \tag{5.6}$$

6. Application to isotropic medium

For the case under consideration, the elasticity tensor denoted by $\boldsymbol{C}$ in terms of its 6×6 representation takes the form [17]

$$\boldsymbol{C} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda_2 & & & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{pmatrix}, \tag{6.1}$$

or in tensorial form

$$
\begin{aligned}
\boldsymbol{C} &= (\lambda + 2\mu)\boldsymbol{e}_k \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_k + \lambda \underbrace{\boldsymbol{e}_i \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_k}_{i \neq k} \\
&+ \mu \underbrace{\left(\boldsymbol{e}_i \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_i + \boldsymbol{e}_i \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_k\right)}_{i \neq k}
\end{aligned}
\tag{6.2}
$$

in Eq.(6.2) summation convention over repeated indices is used. Now, taking into account (6.2), Eq.(3.1), for the case under consideration takes the form

$$
\boldsymbol{Z}_t^{\sim}(\xi) = (2\pi)^2 \nu_y \cdot \boldsymbol{C}_t \cdot\cdot \xi \otimes \boldsymbol{E}_t^{\sim}(\xi) \otimes \xi \cdot\cdot \boldsymbol{C}_t \cdot \nu_x . \tag{6.3}
$$

7. Concluding remarks

A closed form relation between stress intensity factor (SIF) and displacement discontinuity at the crack front, called displacement intensity factor (DIF), is found by applying the method of integro-differential (pseudodifferential) operators connecting the inner asymptote of the displacement discontinuity field with the outer asymptote of the (outer) stress field at the crack front. The constructed relation is valid for an arbitrary shaped plane crack in a medium with arbitrary elastic anisotropy.

Moreover, relations (5.5), (5.6) reveal that the vector-valued SIF ($\boldsymbol{S}_{\boldsymbol{x}_0}$) and DIF($\boldsymbol{D}_{\boldsymbol{x}_0}$) are connected up to a scalar invariant multiplier $2^{-1}(2\pi)^{-1/2}$ by the matrix symbol $\boldsymbol{Z}^{\sim}(\boldsymbol{n})$, which is solely dependent upon elastic properties of the considered medium and direction of the unit normal to the plane crack front.

From the point of view of a researcher it is much more convenient to measure the displacement discontinuity field that does not have infinite values, and whose asymptotic value at the crack tip is finite (actually it is zero), thus considerably simplifyingthe computation procedure. According to the developed methodology, evaluating DIF ($\boldsymbol{D}_{\boldsymbol{x}_0}$) and applying formula (5.6) immediately yields the questioned SIF.

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